Double Graviton Eikonal Dressing and $\mathcal{O}(G_N^2)$ Classical Observables

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ongoing work with Karan Fernandes

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Motivation: Amplitude for GWs

- The binary black hole inspiral dynamics can be analytically addressed through effective field theories (EFTs).
- Scattering amplitudes can provide the matching of Wilson coefficients of EFTs. E.g., Mariana's talk!
- Combining both yields EFTs at high PM and PN orders for predicting precise GW waveforms for observations. E.g., Fillipo's talk!

- Another route: Use scattering amplitudes to evaluate the classical observables such as waveforms, scattering angles, loss of momentum & angular momentum.

KMOC & Eikonal operator

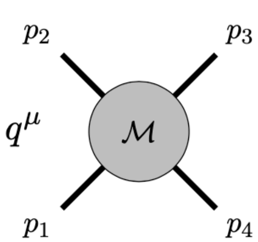
Two ways of evaluating classical observables from amplitudes

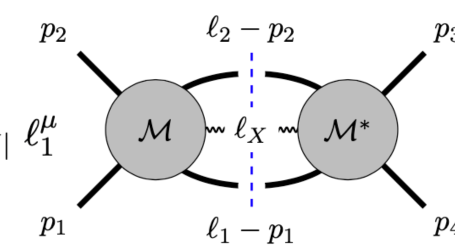
1. KMOC ('18, '19): Relate the observable to scattering amplitudes, $\langle \Delta O \rangle = \langle in \;\; S^{\dagger}[O,S] \;\; in \rangle = i\langle in \;\; [O,T] \;\; in \rangle + \langle in \;\; T^{\dagger}[O,T] \;\; in \rangle$, and take the classical (on-shell point particle) limit, i.e., $\ell_c \ll \ell_w \ll b$

This generally leads to the expression as Fourier transform of amplitude kernel w.r.t. the transverse/soft momentum exchange.

E.g.,

$$\Delta p_1^{\mu,(1)} = i\frac{g^4}{4}\left\langle\!\!\left\langle \int \hat{d}^4\bar{q}\;\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\;e^{-ib\cdot\bar{q}}\;\mathcal{I}^{\mu}_{p_1}\right\rangle\!\!\right\rangle$$

$\mathcal{I}_{p_1,\mathrm{v}} = q^{\mu}$ 

$,\mathcal{I}_{p_1,\mathrm{r}} = -\mathrm{i}\sum_X \int \mathrm{d}\widetilde{\Phi}_{2+|X|}\;\ell_1^{\mu}$ 

2. Eikonal operator (Di Vecchia et al. '22, '23): Introduce the soft-graviton dressing S-matrix to evaluate vev of observable.

$$S \longrightarrow S(\sigma, q; a, a^{\dagger}) = e^{-\hat{\Delta}_{\mathrm{s.d.}}} e^{2i\mathrm{Re}\delta}\,,$$

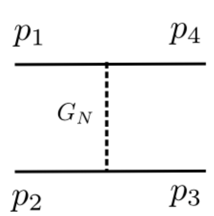
$\sigma = -\frac{p_1\cdot p_2}{m_1 m_2} = \frac{1}{\sqrt{1-v_{12}^2}}$, b = impact parameter

Eikonal Approach

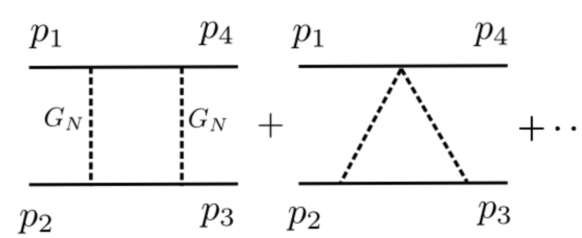
- Classical physic from a 2-2 scattering amplitude $A_4(\sigma, q^2)$ is encoded in the eikonal phase: $e^{2i\delta(\sigma,b)} \overset{\hbar\to 0}{=} 1 + i\int d^4q\delta(2p_1\cdot q)\delta(2p_2\cdot q)e^{-iq\cdot b/\hbar}A_4(\sigma, q^2).$

  Scattering angle or Shapiro time delay can be derived from $\delta(\sigma, b)$.

