Lieb-Schultz-Mattis type theorems for Majorana models with discrete symmetries

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"Entanglement in Strongly Correlated Systems"
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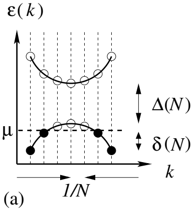
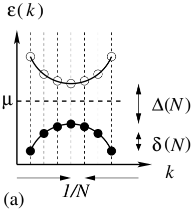
Example 1: Free spinless fermions

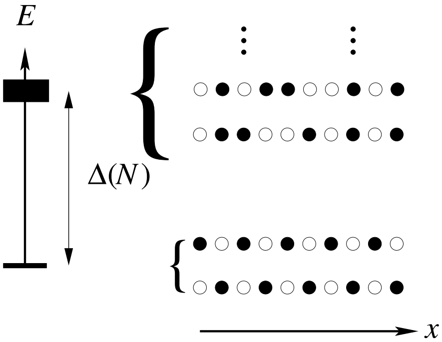


The ground states are gapless and non-degenerate in the thermodynamic limit $N \to \infty$.



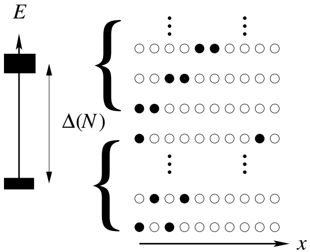
The ground states are gapped and non-degenerate in the thermodynamic limit $N \to \infty$.

Example 2: Nearest-neighbor repulsive interactions between spinless fermions at the filling fraction $\nu = 1/2$



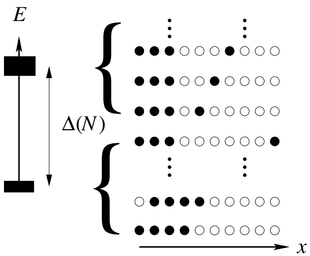
The ground states are gapped, finitely degenerate, and long-range ordered in the thermodynamic limit N even $\to \infty$.

Example 3: **Strong** nearest-neighbor repulsive interactions between spinless fermions at the filling fraction $0 < \nu < 1/2$.



The ground states are gapped and infinitely degenerate in the thermodynamic limit $N \to \infty$. There is no long-range order.

Example 4: **Strongly** attractive fermions



The ground states are gapped, infinitely degenerate, and violate the clustering property in the thermodynamic limit $N \to \infty$.

In general, a quantum many body state $|\Phi\rangle$ is said to satisfy the cluster decomposition property or clustering property if, for any pair of local operators $\widehat{O}_1(\boldsymbol{r})$ and $\widehat{O}_2(\boldsymbol{r})$, the identity

$$\lim_{|\boldsymbol{r}-\boldsymbol{r}'|\to\infty} \left\langle \Phi \,\middle|\, \widehat{O}_1(\boldsymbol{r})\,\widehat{O}_2(\boldsymbol{r}') \,\middle|\, \Phi \right\rangle = \lim_{|\boldsymbol{r}-\boldsymbol{r}'|\to\infty} \left\langle \Phi \,\middle|\, \widehat{O}_1(\boldsymbol{r}) \,\middle|\, \Phi \right\rangle \left\langle \Phi \,\middle|\, \widehat{O}_2(\boldsymbol{r}') \,\middle|\, \Phi \right\rangle \tag{1}$$

holds.

Comments:

There were no examples in one-dimensional space of local and $G_{\text{trsl}} \times G_f$-symmetric Hamiltonian with

1. gapped,
2. n-fold degenerate with $1 < n < \infty$,
3. and $G_{\text{trsl}} \times G_f$-symmetric

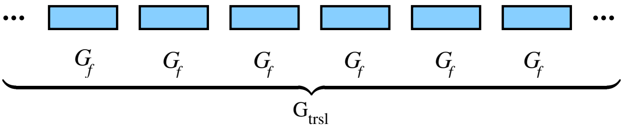
ground states.

