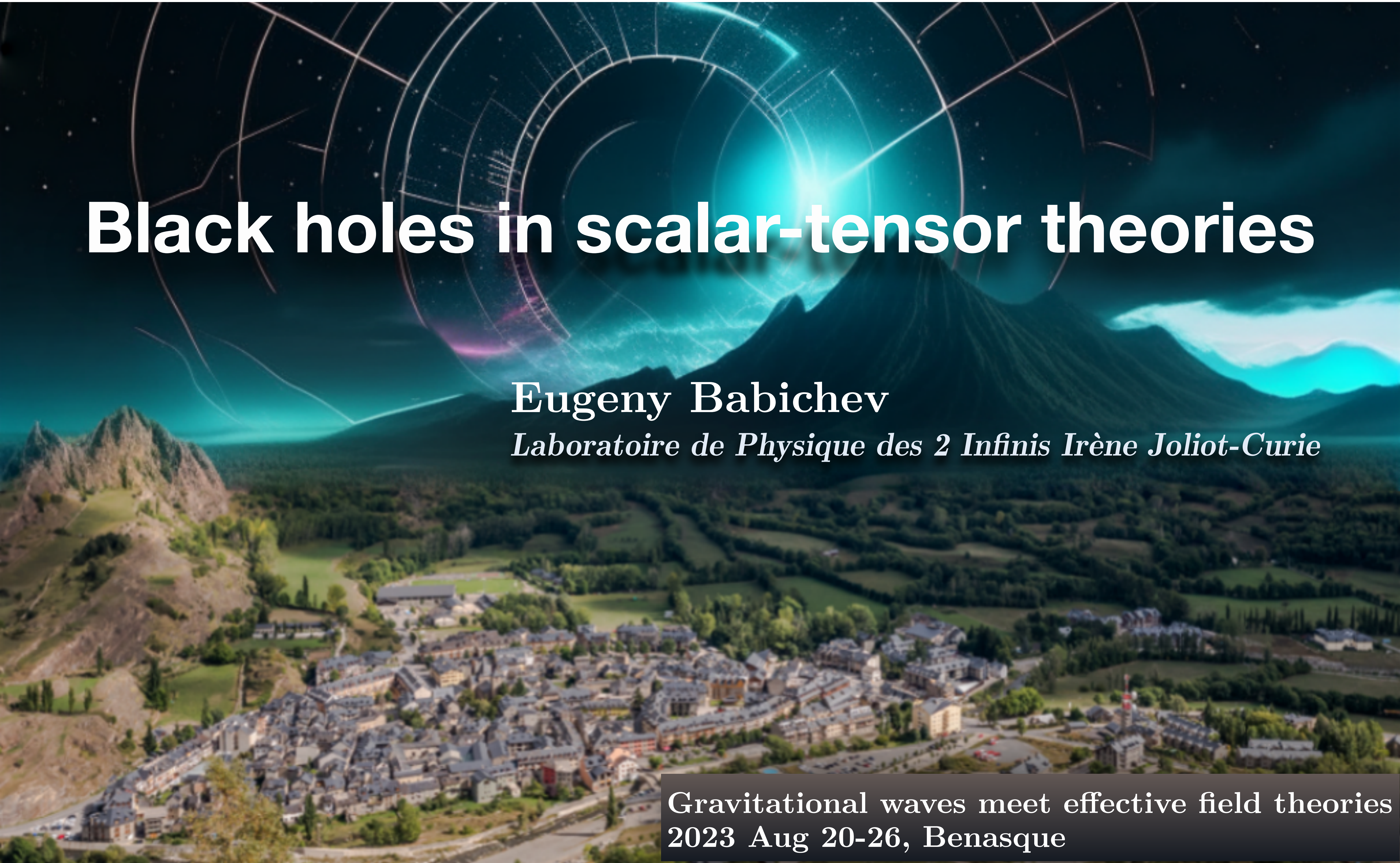


Black holes in scalar-tensor theories

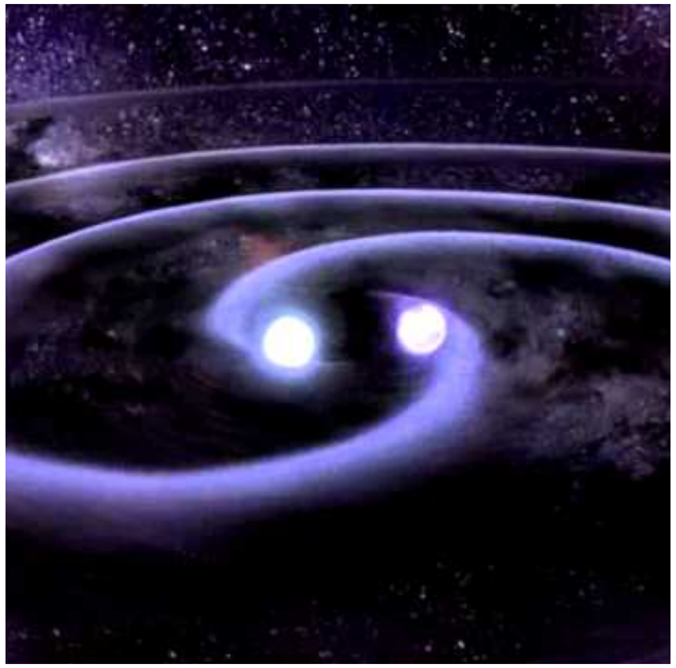
Eugeny Babichev

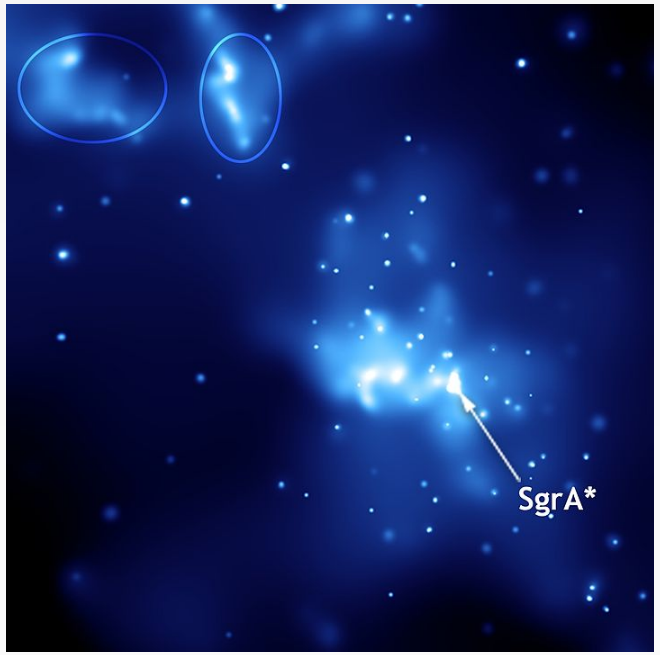
Laboratoire de Physique des 2 Infinis Irène Joliot-Curie