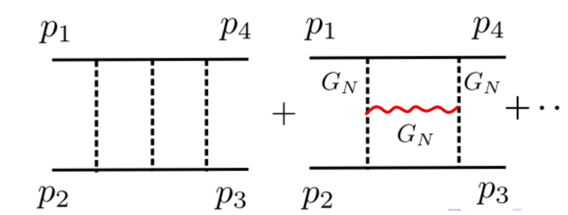
$\delta_0 \sim G_N$



$\delta_1 \sim G_N^2$

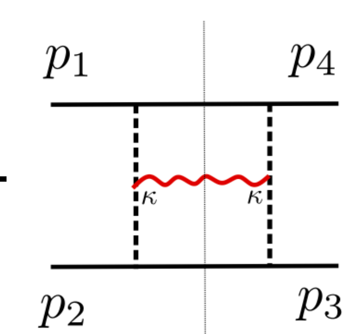


$\delta_2 \sim G_N^3$



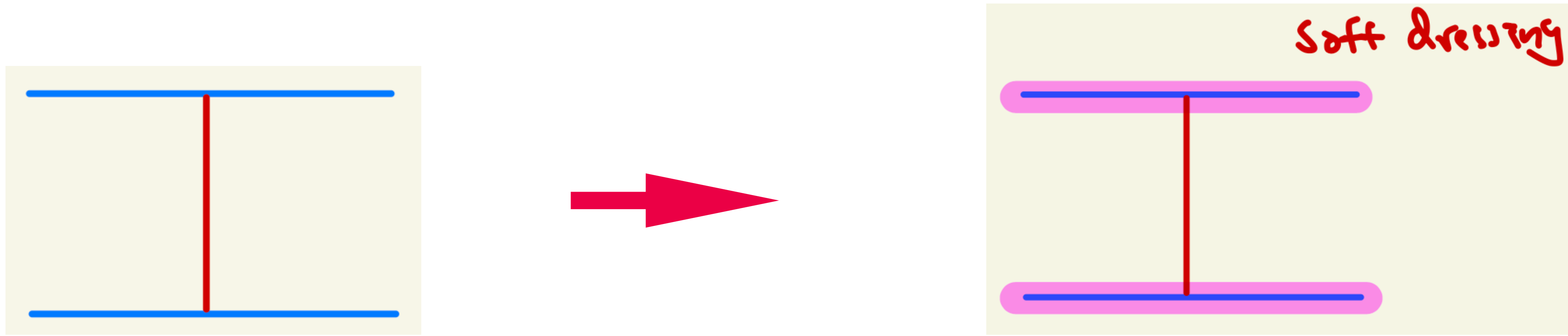
- At 2-loop, $\mathrm{Im}\delta_2 \neq 0$ and divergent in the ultra relativistic limit due to radiative exchange. This implies that the elastic process is suppressed when the inelastic channels (cut diagram) open up.

- This can be fixed by including the emission of soft gravitons, i.e., Di Vecchia et al. '21: $\lim_{\epsilon\to 0}\mathrm{Re}\ 2\delta_2^{\mathrm{r.r.}} = \lim_{\epsilon\to 0}[-\epsilon\pi\mathrm{Im}\ 2\delta_2] = \frac{dE^{\mathrm{rad}}}{2\hbar d\omega}(\omega\to 0)$



ϵ = Dim . Reg . cutoff .

Soft-dressing states



- Due to the IR/macroscopic feature of soft emission, it was proposed by Veneziano et al. to promote $e^{\mathrm{Im}2\delta}$ into a coherent-state operator: $e^{-\hat{\Delta}_{\text{s.d.}}}$ with $\hat{\Delta}_{\text{s.d.}} \sim \frac{1}{\hbar}\int d^3k[a_i(k)f_i^*(k) - a_i^\dagger f_i(k)]$.
- As suggested by Weinberg's classic discussion of canceling the IR divergence, $f_i(k)$ should take the form of Weinberg's soft factor.
- The classical observables can be obtained as $\langle \hat{Q} \rangle_c = \langle 0 \;\; e^{\hat{\Delta}_{\text{s.d.}}} \hat{Q} e^{-\hat{\Delta}_{\text{s.d.}}} \;\; 0 \rangle$, Di Vecchia et al, '22 & '23.
- In this talk, we will extend the eikonal operator by including the next-leading correction by double gravitons.

Classical Observables with soft dressing

- Few such observables we will consider, up to $\mathcal{O}(G_N^2)$:

1. Asymptotic Waveform $\sim \dfrac{1}{r}\sum_n \left[\mathcal{K}_1 + \dfrac{\mathcal{K}_2}{u}\right]\Theta(\eta_n u)$, $\mathcal{K}_{1,2}$ are the Lorentz-covariant kinematic factors at $\mathcal{O}(G_N^{1,2})$, respectively.
2. Radiation-reaction: $\lim_{\epsilon\to 0}[-\epsilon\pi \mathrm{Im}\ 2\delta_{2,3}] \stackrel{?}{=} \dfrac{dE^{\mathrm{rad}}}{2\hbar d\omega}(\omega \to 0)$. These can be further adopted to calculate the scattering angle.
3. Radiated linear momentum and angular momentum (in progress).