This is not an accident!

Question:

Can one make any general statement about the degeneracy of gapped superconducting ground states of local lattice Hamiltonians?

Main results: No-go Theorem I



Theorem (no-go theorem I)

Any one-dimensional lattice Hamiltonian that is

(i) local,

(ii) admits the global symmetry group $\mathrm{G}_{\mathrm{trsl}} \times \mathrm{G}_f$*,*
where the fermionic symmetry group G_f *is an on-site symmetry that is realized locally by a nontrivial projective representation,*

cannot have nondegenerate, gapped, and $\mathrm{G}_{\mathrm{trsl}} \times \mathrm{G}_f$*-symmetric ground states that can be described by even- or odd-parity injective fermionic matrix product states (FMPS).*

Fermionic symmetry group G_f

A symmetry group is called fermionic if it contains a normal cyclic subgroup of order two

$$\mathbb{Z}_2^{\mathrm{F}} \equiv \left\{ p,\ p^2 \equiv \mathrm{id} \right\} \tag{2}$$

that **cannot be broken** (neither explicitly nor spontaneously).

On-site symmetry group G_f

A symmetry group G_f **is on-site** if any one of its elements g is represented quantum mechanically by an operator $\widehat{U}(g)$ obeying the factorization

$$\widehat{U}(g) = \begin{cases} \prod_{j\in\Lambda} \widehat{u}_j(g), & \text{if unitary,} \\ \\ \left[\prod_{j\in\Lambda} \widehat{u}_j(g)\right] \mathsf{K}, & \text{if non-unitary,} \end{cases} \tag{3a}$$

where

1. Λ denotes the lattice,
2. $\widehat{u}_j(g)$ only acts non-trivially on site j,
3. the pair $\widehat{u}_j(g)$ and $\widehat{u}_{j'}(g')$ for any two distinct sites $j, j' \in \Lambda$ and any two elements $g, g' \in \mathrm{G}_f$ either commute or anticommute,

$$\begin{aligned} &\widehat{u}_j(g)\,\widehat{u}_{j'}(g') = \eta(g,g')\,\widehat{u}_{j'}(g')\,\widehat{u}_j(g), \\ &\eta(g,g') = (-1)^{\rho(g)\,\rho(g')}, \qquad \rho(g), \rho(g') = 0, 1. \end{aligned} \tag{3b}$$

Here, g is represented by a bosonic (fermionic) operator if $\rho(g) = 0$ [$\rho(g) = 1$].

Projective representations

Let G be a group with the composition rule $\star$ by which

$$g_1 \star g_2 = g_{12}, \qquad g_1, g_2, g_{12} \in \mathrm{G}. \tag{4}$$

An *n*-dimensional representation of G is a group homomorphism on the space of linear maps acting on an *n*-dimensional vector space.
In quantum mechanics, any group element g is assigned an operator $\widehat{U}_g$ acting on an Hilbert space. This assignemnt is not unique, since physical states are rays in the Hilbert space. Hence,

$$\widehat{U}_{g_1}\,\widehat{U}_{g_2} = e^{\mathrm{i}\phi_{g_1,g_2}}\,\widehat{U}_{g_{12}}, \qquad g_1, g_2, g_{12} \in \mathrm{G}. \tag{5}$$

Here, the function

$$\phi : \mathrm{G} \times \mathrm{G} \to [0, 2\pi[\tag{6}$$

must be compatible with the associativity of $\star$. The presence of the phase factor on the right-hand side of Eq. (5) **defines a projective representation of the group** G. This projective representation is trivial if it is possible to choose

$$\phi_{g_1,g_2} = 0 \bmod 2\pi, \qquad \forall g_1, g_2 \in \mathrm{G}. \tag{7}$$

Otherwise, this projective representation of G is nontrivial.