Gravitational waves meet effective field theories
2023 Aug 20-26, Benasque

Motivation

Observation of black holes and neutron stars: a breakthrough





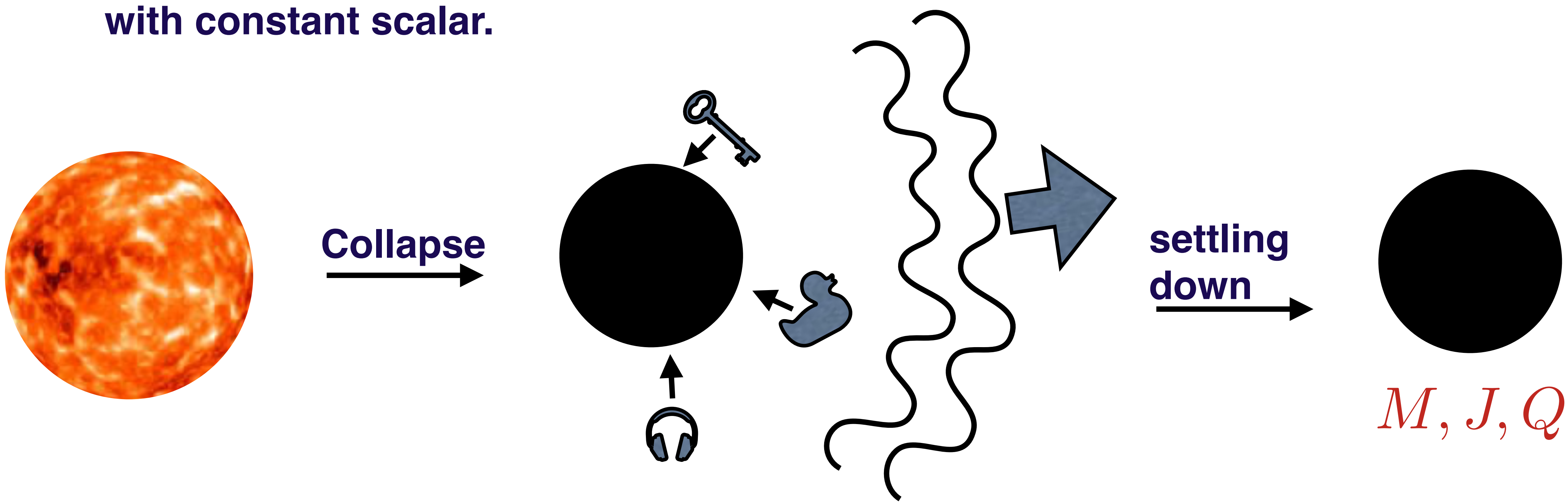
GW signals from binaries at their ringdown phase (LIGO/Virgo)	Image of M87 black hole with its light ring (from array of radio telescopes, EHT)	Observation of star trajectories orbiting SgrA central black hole (GRAVITY)

- Alternatives to GR black holes and stars as precise rulers of departure from GR?
- Other compact objects like wormholes?

No hairs in GR

- Gravitational collapse ->
- Black holes eat or expel surrounding matter
- Their stationary phase is characterised by a limited number of charges
- No details about collapse
- Black holes are bald

- **No hair theorems/arguments dictate that adding degrees of freedom lead to trivial (General Relativity) or singular solutions.**
- **E.g. in the standard scalar-tensor theories BH solutions are GR black holes with constant scalar.**

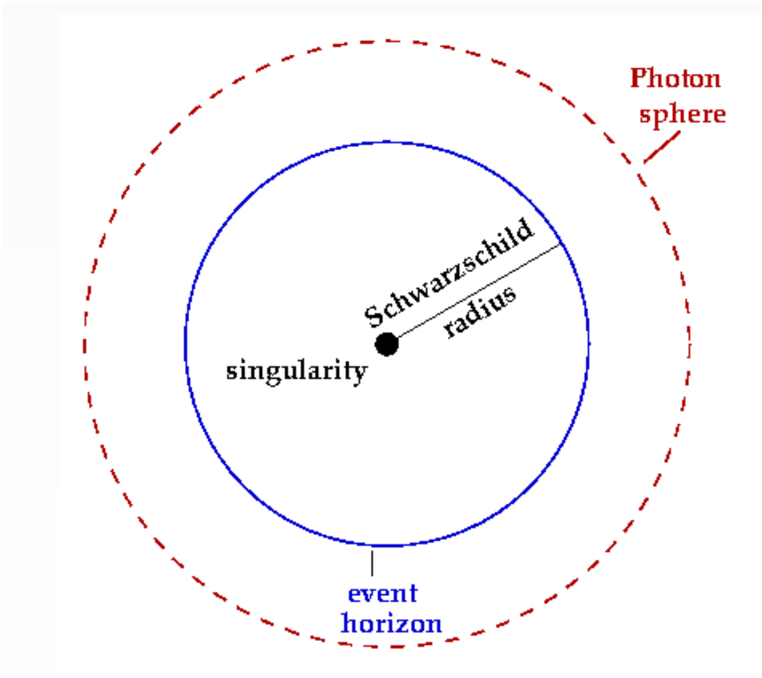


Schwarzschild solution

- Schwarzschild solution (static and spherically symmetric):

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad f(r) = 1 - \frac{2M}{r}$$

- The zero of $f(r)$ is the horizon of the black hole ($r_g = 2M$).
- An event horizon is a surface of no return. Nothing can escape the event horizon.
- An interior of the event horizon hides the curvature singularity at $r = 0$.



Kerr solution

Kerr'63

- Rotating vacuum black holes in General Relativity are described by the Kerr metric.

- In Boyer-Lindquist coordinates:

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 - \frac{4aMr\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\varphi + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\varphi^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2$$

where M is the mass, a is the angular momentum and

$$\rho^2 = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 + a^2 - 2Mr$$

- A ring singularity at $\rho = \sqrt{r^2 + a^2\cos^2\theta} = 0$, i.e.

$$r = 0 \qquad \text{and} \qquad \theta = \frac{\pi}{2}$$

Properties of the Kerr metric

- The metric is stationary and axi-symmetric, which corresponds to 2 Killing directions

$$\xi_{(t)} = \partial_t \qquad \text{and} \qquad \xi_{(\varphi)} = \partial_\varphi$$

- The spacetime is circular, i.e. symmetric under the reflection $(t, \varphi) \to (-t, -\varphi)$, because the Killing fields verify the condition