Outline

1. Eikonal operator as Generalized Wilson Line (GWL)
2. Gauge Issue of GWL & Collinear Limit
3. Factorization of 2-graviton eikonal operator
4. $\mathcal{O}(G_N^2)$ waveform & its implications
5. Radiation-Reaction at $\mathcal{O}(G_N^2)$

Eikonal operator

- Leading term of $\hat{\Delta}_{\text{s.d.}}$ takes the form of Weinberg's soft factor: $\hat{\Delta}_{\text{W}}^{\kappa} = \frac{1}{\hbar}\int d^3k[a_i(k)f_i^*(k) - a_i^\dagger f_i(k)]\,,$ with

Dressing external legs

$$f_i(k) = \varepsilon_i^{*\mu\nu}F_{\mu\nu}(k), \quad \text{with} \quad F^{\mu\nu} = \kappa\sum_n \frac{p_n^\mu p_n^\nu}{k\cdot p_n - i\eta_n 0_k}, \quad \eta_n = \pm 1 \longrightarrow h_{\mu\nu}^{\text{soft}} \sim \Theta(\eta_n u)$$

$$\left[a_i(k)\,, a_j^\dagger(k')\right] = 2\hbar\omega_k(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')\delta_{ij}, \qquad \int d^3k = \frac{1}{2(2\pi)^3}\int_0^\infty \Theta\left(\omega^* - \omega_k\right) d\omega_k\omega_k \oint d\Omega_k \ \text{ with } \omega^* = \text{IR scale.}$$

- Gauge transformation $\epsilon_i^{\mu\nu}(k) \to \epsilon^{\mu\nu}(k) + \xi^\mu k^\mu + \xi^\nu k^\mu$, with $\xi\cdot k = 0$,, $\hat{\Delta}_W^\kappa$ is gauge invariant by the conservation of external momenta: $k_\mu F^{\mu\nu}(k) = \kappa\sum_n p_n^\nu = 0!$

- This soft dressing can be derived and generalized as a Generalized Wilson Line operator (GWL) (White '11, Bonocore et al. '22) : Consider a canonical scalar coupled to gravity with the action $S = \int d^4x\phi(2\hat{H})\phi,$ with $-i\partial_\mu \to p_\mu,$

$$\begin{aligned}
2\hat{H}(x\,,p) &= p^2 + m^2 + 2\kappa\left[-p^\mu p^\nu h_{\mu\nu} + ip^\nu\partial^\mu h_{\mu\nu} + \frac{1}{2}\left(p^2+m^2\right)h - \frac{i}{2}p^\mu\partial_\mu h\right] \\
&+ 4\kappa^2\left[\left(p^2+m^2\right)\left(\frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right) - ip^\mu\partial_\mu\left(\frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right)\right. \\
&\left. + p^\mu p^\nu\left(h_{\mu\rho}h_\nu^\rho - \frac{1}{2}h_{\mu\nu}h\right) - ip^\mu\partial^\nu\left(h_{\mu\rho}h_\nu^\rho - \frac{1}{2}h_{\mu\nu}h\right)\right] + \mathcal{O}(\kappa^3)\,,
\end{aligned}$$

GWL

- The dressed propagator in the Schwinger formalism:

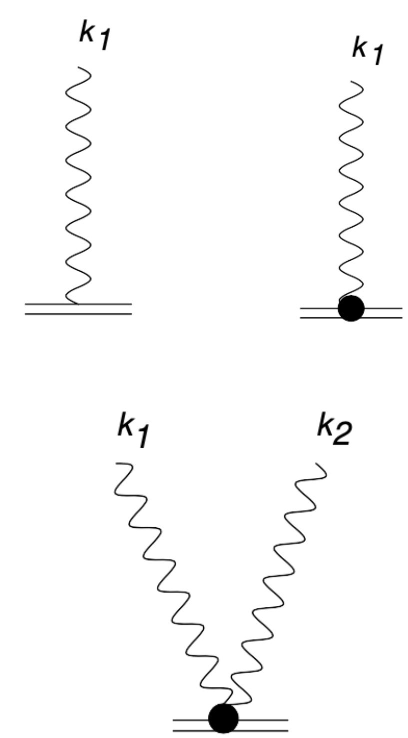
$$\left\langle p_f \middle| \left(\hat{H} - i\epsilon\right)^{-1} \middle| x_i \right\rangle = \int_0^\infty dT \int_{x(0)=x_i}^{p(T)=p_f} \mathscr{D}p\mathscr{D}x \; e^{\left[-\frac{i}{\hbar} p(T).x(T) + \frac{i}{\hbar}\int_0^T dt\left(p\dot{x} - \hat{H}(x\,,p) + i\epsilon\right)\right]},$$