Examples of nontrivial projective representations

1. For any half-integer spin (spinor) representation of $\mathrm{SO}(3)$, a 2π-rotation squares to minus the identity.

2. Reversal of time

$$\begin{aligned} \mathcal{T}:\ &\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d\times\mathbb{R},\\ &(\boldsymbol{x},t)\mapsto(\boldsymbol{x},-t), \end{aligned} \tag{8a}$$

generates the cyclic group

$$\mathrm{G}:=\left\{\mathcal{T},\mathcal{T}^2\equiv\mathrm{id}\right\}. \tag{8b}$$

In quantum mechanics, it is represented on the Hilbert space

$$\mathcal{H}_t:=L^2_t(\mathbb{R}^d;\mathbb{C}^2) \tag{9a}$$

for square-integrable spinors by the antilinear transformation

$$\begin{aligned} &\widehat{\mathcal{T}}:\mathcal{H}_t\to\mathcal{H}_{-t},\\ &\Psi(\boldsymbol{x},t)\mapsto\mathrm{i}\sigma_2\,\Psi^*(\boldsymbol{x},-t). \end{aligned} \tag{9b}$$

Because of

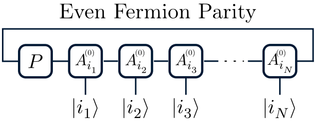
$$\widehat{\mathcal{T}}^2=(\mathrm{i}\sigma_2\,\mathsf{K})\,(\mathrm{i}\sigma_2\,\mathsf{K})=(\mathrm{i}\sigma_2)^2\,(\mathsf{K})^2=(\mathrm{i})^2\,\sigma_0\equiv-\begin{pmatrix}1&0\\0&1\end{pmatrix}, \tag{9c}$$

$\widehat{\mathcal{T}}$ squares to minus the identity and not to the identity. This is why $\mathcal{T}$ is represented projectively on $\mathcal{H}_t$.

Fermionic matrix product states (FMPS)

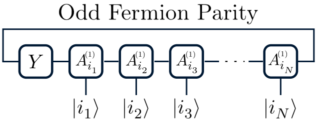
An even parity FMPS obeying periodic boundary conditions is defined by

$$\left|\left\{A^{(0)}_{\sigma_j}\right\}\right\rangle := \sum_{\boldsymbol{\sigma}} \mathrm{tr}\left(P\,A^{(0)}_{\sigma_1}\cdots A^{(0)}_{\sigma_N}\right)|\Psi_{\boldsymbol{\sigma}}\rangle\,. \tag{10}$$

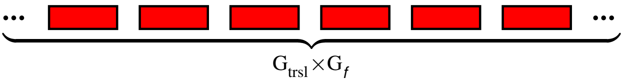


An odd parity FMPS obeying periodic boundary conditions is defined by

$$\left|\left\{A^{(1)}_{\sigma_j}\right\}\right\rangle := \sum_{\boldsymbol{\sigma}} \mathrm{tr}\left(Y\,A^{(1)}_{\sigma_1}\cdots A^{(1)}_{\sigma_N}\right)|\Psi_{\boldsymbol{\sigma}}\rangle\,. \tag{11}$$



Main results: No-go Theorem II



Theorem (no-go theorem II)

Any translationally invariant and local d-dimensional lattice Majorana Hamiltonian with an odd number of Majorana degrees of freedom per repeat unit cell that is invariant under the symmetry group $G_{trsl} \times G_f$, cannot have nondegenerate, gapped, and $G_{trsl} \times G_f$-symmetric ground states.