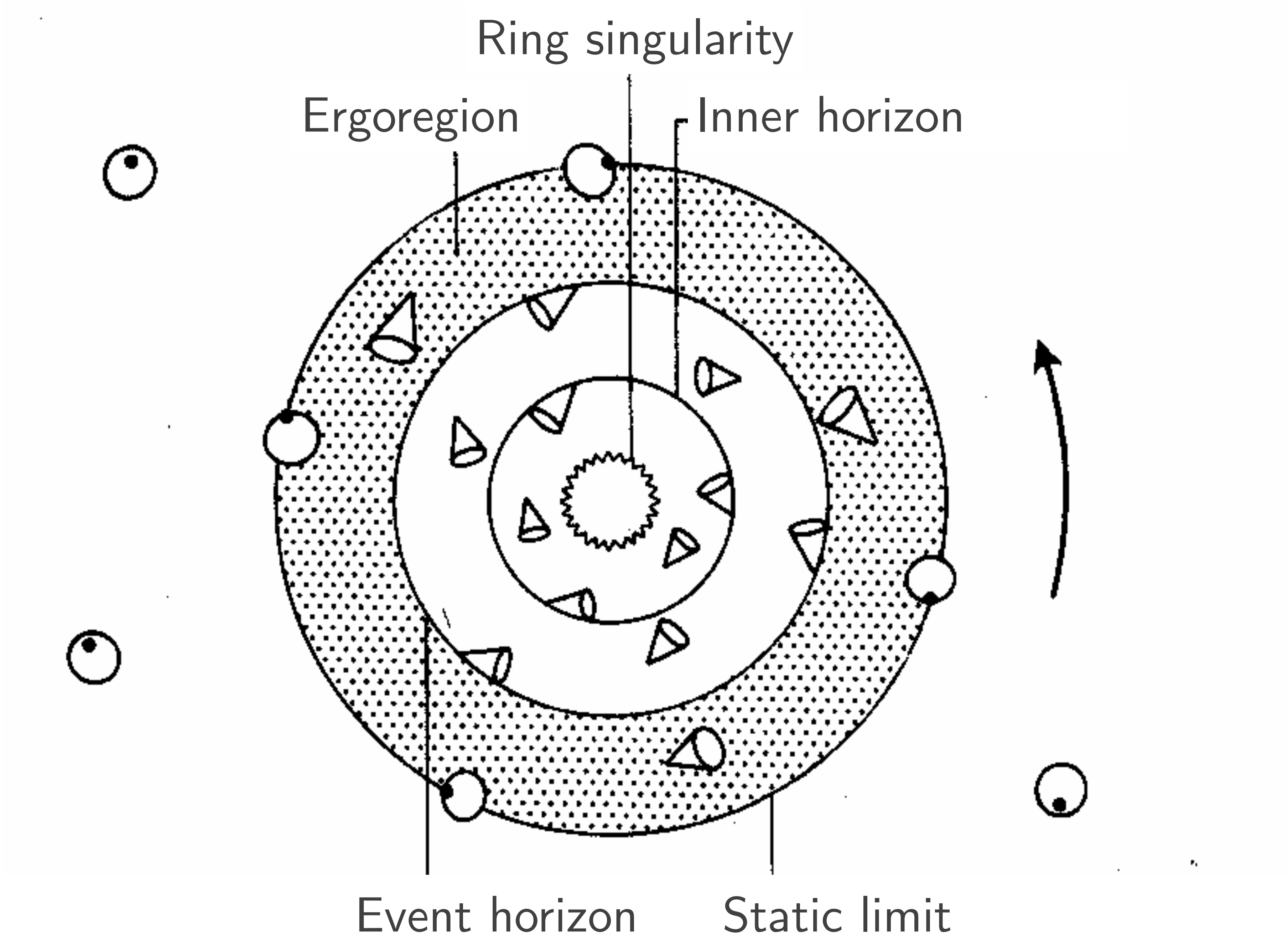
$$\xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(t)} = \xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(\varphi)} = 0 \ .$$

- The Kerr spacetime also admits a nontrivial Killing 2-tensor K verifying the equation

$$\nabla_{(\mu} K_{\nu\sigma)} = 0 \ .$$

- This defines a third nontrivial constant of motion along geodesics (Carter's constant). The geodesic equations thus reduce to a first order system.

Important surfaces in the Kerr metric



from d'Inverno's book

How to construct exact black hole solutions in modified gravity?

Shift-symmetry

Conformal symmetry

Stability of DHOST theory under conf/disformal transformation

Horndeski theory

Horndeski'1974

$$S = \int d^4x\, F\left[g, \partial g, \partial^2 g, \partial^3 g, ...\varphi, \partial\varphi, \partial^2\varphi, \partial^3\varphi, ...\right] \quad \Rightarrow \quad E[g, \partial g, \partial^2 g, \varphi, \partial\varphi, \partial^2\varphi] = 0$$

$$G_2(X,\phi),\ G_3(X,\phi),\ G_4(X,\phi),\ G_5(X,\phi)$$

$X = \partial_\mu\phi\partial^\mu\phi$

$$\mathcal{L}_2 = G_2\left(X,\phi\right)$$

$$\mathcal{L}_3 = G_3\left(X,\phi\right)\Box\phi$$

$$\mathcal{L}_4 = G_4(X,\phi)\,R + G_{4,X}(X,\phi)\left[(\Box\phi)^2 - (\nabla\nabla\phi)^2\right]$$

$$\mathcal{L}_5 = G_{5,X}\left(X,\phi\right)\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla\nabla\phi\right)^2 + 2\left(\nabla\nabla\phi\right)^3\right] - 6G_5\left(X,\phi\right)\, G_{\mu\nu}\nabla^\mu\nabla^\nu\phi$$

Deffayet+'09'11

Kobayashi+'11

Degenerate higher order Scalar-Tensor theories (DHOST)

Langlois&Noui, Crisostomi+'16

$$S = M_P^2 \int \mathrm{d}^4x\sqrt{-g}\left(f(\phi,X)R + K(\phi,X) - G_3(\phi,X)\Box\phi + \sum_{i=1}^{5} A_i(\phi,X)\mathcal{L}_i\right) + S_\mathrm{m}\left[g_{\mu\nu},\psi_\mathrm{m}\right]$$

$$\mathcal{L}_1 = \phi_{\mu\nu}\phi^{\mu\nu}, \quad \mathcal{L}_2 = (\Box\phi)^2, \quad \mathcal{L}_3 = \phi_{\mu\nu}\phi^\mu\phi^\nu\Box\phi,$$
$$\mathcal{L}_4 = \phi_\mu\phi^\nu\phi^{\mu\alpha}\phi_{\nu\alpha}, \quad \mathcal{L}_5 = (\phi_{\mu\nu}\phi^\mu\phi^\nu)^2$$
$$X = \phi^\mu\phi_\mu$$

- One subclass of DHOST (subclass Ia) is phenomenologically interesting [Langlois, Noui; Crisostomi+'16]:

$$A_2 = A_2\left(A_1, A_3\right)$$
$$A_4 = A_4\left(A_1, A_3\right)$$
$$A_5 = A_5\left(A_1, A_3\right)$$

From DHOST to DHOST

Under a disformal transformation

$$g_{\mu\nu} \to C(\phi, X) g_{\mu\nu} + D(\phi, X) \partial_\mu \phi \partial_\nu \phi$$