- Take the on-shell limit, and integrate out the fluctuation around a straight-line trajectory (soft-limit): $x^\mu(t) = x_i^\mu + p_f^\mu t + \tilde{x}^\mu(t)$; the amputated dressed propagator $-i(p_f^2 + m^2)\left\langle p_f \middle| \left(\hat{H} - i\epsilon\right)^{-1} \middle| x_i \right\rangle$ takes the form of GWL:

$$W_p\,(0\,,\infty) = \exp\Bigg\{ i\kappa \int_0^\infty dt \left[\overbrace{-p^\mu p^\nu}^{\text{Eikonal}} + \overbrace{ip^{(\mu}\partial^{\nu)} - \frac{i}{2}\eta^{\mu\nu}p^\alpha\partial_\alpha + \frac{i}{2}tp^\mu p^\nu \partial^2}^{1-\text{graviton next}-\text{to}-\text{Eikonal}} \right] h_{\mu\nu}(pt)$$

$$\underbrace{+2i\kappa^2 \int_0^\infty dt \int_0^\infty ds \Big[\frac{p^\mu p^\nu p^\rho p^\sigma}{4} \min(t,s)\partial_\alpha h_{\mu\nu}(pt)\partial^\alpha h_{\rho\sigma}(ps)}_{2-\text{graviton next}-\text{to}-\text{Eikonal}}$$

$$\underbrace{+p^\mu p^\nu p^\rho \Theta(t\,,s)h_{\rho\sigma}(ps)\partial^\sigma h_{\mu\nu}(pt) + p^\nu p^\rho \delta(t-s)\eta^{\mu\sigma}h_{\rho\sigma}(ps)h_{\mu\nu}(pt)\Big]}_{2-\text{graviton next}-\text{to}-\text{Eikonal}} \Bigg\}$$



Eikonal Operator from GWL

- Introduce real graviton modes: $x_n^\mu = p_n^\mu t,$

$$h_{\mu\nu}(p_n t) = \int_{\vec{k}} d^3k \left[a_i(k)\varepsilon_{i\,,\mu\nu}(k)e^{ik.p_n t} + a_i^\dagger(k)\varepsilon^*_{i\,,\mu\nu}(k)e^{-ik.p_n t}\right]; \quad h_{\rho\sigma}(p_n s) = \int_{\vec{l}} d^3l \left[a_j(l)\varepsilon_{j\,,\rho\sigma}(l)e^{il.p_n s} + a_j^\dagger(l)\varepsilon^*_{j\,,\rho\sigma}(l)e^{-il.p_n s}\right]$$

- The GWL line turns into an eikonal operator up to $\mathcal{O}(\kappa^2)$:

$$W_p(0,\infty) = e^{-\hat{\Delta}_{s.d.}} = e^{-\hat{\Delta}^{\kappa}_{\mathrm{W}} - \hat{\Delta}^{\kappa^2}_{\mathrm{NE}}}, \qquad \hat{\Delta}^{\kappa^2}_{\mathrm{NE}} = \frac{1}{2\hbar}\int_{\vec{k}} d^3k \int_{\vec{l}} d^3l \left[a_i^\dagger(k)a_j^\dagger(l)A_{ij}(k,l) - a_j^\dagger(l)a_i(k)B_{ij}(k,l) - \mathrm{h.c.}\right]$$

$$A_{ij}(k,l) = \varepsilon^*_{i\,,\mu\nu}(k)A^{\mu\nu\rho\sigma}(k,l)\varepsilon^*_{j\,,\rho\sigma}(l)\ , \quad B_{ij}(k,l) = \varepsilon_{i\,,\mu\nu}(k)B^{\mu\nu\rho\sigma}(j,l)\varepsilon^*_{j\,,\rho\sigma}(l)$$

$$A^{\mu\nu\rho\sigma}(k,l) = \sum_n \frac{\kappa^2}{p_n\,.\,(k+l) - i\eta_n 0_k - i\eta_n 0_l}\left[\left(\frac{p_n^\mu p_n^\nu}{k\,.\,p_n - i\eta_n 0_k}\right)(p_n^\rho k^\sigma + p_n^\sigma k^\rho) + \left(\frac{p_n^\rho p_n^\sigma}{l\,.\,p_n - i\eta_n 0_l}\right)(p_n^\mu l^\nu + p_n^\nu l^\mu) - (p_n^\rho p_n^\mu \eta^{\nu\sigma} + p_n^\rho p_n^\nu \eta^{\mu\sigma} + p_n^\sigma p_n^\mu \eta^{\nu\rho} + p_n^\sigma p_n^\nu \eta^{\mu\rho})\right]$$

$$B^{\mu\nu\rho\sigma}(k,l) = \sum_n \frac{\kappa^2}{p_n\,.\,(k-l) + i\eta_n 0_k + i\eta_n 0_l}\left[\left(\frac{p_n^\mu p_n^\nu}{k\,.\,p_n + i\eta_n 0_k}\right)(p_n^\rho k^\sigma + p_n^\sigma k^\rho) + \left(\frac{p_n^\rho p_n^\sigma}{l\,.\,p_n - i\eta_n 0_l}\right)(p_n^\mu l^\nu + p_n^\nu l^\mu) - (p_n^\rho p_n^\mu \eta^{\nu\sigma} + p_n^\rho p_n^\nu \eta^{\mu\sigma} + p_n^\sigma p_n^\mu \eta^{\nu\rho} + p_n^\sigma p_n^\nu \eta^{\mu\rho})\right]$$