Main results: Comments

1. The thermodynamic limit is implicit in both theorems.
2. Theorem I is only predictive when G_f is realized by a nontrivial projective representation on the local Fock space. When G_f is a Lie group with a trivial projective representation on the local Fock space, then Theorem I is not predictive. However, one can use complementary arguments, such as the adiabatic threading of a gauge flux (Laughlin 1981, Oshikawa 2000), to decide if a symmetric and gapped ground state is degenerate. It is when G_f is a finite group that the full power of Theorem I is unleashed.
3. Theorem II for one-dimensional lattices can be proved with the help of Theorem I.
4. A weaker form of Theorem I holds in any dimension if it is assumed that G_f is Abelian and can be realized locally using unitary operators.
5. Theorem II holds in any dimension without any restriction on G_f.
6. The proof of Theorems I makes use of the direct product in the symmetry group $\mathrm{G}_{\mathrm{trsl}} \times \mathrm{G}_f$.

A brief review of the Lieb-Schultz-Mattis theorem

One motivation by Lieb, Schultz, and Mattis in 1961was to find an analytical argument that could decide if the nearest-neighbor antiferromagnetic quantum spin-1/2 Heisenberg chain supports **antiferromagnetic long-range order at zero temperature.**

Although they could not answer this question rigorously (Mermin and Wagner in 1966 proved rigorously that the ground state does not support antiferromagnetic long-range order), they could show rigorously that **the antiferromagnetic quantum spin-1/2 *XY* Hamiltonian has a gapless spectrum with all correlation functions of spins decaying algebraically in space.**

In modern (1980's onward) parlance, the antiferromagnetic quantum spin-1/2 *XY* chain realizes the Gaussian conformal field theory with central charge $c = 1$.

In their appendices, Lieb, Schultz, and Mattis also established two theorems, the second of which is now called the Lieb-Schultz-Mattis theorem:

Theorem (3)

The ground state of the nearest-neighbor antiferromagnetic quantum spin-1/2 Heisenberg chain made of N sites is annihilated by the total spin operator

$$\widehat{\boldsymbol{S}} := \sum_{j=1}^{N} \widehat{\boldsymbol{S}}_j \tag{12}$$

for any even integer N.

Theorem (Lieb-Schultz-Mattis)

The nearest-neighbor antiferromagnetic quantum spin-1/2 Heisenberg chain made of N sites and obeying periodic boundary conditions supports an excited eigenstate with an energy of order $1/N$ above the nondegenerate ground state for ***any even integer*** N.

Remark (1)

The (original) proof of the LSM theorem **does not apply** to the nearest-neighbor antiferromagnetic quantum spin-1 Heisenberg chain. It is still possible to show for $S = 1$ that there exists a state with an energy of order $1/N$ above the ground state for any N, however, it is not possible to show that this state is orthogonal to the ground state.

Remark (2)

The LSM Theorem only proves the existence of at least one excited state with an energy that collapses like $1/N$ to that of the ground state in the thermodynamic limit $N \to \infty$. This state could be isolated and there is no guarantee that the thermodynamic limit of the ground state and this excited state are distinct. Hence, **neither** the existence of a gapless continuum of states above the ground state **nor** the nondegeneracy of the ground state in the thermodynamic limit have been shown (one would need the Bethe Ansatz).

Remark (3)

The (original) proof of the LSM Theorem makes use of the global $\mathrm{SU}(2)$ symmetry, time-reversal symmetry, and of Theorem 3 (i.e., the fact that the ground state is nondegenerate).

Remark (4)

The same constructive proof applied to a dimension of space d larger than one would imply the bound N^{d-2} between the ground state and the excited states. This bound is thus useless when $d > 1$.

The qualitative difference (Haldane 1983) between half-integer and integer antiferromagnetic quantum spin-S Heisenberg chains motivated the following important refinements of the Lieb-Schultz-Mattis Theorem:

1. Affleck and Lieb in 1986 showed that any half-integer antiferromagnetic quantum spin-S Heisenberg chain with translation and global internal $\mathrm{U}(1)$ spin invariance has a nondegenerate ground state for any finite chain made of an even number N of sites with a gap of order $1/N$. In the thermodynamic limit, the ground states are **either** degenerate **or** nondegenerate and gapless.
2. Oshikawa, Yamanaka, and Affleck in 1997 replaced the condition of Affleck and Lieb that S is a half integer with the condition that $\nu := \frac{M^z}{N} + S$ is not an integer.
3. An analogous theorem was proven by Yamanaka, Oshikawa, and Affleck in 1997 for any local lattice models of interacting electrons for which the electronic charge is conserved, translation symmetry holds, and the ratio ν between the (conserved) total number of electrons N_{f} and the number of sites N on the ring is not an integer.
4. All these papers had always chosen Hamiltonians for which **either** reversal of time **or** inversion in space were symmetries. This assumption was shown by Koma in 2000 to be superfluous.