DHOST ⇒ DHOST

[Achour+,Crisostomi+'16]

More precisely,

$$S_{DHOST}[\tilde{g}_{\mu\nu}, \phi] = S_{DHOST}[g_{\mu\nu} + D(X)\partial_\mu \phi \partial_\nu \phi, \phi] = \bar{S}_{DHOST}[g_{\mu\nu}, \phi]$$

Theory 1

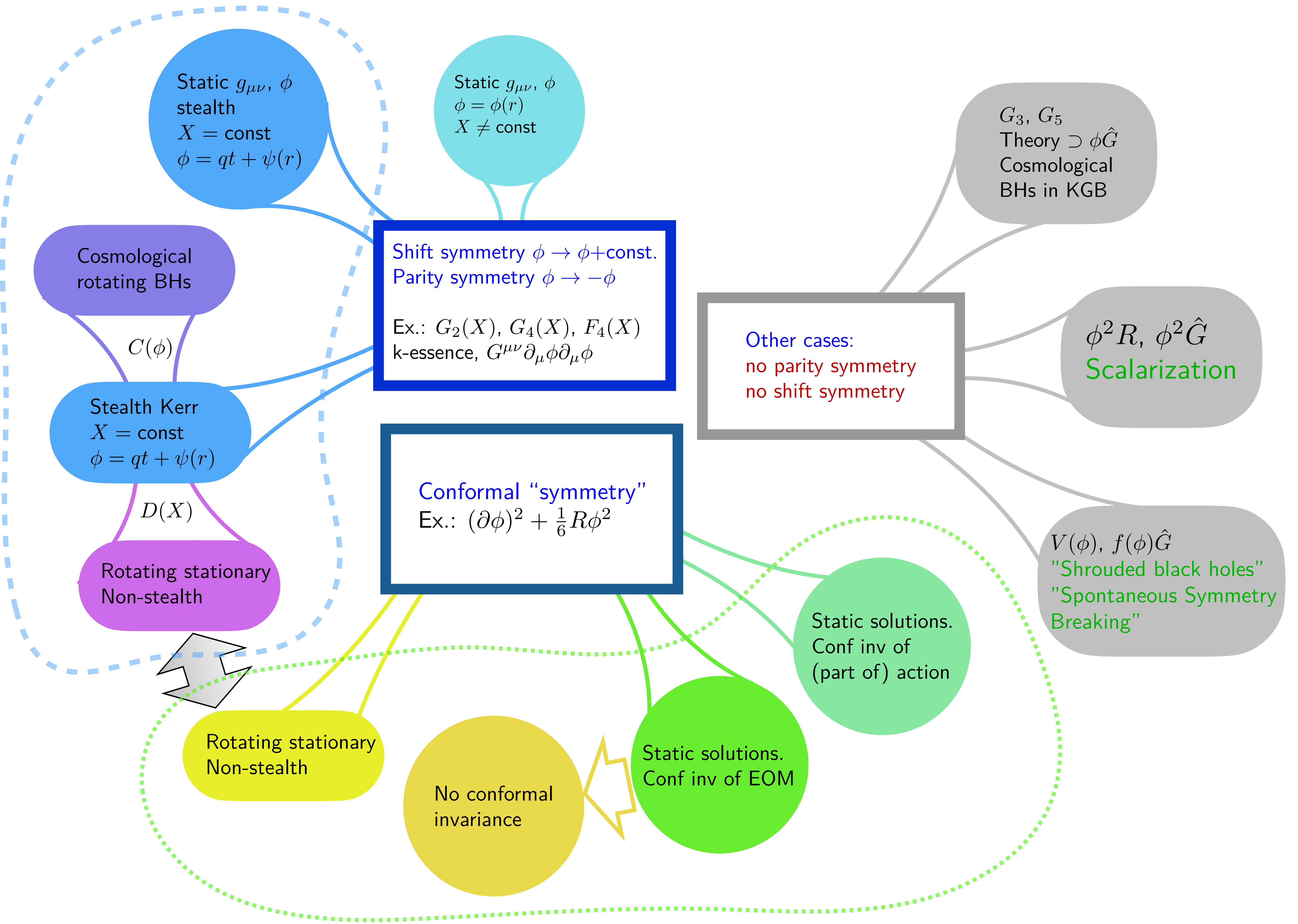
$$\bar{S}_{DHOST}[g_{\mu\nu}, \phi] + S_{\text{matter}}[g_{\mu\nu}, \Psi_{\text{matter}}]$$

Theory 2

$$S_{DHOST}[\tilde{g}_{\mu\nu}, \phi] + S_{\text{matter}}[\tilde{g}_{\mu\nu}, \Psi_{\text{matter}}]$$

Coupling to matter is different: we get a different theory

Hairy solutions in ST theories



Shift symmetry

Static configurations in shift symmetric case

- Shift symmetry of the theory implies conserved current

$$\nabla_\mu J^\mu = 0,$$

where $J^\mu = \frac{\delta S}{\delta(\partial_\mu \phi)}$.

- Assumption of staticity of both metric and scalar field then leads to automatic first integral

$$J^r = \text{const},$$

where const$=$0 usually to avoid divergence of the norm of the current J^2 at the horizon.

$X \neq$ const

- Static solutions in theory with "John" term $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ [Rinaldi'12; Anabalon+'13; Minamitsuji'13; EB, Charmousis'13] divergence of the scalar at the horizon, inside the horizon the scalar solution does not exist.

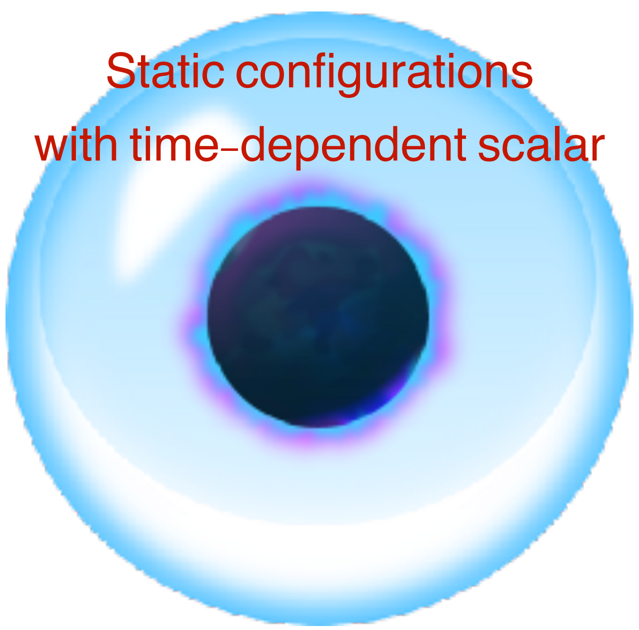
- Static and time-dependent solutions in the theories that evade a no-hair theorem [EB, Charmousis & Lehébel'17].