Gauge invariance vs Collinear limit

- Under gauge transformation $\varepsilon_i^{\mu\nu}(k) \to \varepsilon_i^{\mu\nu}(k) + \xi^\mu k^\mu + \xi^\nu k^\mu, \quad \varepsilon_i^{\rho\sigma}(\ell) \to \varepsilon_i^{\rho\sigma}(\ell) + \zeta^\rho \ell^\sigma + \zeta^\sigma k^\rho,$ with $\xi \cdot k = \zeta \cdot \ell = 0\,,\ \hat{\Delta}_{\rm NE}^{\kappa^2}$ is not gauge invariant as

$$k_\mu \xi_\nu A^{\mu\nu\rho\sigma}(k,l) = \kappa^2 \sum_n \frac{p_n \cdot k}{p_n \cdot (k+l)} \left[\left(\frac{p_n^\rho p_n^\sigma}{p_n \cdot l} \right) \xi \cdot l - \left(p_n^\rho \xi^\sigma + p_n^\sigma \xi^\rho \right) \right], \quad k_\mu \xi_\nu B^{\mu\nu\rho\sigma}(k,l) = \kappa^2 \sum_n \frac{p_n \cdot k}{p_n \cdot (k-l)} \left[\left(\frac{p_n^\rho p_n^\sigma}{p_n \cdot l} \right) \xi \cdot l - \left(p_n^\rho \xi^\sigma + p_n^\sigma \xi^\rho \right) \right]$$

$$l_\rho \zeta_\sigma A^{\mu\nu\rho\sigma}(k,l) = \kappa^2 \sum_n \frac{p_n . k}{p_n \cdot (k+l)} \left[\left(\frac{p_n^\mu p_n^\nu}{p_n \cdot k} \right) \zeta \cdot k - \left(p_n^\mu \zeta^\nu + p_n^\nu \zeta^\mu \right) \right], \quad l_\rho \zeta_\sigma B^{\mu\nu\rho\sigma}(k,l) = \kappa^2 \sum_n \frac{p_n \cdot k}{p_n \cdot (k-l)} \left[\left(\frac{p_n^\mu p_n^\nu}{p_n \cdot k} \right) \zeta \cdot k - \left(p_n^\mu \zeta^\nu + p_n^\nu \zeta^\mu \right) \right]$$

- We have no fundamental way to fix it, but observe that the gauge invariance can be recovered in the collinear limit, i.e., $k \cdot \ell = 0$, or $k^\mu/\omega_k = \ell^\mu/\omega_\ell = q^\mu = (1,\hat{q})$, with $\xi \cdot q = \zeta \cdot q = 0$, e.g., $k_\mu \xi_\nu A^{\mu\nu\rho\sigma}(k,l)\Big|_{k\cdot\ell=0} = \dfrac{\kappa^2 \omega_k}{\omega_k + \omega_l} \sum_n \left[\left(\dfrac{p_n^\rho p_n^\sigma}{p_n \cdot q} \right) \xi \cdot q - \left(p_n^\rho \xi^\sigma + p_n^\sigma \xi^\rho \right) \right] = 0.$

- Below, we will impose the collinear limit: $\delta_{\hat{k},\hat{\ell}}$, s.t. $\oint d\Omega_k \oint d\Omega_l \to \oint d\Omega_{q=k=\ell}$ in the angular integrations in the Eikonal operator. Thus, the resultant classical observables are gauge invariant.

- Need to find a 1st-principle way of ensuring gauge invariance of GWL by collinear limit.

Factorizing Eikonal Operator

- The above expression of the Eikonal operator involves both 1- & 2-graviton modes. This, however, can be factorized by applying the BCH formula: $W_p(0,\infty) = e^{-\Delta_0^{\kappa^4}} e^{-\hat{\Delta}_{\mathrm{W}}^{\kappa} - \hat{\Delta}_1^{\kappa^3} + \mathcal{O}(\kappa^5)} e^{-\hat{\Delta}_2^{\kappa^2}}$, with

$$\hat{\Delta}_1^{\kappa^3} = \frac{1}{2\hbar} \int_{\vec{k}} d^3k \int_{\vec{l}} d^3l \, \delta_{\hat{k},\hat{\ell}} \left[a_i^\dagger(k) \Big(A_{ij}(k,l) f_j^*(l) + B_{ij}^*(k,l) f_j(l) \Big) - \mathrm{h.c.} \right]$$

- Reshuffles $\hat{\Delta}_{\mathrm{NE}}^{\kappa^2}$ into $\Delta_0^{\kappa^4}$ (a pure phase $\stackrel{!}{=} 0$), $\hat{\Delta}_1^{\kappa^3}$, $\hat{\Delta}_2^{\kappa^3}$.
- $\hat{\Delta}_1^{\kappa^3}$ is the correction to Weinberg's factor, and can be understood as a re-sum of ladder diagrams with the 2-graviton vertex included.