The most recent and general extension of the Lieb-Schultz-Mattis Theorem is due to Tasaki in 2018. Its proof relies on three steps:

1. First, **variational** state**s** are constructed.
2. Second, the energy expectation value**s** for these variational states are shown to **collapse** to the ground-state energy in the thermodynamic limit.
3. Third, the variational state**s** are shown to be **orthogonal** with each other and with the ground state for any fixed number of degrees of freedom.
4. Finally, the conditions are given (no spontaneous symmetry breaking of translation symmetry) under which th**ese** orthogonaliti**es** survive the thermodynamic limit.

The variational state**s** in the first step are constructed by deformations in position space of the ground state that are **local** and **smooth**. Here, the existence of an on-site **global continuous symmetry** of the Hamiltonian is crucial.
The degree of smoothness of these local deformation**s** is controlled by the length of the one-dimensional lattice hosting the quantum degrees of freedom and the continuity of the on-site global symmetry of the Hamiltonian. The longer the length of the one-dimensional lattice, the smoother the local deformation**s** in position space of the ground state are and the closer the energy expectation value**s** of the variational state**s** relative to the ground state energy are. The **locality** of the Hamiltonian is needed to control the separation in energy between the variational and ground states. The conditions for degenerate ground states in the thermodynamic limit are given: **either** infinite degeneracy **or** spontaneous symmetry breaking of translation symmetry.
The proof of orthogonality in the third step hinges on the filling fraction ν not being integer valued and the existence of translation symmetry in addition to a continuous symmetry. **No more information from the Hamiltonian is needed to complete this step of the proof.**

Question 1: Can the condition that the Hamiltonian is invariant under an on-site continuous symmetry be weaken by demanding that the on-site symmetry group is no more than a discrete group?

Question 2: Can this discrete group accomodate the conservation of fermion parity?

To answer these questions, we start from the **logical contraposition** of Tasaki's extension of the Lieb-Schultz-Mattis Theorem. It applies to a local lattice Hamiltonian that is invariant under translation of the lattice repeat unit cell by one lattice spacing and invariant under a continuous on-site symmetry group. It also presumes the existence of a positive real-valued number ν, the filling fraction of the repeat unit cell. It states that, **if** the ground-states are finitely degenerate and separated by a gap from all excited states in the thermodynamic limit, **then** either translation symmetry is spontaneously broken or ν is an integer.

Theorem I revisited

If

1. space is *d*-dimensional,
2. Hamiltonian is local,
3. Hamiltonian has $\mathrm{G}_{\mathrm{trsl}} \times \mathrm{G}_f$ symmetry, with G_f **Abelian** an on-site fermionic symmetry **that is represented unitarily**,
4. Hamiltonian has gapped ground-states,
5. which are nondegenerate and $\mathrm{G}_{\mathrm{trsl}} \times \mathrm{G}_f$ symmetric,

then G_f must have a trivial projective representation.

Our method is inspired by the one used by Yao and Oshikawa 2021 for quantum spin Hamiltonians.