Shift symmetry and time dependence

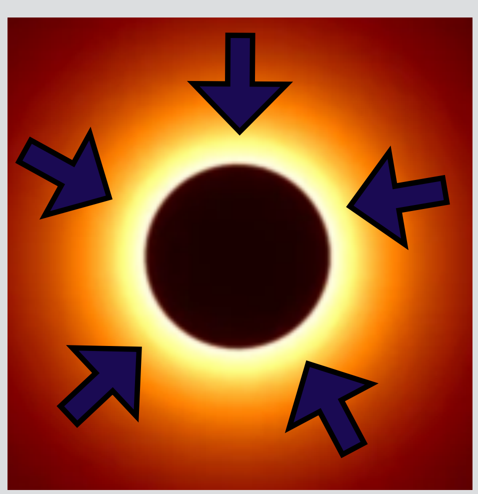
$$S = \int d^4 \mathcal{L}\left(g_{\mu\nu}, \partial g_{\mu\nu}, ..., \cancel{\phi}, \partial\phi, \partial^2\phi, ...\right) \Rightarrow$$

The ansatz $\phi = qt + ...$ goes through EOMs, leaving no t-dependence (only q).

$T_{\mu\nu}$ is t-independent.



Similar ideas



Accretion of perfect fluid in test-field approximation $\phi = qt + ...$, where ϕ is "scalar potential"

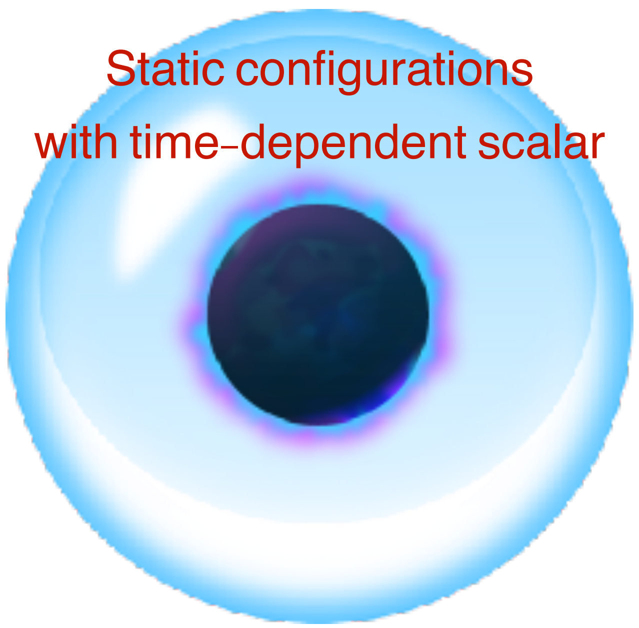
Boson stars, Kerr BHs with complex scalar hair $\phi = f(r, \theta) e^{-i(\omega t - n\varphi)}$

Shift symmetry and time dependence

$$S = \int d^4 \mathcal{L}\left(g_{\mu\nu}, \partial g_{\mu\nu}, \ldots, \cancel{\phi}, \partial\phi, \partial^2\phi, \ldots\right)$$

⇨ The ansatz $\phi = qt + \ldots$ goes through EOMs, leaving no t-dependence (only q).

$T_{\mu\nu}$ is t-independent.



Shift symmetry of the theory implies conserved current $\nabla_\mu J^\mu = 0$. Need to impose

$$J^r = 0$$

because $J^r \propto E^t_r$.

Linear time-dependence $\phi = qt + \psi(r, \theta)$:

- **Possibility to build non-trivial solutions**
- **Matching to cosmology**
- **Static (stationary) metric**

Example of exact solution

EB, Charmousis'13

- Subclass of Horndeski theory:

$$S = \int d^4x\sqrt{-g}\,(R - 2\Lambda - \eta X + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)$$

- Simple (stealth) solution reads

$$f = h = 1 - \frac{2M}{r} + \frac{\eta}{3\beta}r^2, \quad \phi = qt \pm \int dr\frac{q}{h}\sqrt{1-h}.$$

Secondary hair $q^2 = \frac{\zeta\eta + \Lambda\beta}{\beta\eta}$

- $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -q^2$ is constant for such solutions [Kobayashi&Tanahashi'14]. Leads to nice generalization to include arbitrary G_2 and G_4.

- Also there are further generalisations to beyond Horndeski, DHOST.

Example of exact solution

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Secondary hair $q^2 = \frac{\zeta\eta+\Lambda\beta}{\beta\eta}$

- Disformal transformation $g_{\mu\nu} \to g_{\mu\nu} + D(X)\phi_\mu\phi_\nu$, e.g. to get the speed of gravity = speed of light [EB, Charmousis, Esposito-Farèse, Lehébel]:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{\beta}{\zeta + \frac{\beta}{2}\varphi_\lambda^2}\varphi_\mu\varphi_\nu$$

A coordinate change shows that $\mathcal{D}$(spherical stealth) = spherical stealth

Geodesics in Kerr and Carter constant

- Hamilton-Jacobi equation for geodesics:

$$\frac{dS}{d\lambda} = g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = -m^2$$

- We have 3 obvious constants of motion, energy E, angular momentum L_z and the mass of the particle m.
  The HJ functional is written: $S = -Et + L_z \varphi + S(r, \theta)$.

- B. Carter demonstrated that $S(r,\theta) = S_r(r) + S_\theta(\theta)$ and:

$$S_r = \pm \int \frac{\sqrt{R}}{\Delta} dr, \quad S_\theta = \pm \int \sqrt{\Theta} d\theta$$

$$R(r) = \left[E\left(r^2 + a^2\right) - aL_z\right]^2 - \Delta\left[\mathcal{Q} + (aE - L_z)^2 + m^2 r^2\right]$$

$$\Theta(\theta) = -\sin^2\theta \left(aE - \frac{L_z}{\sin^2\theta}\right)^2 + \left[\mathcal{Q} + (aE - L_z)^2 - m^2 a^2 \cos^2\theta\right]$$

- The 4th constant of integration $\mathcal{Q}$ is Carter's constant.

Rotating solution?

Charmousis+'19

- The idea is to associate the scalar ϕ with the geodesics in Kerr space.
- Hamilton-Jacobi equation

$$g^{\mu\nu}_{\mathsf{Kerr}}\partial_\mu S \partial_\nu S = -m^2$$

- If we assume for the scalar $X = g^{\mu\nu}_{\mathsf{Kerr}}\partial_\mu \phi \partial_\nu \phi = -q^2$ (like in spherical symmetry), one can look for the solution $\phi = S$.
- Ensure that there is no backreaction so Kerr solution remains to be valid. Restricts considerably the class of the DHOST theories.
- Choose geodesics such that ϕ is regular everywhere (at least outside the horizon). Fix constants of integration of geodesics.