- Then, for an observable Q, its classical value is $\langle \hat{Q} \rangle_c = \langle 0 \; e^{\hat{\Delta}_{s.d.}} \hat{Q} e^{-\hat{\Delta}_{s.d.}} \; 0 \rangle$,

E.g., $\langle a_i(k) \rangle_c = f_i(k) - \dfrac{1}{2} \displaystyle\int d^3l \, \delta_{\hat{k},\hat{\ell}} \Big(A_{ij}(k,l) f_j^*(l) + B_{ij}^*(k,l) f_j(l) \Big) + \mathcal{O}\left(\kappa^5\right) = \langle a_i(k) \rangle_{\Delta_W}$

$$\langle a_i^\dagger(k) a_i(k) \rangle_c = f_i^*(k) f_i(k) - \frac{1}{2} \int d^3l \, \delta_{\hat{k},\hat{\ell}} \Big(f_i(k) A_{ij}^*(k,l) f_j(l) + f_i(k) B_{ij}(k,l) f_j^*(l) + h.c. \Big) + \mathcal{O}(\kappa^5)$$

Memory Effect from $\mathcal{O}(G_N)$ Waveform

Di Vecchia et al. '22

- GW waveform from the eikonal approach:

$$W_{\mu\nu} = 2\kappa\langle \int d^3k \Big[\langle a_i(k)\rangle_c \varepsilon_{i,\mu\nu}(k) e^{ik.x} + \langle a_i^\dagger(k)\rangle_c \varepsilon^*_{i,\mu\nu}(k) e^{-ik.x} \Big] = W^{G}_{\mu\nu} + W^{G^2}_{\mu\nu}$$

$$W^{G_N}_{\mu\nu}(x) = 2\kappa \int d^3k\, \Theta\left(\omega^* - \omega_k\right) \left[e^{-i\omega_k u - i\omega_k r(1-\hat{k}\cdot\hat{x})} f_{\mu\nu}(\omega_k, \hat{k}) + \mathrm{c.c.} \right]$$

$$= 2G_N \int_{-\omega^*}^{\omega^*} \frac{d\omega_k}{2\pi} \omega_k e^{-i\omega_k u} \oint d\Omega_k e^{-i\omega_k r(1-\hat{k}.\hat{x})} \Pi_{\mu\nu\rho\sigma}(\hat{k}) \sum_n \frac{p_n^\rho p_n^\sigma}{(\eta_n \omega_k - i0_k)(E_n - \vec{p}_n\cdot\hat{k})} + \mathrm{c.c.}$$

$$\xrightarrow[\oint d\Omega_k e^{\mp i\omega_k r(1-\hat{k}\cdot\hat{x})} \overset{\hat{k}\cdot\hat{x}\approx 1}{\approx} \pm\frac{2\pi}{i\omega_k r}]{\text{saddle point approx at large r}} \frac{2G_N}{r} \Pi_{\mu\nu\rho\sigma}(\hat{x}) \sum_n \frac{p_n^\rho p_n^\sigma}{E_n - \vec{p}_n\cdot\hat{k}} \int_{-\omega^*}^{\omega^*} \frac{d\omega_k}{2\pi i} \frac{e^{-i\omega_k u}}{-\eta_n\omega_k - i0_k}$$

$$= \frac{2G_N}{r} \Pi_{\mu\nu\rho\sigma}(\hat{x}) \sum_n \frac{p_n^\rho p_n^\sigma}{E_n - \vec{p}_n\cdot\hat{k}} \Theta(\eta_n u), \qquad \Pi^{\mu\nu\rho\sigma}(\hat{k}) = \text{TT projector} = \varepsilon_i^{\mu\nu}(\hat{k})\varepsilon_i^{\rho\sigma}(\hat{k})$$

$$\xrightarrow[\vec{v}_n = \frac{\vec{p}_n}{E_n},\quad E_n = \frac{m_n}{\sqrt{1-v_n^2}}]{\text{de Donder gauge}} W^{ij}(u>0) - W^{ij}(u<0) = \frac{4G}{r} \sum_n \frac{m_n}{\sqrt{1-v_n^2}} \frac{v_n^i v_n^j}{1 - \vec{v}\cdot\hat{x}}$$

Memory Effect!

Braginsky-Thorne '87

Obtain $\mathcal{O}(G_N^2)$ Waveform

$$W_{\mu\nu}^{G_N^2}(x) = -\frac{\kappa}{(2\pi)^6}\int_{-\infty}^{\infty} d\omega_k \omega_k \Theta(\omega_k)\int_{-\infty}^{\infty} d\omega_l \omega_l \Theta(\omega_l)\,\Theta\left(\omega^* - \omega_k - \omega_l\right) e^{-i\omega_k u}$$