Step 1 proof Theorem I revisited

By assumption, the combined symmetry group is the direct product

$$\mathrm{G}_{\mathrm{total}} \equiv \mathrm{G}_{\mathrm{trsl}} \times \mathrm{G}_{f}. \tag{13}$$

Any translationally- and G_f-invariant and local Hamiltonian can be written in the form

$$\widehat{H}_{\mathrm{pbc}} := \sum_{\hat{\mu}=\hat{1}}^{\hat{d}} \sum_{n_{\hat{\mu}}=1}^{N_{\hat{\mu}}} \left(\widehat{T}_{\hat{\mu}}\right)^{n_{\hat{\mu}}} \hat{h}_{\boldsymbol{j}} \left(\widehat{T}^{\dagger}_{\hat{\mu}}\right)^{n_{\hat{\mu}}}, \qquad \left(\widehat{T}_{\hat{\mu}}\right)^{N_{\hat{\mu}}} = \widehat{\mathrm{id}} \text{ by Eq. (13)}, \tag{14a}$$

where $\hat{h}_{\boldsymbol{j}}$ is a local Hermitian operator centered at an **arbitrarily chosen** repeat unit cell $\boldsymbol{j}$. More precisely, it is a finite-order polynomial in the Majorana operators centered at $\boldsymbol{j}$ that is also invariant under all the non-spatial symmetries, i.e.,

$$\hat{h}_{\boldsymbol{j}} = \widehat{U}(g)\, \hat{h}_{\boldsymbol{j}}\, \widehat{U}^{-1}(g) = \left(\hat{h}_{\boldsymbol{j}}\right)^{\dagger} \tag{14b}$$

for any $g \in \mathrm{G}_f$.

Step 2 proof Theorem I revisited

By construction

$$\begin{aligned} &\widehat{H}_{\mathrm{pbc}}, \\ &\widehat{T}_{\hat{\mu}}, \qquad \forall \hat{\mu} = \hat{1}, \cdots, \hat{d}, \\ &\widehat{U}(g), \qquad \forall g \in \mathrm{G}_f, \end{aligned} \tag{15}$$

commute pairwise.
Energy eigenvalues of

$$\widehat{H}_{\mathrm{pbc}}$$

can thus be labeled by the eigenvalues of

$$\widehat{T}_{\hat{\mu}} \quad \forall \hat{\mu} = \hat{1}, \cdots, \hat{d},$$

and of

$$\widehat{U}(g) \quad \forall g \in \mathrm{G}_f.$$

However, nothing can be said about the degeneracies of the eigenvalues of $\widehat{H}_{\mathrm{pbc}}$.

Step 3 proof Theorem I revisited

Yao-Oshikawa conjecture: *When a **local** quantum many-body Hamiltonian with lattice translation invariance and a global (continuous or discrete) symmetry has a **gapped** spectrum with **nondegenerate** ground states under periodic boundary conditions, the same must be true under **any symmetry-twisted boundary conditions**.*

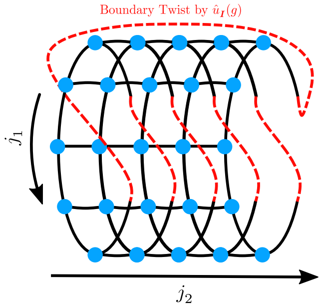


Figure: Example of a path that visits all the sites of a two-dimensional lattice **that decorates the surface of a torus.**

Withoult loss of generality, $d = 1$ and $|\Lambda| = N$. Define

$$\widehat{H}^{\mathrm{tilt}}_{\mathrm{twis}}(g) := \sum_{a=1}^{N} \left(\widehat{T}_{\hat{1}}(g)\right)^{a} \hat{h}^{\mathrm{tilt}}_{1} \left(\widehat{T}^{-1}_{\hat{1}}(g)\right)^{a}, \tag{16a}$$

$$\widehat{T}_{\hat{1}}(g) := \hat{u}_1(g)\,\widehat{T}_{\hat{1}} \implies \widehat{T}_{\hat{1}}(g)\,\hat{\chi}_j\,\widehat{T}^{-1}_{\hat{1}}(g) = \begin{cases} (-1)^{\rho(g)}\,\hat{\chi}_{j+1}, & \text{if } j \neq N \\ & \text{and } \rho(g) = 0(1) \text{ if } \hat{u}_1(g) \text{ commutes (anticommutes)} \\ & \text{with on-site fermion parity,} \\ \hat{u}_1(g)\,\hat{\chi}_1\,\hat{u}^{-1}_1(g), & \text{if } j = N, \end{cases} \tag{16b}$$

then (!!!**key step of the proof**!!!)