Stealth Kerr solution in DHOST

Charmousis+'19

- A stealth Kerr solution, where the metric is Kerr and the scalar field such that

$$g = g_{\text{Kerr}}$$
$$X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = X_0 = \text{const.}$$
$$\phi = q\left[t + \int \frac{\sqrt{2Mr(a^2+r^2)}}{\Delta}\mathrm{d}r\right]$$

- The metric g_{Kerr} is regular everywhere apart from the ring singularity and
- The scalar field is regular at $r > 0$.

Cosmological black holes

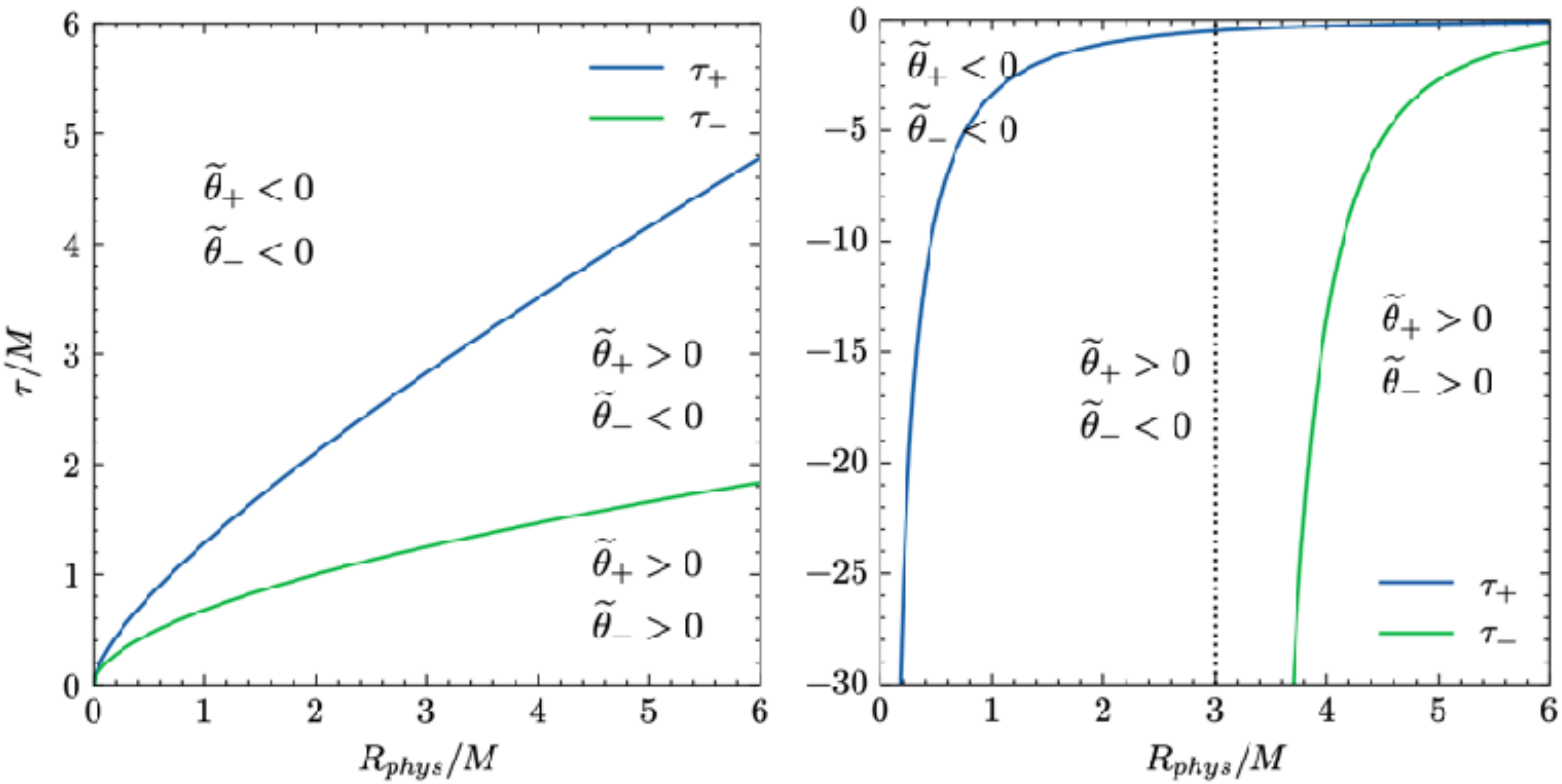
EB, Charmousis, Lecoeur'23

- Time-dependent solutions with $\phi = qt + \psi(r,\theta)$ with flat asymptotic: $g_{\mu\nu} \to \eta_{\mu\nu}$ and $\phi = qt$ as $r \to \infty$.
- Perform a conformal transformation of the solution $g_{\mu\nu} \mapsto \widetilde{g}_{\mu\nu} = C\left(\phi\right) g_{\mu\nu}$. $\mathcal{C}$(DHOST) $=$ DHOST.
- ϕ plays a role of conformal time of expanding universe: asymptotically $\eta_{\mu\nu} \mapsto \widetilde{g}_{\mu\nu} = C\left(\phi\right) \eta_{\mu\nu}$ with $C(\phi) \equiv a^2_{\text{FLRW}}(\phi)$.
- Choice of C corresponds to a cosmological evolution.
- Regular ϕ (at the horizon) leads to regular resulting conformal solution.
- Black hole embedded in FLRW universe.

Cosmological black holes

EB, Charmousis, Lecoeur'23

- Non-stationary metrics, treat in terms of trapping (apparent) horizons.
  Expansions, null geodesics congruences: 2+2 formalism by [Hayward'94].
- Spherically symmetric case: the seed metric is that of [Charmousis+'19] with zero rotation parameter:
  [Culetu'12] spacetime, partially treated in [Sato,Maeda,Harada'22].



Rotating case is more complicated

Seed Kerr-dS is another generalization

Disformed Kerr black hole

Anson, EB, Charmousis, Hassaine'20
[see also Achour+'20]

- Starting from the stealth Kerr solution, we perform the transformation:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu}^{(\text{Kerr})} - \frac{D}{q^2}\,\partial_\mu\phi\,\partial_\nu\phi, \qquad \phi = q\left[t + \int \frac{\sqrt{2Mr(a^2+r^2)}}{\Delta}\mathrm{d}r\right]$$

where D and q are constants.