$$\underbrace{\oint d\Omega_k e^{-i\omega_k r(1-\hat{k}\cdot\hat{x})}}_{\approx\frac{2\pi}{i\omega_k r}}\Big[\Pi_{\mu\nu}^{\alpha\beta}(\hat{k})\underbrace{\Big(B^*_{\alpha\beta\rho\sigma}(\omega_k,\omega_l,\hat{k})f^{\rho\sigma}(\omega_l,\hat{k}) + A_{\alpha\beta\rho\sigma}(\omega_k,\omega_l,\hat{k})f^{*\,\rho\sigma}(\omega_l,\hat{k})\Big)}_{B^* \text{ and } A \text{ contracted with } f \text{ by TT}-\text{projector } \Pi_{\theta\phi\mu\nu}} - \underbrace{\text{c.c.}}_{\omega_{k,\ell}\to-\omega_{k,\ell}}\Big]$$

$$= \frac{8G_N^2}{\pi^2 r}\Pi_{\mu\nu\alpha\beta}(\hat{x})\sum_{a,b}\frac{\eta_a\eta_b m_a m_b\sigma_{ab}(p_a^\alpha p_b^\beta + p_b^\alpha p_a^\beta) - m_b^2 p_a^\alpha p_b^\beta}{(E_a - \vec{p}_a\cdot\hat{x})(E_b - \vec{p}_b\cdot\hat{x})}\int_{-\infty}^{\infty} d\omega_l\omega_l\int_{-\infty}^{\infty}\frac{d\omega_k}{2\pi i}\sum_{\eta=\mp}\left[\frac{\Big(\Theta(\omega_k)\Theta(\omega_l) + \Theta(-\omega_k)\Theta(-\omega_l)\Big)\,\Theta\left(\omega^* - \omega_k - \omega_l\right) e^{-i\omega_k u}}{(-\eta_a(\omega_k + \eta\omega_\ell) - i0_k - i0_\ell)(-\eta_b\omega_\ell + i\eta 0_\ell)}\right]$$

$$= \frac{8G_N^2}{\pi^2 r}\Pi_{\mu\nu\alpha\beta}(\hat{x})\sum_{a,b}\frac{\eta_a\eta_b m_a m_b\sigma_{ab}(p_a^\alpha p_b^\beta + p_b^\alpha p_a^\beta) - m_b^2 p_a^\alpha p_b^\beta}{(E_a - \vec{p}_a\cdot\hat{x})(E_b - \vec{p}_b\cdot\hat{x})}\Theta(\eta_n u)\int_{-\infty}^{\infty} d\omega_l\omega_l\,\Theta\left(\omega^* - 2\,\omega_l\right)\frac{e^{-i\omega_l u}}{(-\eta_b\omega_l - i0_l)}$$

$$= \frac{16G_N^2}{\pi^2 r u}\sin\left(\frac{\omega^* u}{2}\right)\Pi_{\mu\nu\alpha\beta}(\hat{x})\sum_{a,b}\frac{\eta_b m_b^2 p_a^\alpha p_b^\beta - \eta_a m_a m_b\sigma_{ab}(p_a^\alpha p_b^\beta + p_b^\alpha p_a^\beta)}{(E_a - \vec{p}_a\cdot\hat{x})(E_b - \vec{p}_b\cdot\hat{x})}\Theta(\eta_n u)$$

Lorentz factor : $\sigma_{ab} = -\eta_a\eta_b\frac{p_a\cdot p_b}{m_a m_b} \xrightarrow{a=b} 1$

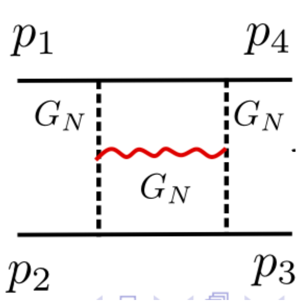
Implication

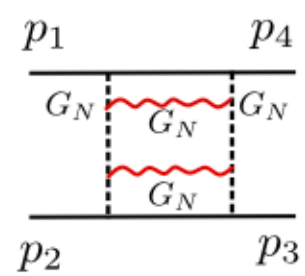
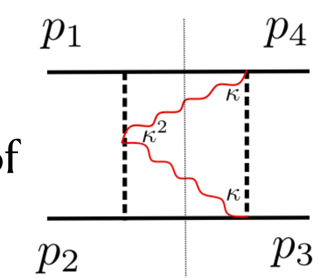
- The collinear limit yields a Lorentz-invariant kinematic factor consistent with gauge invariance.
- Given an IR scale ω^*, the waveform gives a sinusoidal tail $\sim \sin(\omega^* u/2)/u$.
- It vanishes if $\omega^* = 0$, thus yields no (static) memory effect as for the $\mathcal{O}(G_N)$ case.
- In the collinear limit of the external particles (a=b), the kinematic factor collapse into a simple form: $-\frac{\eta_b m_b^2 p_a^\alpha p_b^\beta}{(E_a - \vec{p}_a \cdot \hat{x})^2}$.