$$\left[\widehat{T}_{\hat{1}}(g)\right]^{N} = \widehat{U}(g), \qquad g \in \mathrm{G}_f \qquad \text{and} \qquad \widehat{U}(h)^{-1}\,\widehat{T}_{\hat{1}}(g)\,\widehat{U}(h) = e^{\mathrm{i}\chi(g,h)}\,\widehat{T}_{\hat{1}}(g), \quad h \in \mathrm{G}_f, \tag{16c}$$

where the phase $\chi(g, h) \in [0, 2\pi[$ is gauge invariant and given by

$$\chi(g,h) := \phi(h,g) - \phi(g,h) + \underbrace{(N-1)\pi\,\rho(h)[\rho(g)+1]}_{\text{only} \neq 0 \text{ if } \exists \text{ Majoranas}}. \tag{16d}$$

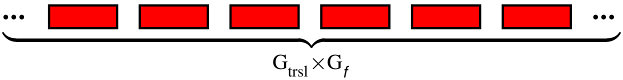
Non obvious fact: The phase $\chi(g, h)$ is vanishing if and only if the second cohomology class $[\phi]$ is trivial.

If $\chi(g, h)$ cannot be made to vanish for all g, h, then one-dimensional representations of (16c) are not allowed.

The ground states of any Hamiltonian of the form (16a) are then either degenerate or spontaneously break the symmetry in the thermodynamic limit.

We have derived the Theorem I revisited for the Abelian group G_f that is represented unitarily when symmetry-twisted boundary conditions apply.

Theorem II revisitd



If

1. space is *d*-dimensional,
2. Hamiltonian $\widehat{H}_{\text{pbc}}$ is local,
3. Hamiltonian $\widehat{H}_{\text{pbc}}$ is invariant under the symmetry group $G_{\text{trsl}} \times G_f$,
4. each repeat unit cell labeled by $\boldsymbol{j} \in \Lambda$ ($|\Lambda|$ must be **even**) hosts the odd number $2n+1$ of Majorana operators $\hat{\chi}_{\boldsymbol{j},l}$ with $l = 1, \cdots, 2n+1$,

then the Hamiltonian $\widehat{H}_{\text{pbc}}$ cannot have nondegenerate, gapped, and $G_{\text{trsl}} \times G_f$-symmetric ground states.

Step 1 proof Theorem II revisited

Substitute

$$\mathrm{G}_{\mathrm{trsl}} \equiv \mathbb{Z}_{N_{\hat{1}}} \times \cdots \times \mathbb{Z}_{N_{\hat{d}}} \tag{17}$$

in $\mathrm{G}_{\mathrm{trsl}} \times \mathrm{G}_f$ by the cyclic group

$$\mathrm{G}^{\mathrm{tilt}}_{\mathrm{trsl}} \equiv \mathbb{Z}_{N_{\hat{1}} \cdots N_{\hat{d}}} \equiv \mathbb{Z}_{|\Lambda|}. \tag{18}$$

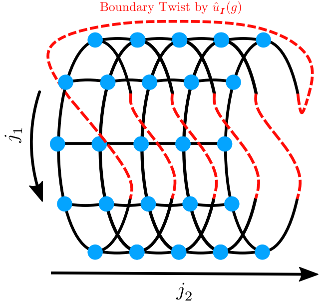


Figure: Example of a path that visits all the sites of a two-dimensional lattice **that decorates the surface of a torus**.