- The line element is now

$$\mathrm{d}\tilde{s}^2 = -\left(1 - \frac{2\tilde{M}r}{\rho^2}\right)\mathrm{d}t^2 - 2D\frac{\sqrt{2\tilde{M}r(a^2+r^2)}}{\Delta}\mathrm{d}t\mathrm{d}r + \frac{\rho^2\Delta - 2\tilde{M}(1+D)rD(a^2+r^2)}{\Delta^2}\mathrm{d}r^2$$
$$- \frac{4\sqrt{1+D}\tilde{M}ar\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\varphi + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\varphi^2 + \rho^2\mathrm{d}\theta^2$$

with $\tilde{M} = M/(1+D)$ and the rescaling $t \to \sqrt{1+D}t$

Disformed Kerr black hole

Anson, EB, Charmousis, Hassaine'20

- The solution is not Ricci-flat, but the only singularity is at $\rho = 0$, like Kerr.
- Non-circular space-time, meaning that the metric cannot be written in a form that is invariant under $(t, \varphi) \to (-t, -\varphi)$
- The spacetime is globally causal, since there is $\phi(t, r)$ which serves as a global time.
- Asymptotically, the disformal metric approaches Kerr

$$\mathrm{d}\tilde{s}^2 = \mathrm{d}s^2_{\mathrm{Kerr}} + \frac{D}{1+D}\left[\mathcal{O}\left(\frac{\tilde{a}^2\tilde{M}}{r^3}\right)\mathrm{d}T^2 + \mathcal{O}\left(\frac{\tilde{a}^2\tilde{M}^{3/2}}{r^{7/2}}\right)\alpha_i \mathrm{d}T\mathrm{d}x^i + \mathcal{O}\left(\frac{\tilde{a}^2}{r^2}\right)\beta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]$$

- There are *three* important surfaces: static limit (egrosphere), stationary limit and the event horizon (in case of Kerr spacetime the two latter coinside).

Disformed Kerr black hole

Anson, EB, Charmousis, Hassaine'20

- *Ergosphere* (static limit): static timelike observers can no longer exist, the Killing vector $l^\mu = (1,0,0,0)$ becomes null. I.e. $\tilde{g}_{tt} = 0$, or

$$\tilde{g}_{tt} = 0 \quad \Rightarrow \quad r_E = \tilde{M} + \sqrt{\tilde{M}^2 - a^2 \cos^2\theta}$$

- Stationary observers, i.e. constant (r,θ), with a 4-velocity $u = \partial_t + \omega \partial_\varphi$. They exist if $u^2 \leq 0$, they cease to exist at the surface $\tilde{g}_{tt}\tilde{g}_{\varphi\varphi} - \tilde{g}_{t\varphi}^2 = 0$, i.e.

$$P(r,\theta) \equiv r^2 + a^2 - 2\tilde{M}r + \frac{2\tilde{M}Da^2 r \sin^2\theta}{\rho^2(r,\theta)} = 0$$

$P(R_0(\theta),\theta) = 0$ is the *stationary limit*. Cannot be the event horizon for since it is not null.

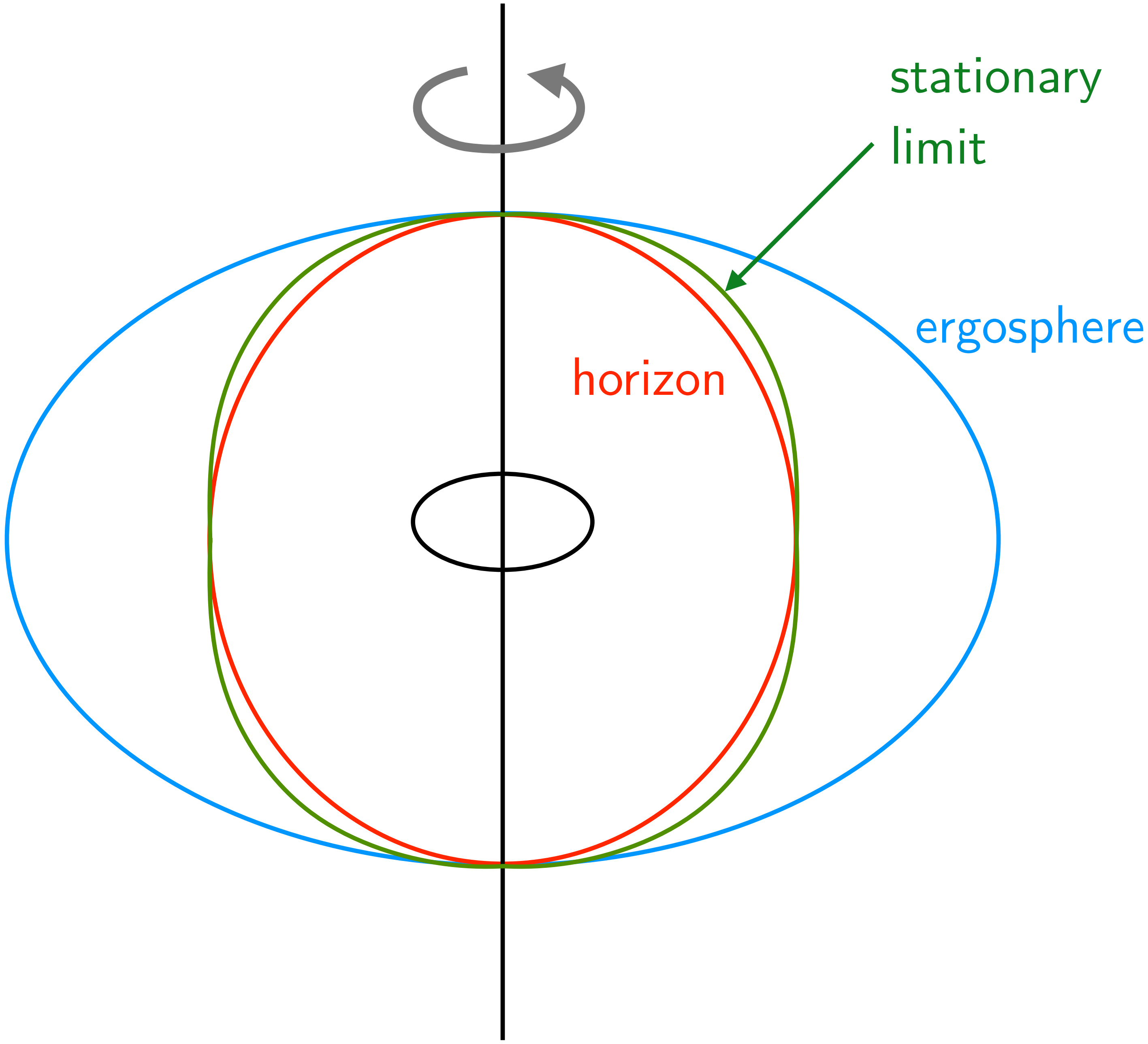
- *Event horizon*

$$R'(\theta)^2 + P(R,\theta) = 0$$

To have a smooth solution, we must impose

$$R'(0) = R'\left(\frac{\pi}{2}\right) = 0.$$

Disformed Kerr metric



Conformal symmetry

BBMB solution

Bocharova, Bronnikov, Melnikov'70; Bekenstein'74

- Scalar field with non-minimal coupling:

$$S = \int d^4x \sqrt{-g} \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} R \phi^2 \right)$$

- The BBMB solution is

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} + r^2 d\Omega^2, \quad \phi = \pm \frac{M}{r - M}$$

- Properties: Metric of the extremal Reissner-Nordstrom; scalar diverges at $r_h = M$; it is unique; hair with the choice $\pm$ due to the discrete symmetry $\phi \to -\phi$.