Radiation-Reaction at $\mathcal{O}(G_N^2)$

- Based on our eikonal operator with 2-graviton dressing, we can check the generalized version of $\lim_{\epsilon\to 0} \mathrm{Re}\ 2\delta_2^{\text{r.r.}} = \lim_{\epsilon\to 0}[-\epsilon\pi \mathrm{Im}\ 2\delta_2] = \frac{dE_{\text{rad}}^{G_N}}{2\hbar d\omega^*}(\omega^* \to 0)$ to $\mathcal{O}(G_N^4)$.

Here, $E_{\text{rad}} = \langle \int d^3k\hbar k^{\mu=0} a_i^\dagger(k) a_i(k)\rangle_c$, and $\mathrm{Im}\delta_3$ is related to the normalization of the eikonal operator, i.e., $\langle 0 \ \ e^{-\hat{\Delta}_{\text{s.d.}}} \ \ 0\rangle = e^{-\Delta_0^{\kappa^4}} \langle 0 \ \ : e^{\hat{\Delta}_W^\kappa - \hat{\Delta}_1^{\kappa^3}} : \ \ 0\rangle$ and $\Delta_0^{\kappa^4} = 0$.

$\delta_2^{\text{r.r.}} \sim G_N^3:$ 

$\delta_3^{\text{r.r.}} \sim G_N^4:$  + Collinear limit of 

(a) (b)

- Radiation-Reaction: $-\mathrm{Im}\ 2\delta_3\Big|_{(b)} = \frac{dE_{\text{rad}}^{G_N^2}}{2\hbar d\omega^*}(\omega^* \to 0)$, not IR divergent.

$$\lim_{\epsilon\to 0}\left[16\epsilon^2(\mathrm{Im}\ 2\delta_2)^2\right]_{(a)} = \frac{1}{\hbar\omega^*}\frac{dE_{\text{rad}}^{G_N,2}}{d\omega^*}, \quad \text{with} \quad E_{\text{rad}}^{G_N,2} = \langle\left(\int d^3k k^0 a_i^\dagger(k) a_i(k)\right)^2\rangle_c .$$

- It needs to be verified by calculating $\mathrm{Im}\delta_3$ from the unitarity cut diagrams.

- This is relevant to determine $\mathrm{Re}\delta_3^{\text{r.r}}$, thus full scattering angle at 4PM.

1. Di Vecchia et al. '22:

$$\lim_{\omega^*\to 0}\frac{dE_{\rm rad}^{G_N}}{d\omega^*}=\frac{2G_N}{\pi}\sum_{a,b} m_a m_b\left(\sigma_{ab}^2-\frac{1}{2}\right)\frac{\eta_a\eta_b\cosh^{-1}\sigma_{ab}}{\sqrt{\sigma_{ab}^2-1}}\sim\mathcal{O}((\omega^*)^0)$$

2. Our result:

$$\lim_{\omega^*\to 0}\frac{dE_{\rm rad}^{G_N^2}}{d\omega^*}=\frac{\omega^* G_N^2}{4(2\pi)^5}\sum_{a,b,c}\frac{m_a^2m_b^2m_c^2\left(2\sigma_{ac}^2+2\sigma_{bc}^2-4\sigma_{ac}\sigma_{bc}\sigma_{ab}-1\right)}{\sqrt{2\left(m_am_c\sigma_{ac}+m_bm_c\sigma_{bc}+m_am_b\sigma_{ab}\right)-\left(m_a^2+m_b^2+m_c^2\right)}}$$

$$\times\left[\frac{\cosh^{-1}\left(\sqrt{2\sigma_{ab}^2-1}\right)}{m_am_b\sqrt{\sigma_{ab}^2-1}}+\frac{\cosh^{-1}\left(\sqrt{2\sigma_{ac}^2-1}\right)}{m_am_c\sqrt{\sigma_{ac}^2-1}}+\frac{\cosh^{-1}\left(\sqrt{2\sigma_{bc}^2-1}\right)}{m_bm_c\sqrt{\sigma_{bc}^2-1}}\right]\xrightarrow{\omega^*\to 0}0$$

3. Damour '20: $P^\mu_{\rm rad}=\dfrac{1}{32\pi G_N}\displaystyle\int du\,d\Omega\; n^\mu(\partial_u f_{ij})^2$, $\qquad h_{ij}^{\rm TT}=\dfrac{f_{ij}(u,\theta,\phi)}{r}+\mathcal{O}(\dfrac{1}{r^2}),$

$$f_{ij}(u,\theta,\phi)=G_Nf^{(1)}(\theta,\phi)+G_N^2f^{(2)}(\theta,\phi)+\mathcal{O}(G_N^3)\longrightarrow P^\mu_{\rm rad}\sim\mathcal{O}(G_N^3)$$

E.g., Fillipo's talk!

$$\Big\downarrow\ \text{Soft dressing}\ \sim\Theta(\eta_a u)$$

$$G_Nf^{(1)}_{soft}(u,\theta,\phi)\longrightarrow P^\mu_{\rm rad}\sim\mathcal{O}(G_N)$$