BBMB solution

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- Properties: Metric of the extremal Reissner-Nordstro scalar diverges at $r_h = M$; it is unique; hair with the choice $\pm$ due to the discre metry $\phi \to -\phi$.

- The key in finding the solution is in the conformal invariance of the scalar part of the action, $g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}$ and $\phi \to e^{-\sigma}\phi \Rightarrow S_\phi \to S_\phi +$ b.t.
As a consequence of the invariance

$$R = 0 \quad \text{(pure geometric constraint)}$$

$$\Box\phi = \frac{1}{6}R\phi \quad \Rightarrow \Box\phi = 0 \quad \text{(first integral)}$$

- This allows to derive the most general asymptotically flat solution [Xanthopoulos & Zannias'91]

MTZ solution

Martinez, Troncoso, Zanelli'03

- Self-interacting scalar:

$$S = \int d^4x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}R\phi^2 - \alpha\phi^4\right)$$

- The MTZ solution is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad f(r) = \left(1 - \frac{M}{r}\right)^2 - \frac{\Lambda}{3}r^2, \quad \phi = \pm\frac{M}{r-M}$$

- BH solution for $\Lambda > 0$ provided that $\alpha = -\Lambda/72$.

- There is a geometric constraint as well:

$$R = 4\Lambda \quad \text{(pure geometric constraint)}$$

$$\Box\phi = \frac{1}{6}R\phi + 4\alpha\phi^3 \quad \Rightarrow \Box\phi \neq 0 \quad \text{(not a first integral)}$$

Lessons from above solutions

Two key ingredients that help to find exact solutions:

1. Pure geometric constraint (thanks to conformal invariance of the scalar field action). Restricts the allowed possible spacetimes

2. Scalar equation is simple to integrate

However the requirement of conformal invariance can be relaxed to ask for conformal invariance of scalar EOM.

Generalization of the action

Lu-Pang'20, Fernandes'21

- Generalized action:

$$S = \int d^4x\sqrt{-g}\left\{R - 2\Lambda - 6\beta\left((\partial\phi)^2 + \frac{1}{6}R\phi^2\right) - 2\lambda\phi^4 \right.$$
$$\left. -\alpha\left[\ln(\phi)\mathcal{G} - \frac{G^{\mu\nu}\phi_\mu\phi_\nu}{\phi^2} - \frac{4\Box\phi\,(\partial\phi)^2}{\phi^3} + \frac{2\,(\partial\phi)^4}{\phi^4}\right]\right\}$$

where $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ is the Gauss-Bonnet invariant.

- The $\alpha-$ contribution breaks the conformal invariance of the action for the scalar. The scalar field equation remains conformally invariant.

- Look for the solution

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad \phi = \phi(r).$$

Geometric constraint from conformal EOM

1 Conformal invariance of the scalar EOM $\Rightarrow$ pure geometric constraint:

$$R - 2\Lambda + \frac{\alpha}{2}\mathcal{G} = 0$$

From which the solution for $f(r)$ immediately follows:

$$f(r) = 1 + \frac{r^2}{2\alpha}\left[1 \pm \sqrt{1 + 4\alpha\left(\frac{2M}{r^3} - \frac{q}{r^4} + \frac{\Lambda}{3}\right)}\right]$$

Geometric constraint comes from conformal symmetry of the scalar EOM, without conformal invariance of the scalar action.

Non-Noetherian scalar field
[Ayon-Beato & Hassaine'23]

Geometric constraint from conformal EOM

2 Scalar field equation is has a "simple" form to integrate (assuming $\alpha \neq 0$):

$$\left(\frac{\phi'}{\phi^2}\right)'\left(f\left[(r\phi)'\right]^2 - \phi^2\left(1+\frac{\beta}{2\alpha}r^2\phi^2\right)\right) = 0.$$

Two disconnected branches of solutions [Fernandes'21]

Extensions: [Babichev, Charmousis, Hassaine Lecoeur'22]

- Slowly rotating solutions
- Radiating solutions (Vaidya-like)
- Wormholes by disformal transformation
- Gravitational monopole-like solution

Rotating solution

Fernandes'23

- Kerr-Schild ansatz:

$$ds^2 = ds^2_{\text{flat}} + H(\mathbf{x}) \left(l_\mu dx^\mu\right)^2 ,$$

where H is a scalar (to look for) and l^μ is the tangent vector to a geodesic null congruence.

- The solution contains arbitrary functions $M(\theta)$ and $q(\theta)$ (a sign of strong coupling?)

Very similar to the disformed Kerr solution:

- Non-circular
- The horizon is given by a similar equation.

No symmetry (but simple scalar EOM?)

EB, Charmousis, Hassaine & Lecoeur '23

Give up the requirement of the symmetries?
But construct a theory that yields a similar scalar field equation with factorization.

$$S=\int \mathrm{d}^4x\sqrt{-g}\left\{(1+W(\phi))R-\frac{1}{2}V_k(\phi)(\nabla\phi)^2+Z(\phi)+V(\phi)\mathcal{G}+V_2(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right.$$
$$\left.+V_3(\phi)(\nabla\phi)^4+V_4(\phi)\Box\phi(\nabla\phi)^2\right\}.$$

The case before corresponds to

$$W=-\beta\mathrm{e}^{2\phi},\quad V_k=12\beta\mathrm{e}^{2\phi},\quad Z=-2\lambda\mathrm{e}^{4\phi}-2\Lambda,\quad V=-\alpha\phi,\quad V_2=4\alpha=V_4,\quad V_3=2\alpha.$$

Look for the solution

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\Omega^2,\quad \phi=\phi(r).$$

No symmetry (but simple scalar EOM?)

EB, Charmousis, Hassaine & Lecoeur '23

Give up the requirement of the symmetries?
But construct a theory that yields a similar scalar field equation with factorization.

$$S=\int \mathrm{d}^4x\sqrt{-g}\left\{(1+W(\phi))R-\frac{1}{2}V_k(\phi)(\nabla\phi)^2+Z(\phi)+V(\phi)\mathcal{G}+V_2(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right.$$
$$\left.+V_3(\phi)(\nabla\phi)^4+V_4(\phi)\Box\phi(\nabla\phi)^2\right\}.$$

- The combination $E_t^t-E_r^r=0$ can be factorized:

$$\left[\frac{\phi''}{(\phi')^2}-1\right]\left[r^2W_\phi+4(1-f)V_\phi+2frV_2\phi'+fr^2V_4(\phi')^2\right]=0,$$

provided specific relations between the potentials (still leaving 3 arbitrary potentials at this step Z, V and W).

- Fix the potentials Z, V and W so that the remaining 2 equations admit the solution for $f=f(r)$

Conclusions

- Use symmetries of gravity theories to construct analytic solutions.
- Shift symmetry of a theory leads to a conserved current.
- Conformal symmetry leads to a geometric constraint.
- General disformal transformation as a way to construct new solutions.