Step 2 proof Theorem II revisited

At the quantum level, substitute

$$\widehat{T}_{\hat{\mu}}\,\hat{\chi}_{\boldsymbol{j}}\left(\widehat{T}_{\hat{\mu}}\right)^{-1}=\hat{\chi}_{\boldsymbol{j}+\boldsymbol{e}_{\hat{\mu}}},\qquad\left(\widehat{T}_{\hat{\mu}}\right)^{N_{\hat{\mu}}}=\widehat{id},\tag{19a}$$

and

$$\widehat{H}_{\mathrm{pbc}}:=\sum_{\hat{\mu}=\hat{1}}^{\hat{d}}\sum_{n_{\hat{\mu}}=1}^{N_{\hat{\mu}}}\left(\widehat{T}_{\hat{\mu}}\right)^{n_{\hat{\mu}}}\hat{h}_{\boldsymbol{j}}\left(\widehat{T}^{\dagger}_{\hat{\mu}}\right)^{n_{\hat{\mu}}}\tag{19b}$$

by

$$\widehat{T}_{\hat{\mu}}\,\hat{\chi}_{\boldsymbol{j}}\,\widehat{T}^{-1}_{\hat{\mu}}=\hat{\chi}_{t_{\hat{\mu}}(\boldsymbol{j})},\qquad\left(\widehat{T}_{\hat{\mu}}\right)^{N_{\hat{\mu}}}=\begin{cases}\widehat{T}_{\hat{\mu}+\hat{1}}, & \text{if } \hat{\mu}=\hat{1},\cdots,\hat{d}-\hat{1},\\[2ex] \widehat{id}, & \text{if } \hat{\mu}=\hat{d},\end{cases}\tag{20a}$$

where $t_{\hat{\mu}}(\boldsymbol{j})$ is the action of the generator of $\mathrm{G}^{\mathrm{tilt}}_{\mathrm{trsl}}$ on the repeat unit cell $\boldsymbol{j}\in\Lambda$, and

$$\widehat{H}^{\mathrm{tilt}}:=\sum_{a=1}^{|\Lambda|}\left(\widehat{T}_{\hat{\mu}}\right)^{a}\hat{h}^{\mathrm{tilt}}_{\boldsymbol{j}}\left(\widehat{T}^{\dagger}_{\hat{\mu}}\right)^{a}\qquad\text{(choices of } \hat{\mu} \text{ and } \boldsymbol{j} \text{ are arbitrary)}\,,\tag{20b}$$

respectively.

Step 3 proof Theorem I revisited is not needed for Theorem II

We define for any $g \in \mathrm{G}_f$ with $\mathfrak{c}(g) = +1$ the generator of symmetry twisted translation

$$\widehat{T}_{\hat{1}}(g) := \hat{u}_{\boldsymbol{I}}(g)\,\widehat{T}_{\hat{1}}, \qquad \hat{u}_{\boldsymbol{I}}^{-1}(g) = \hat{u}_{\boldsymbol{I}}^{\dagger}(g), \tag{21a}$$

through its action

$$\widehat{T}_{\hat{1}}(g)\,\hat{\chi}_{\boldsymbol{j}}\,\widehat{T}_{\hat{1}}^{-1}(g) = \begin{cases} (-1)^{\rho(g)}\,\hat{\chi}_{t_{\hat{1}}(\boldsymbol{j})}, & \text{if } \boldsymbol{j} \neq \boldsymbol{N}, \\ \widehat{U}_{\boldsymbol{I}}(g)\,\hat{\chi}_{\boldsymbol{I}}\,\widehat{U}_{\boldsymbol{j}}^{-1}(g), & \text{if } \boldsymbol{j} = \boldsymbol{N}, \end{cases} \tag{21b}$$

on any Majorana operator labeled by $\boldsymbol{j} \in \Lambda$. Here, $\boldsymbol{I} \equiv (1, \cdots, 1) \in \Lambda$, $\boldsymbol{N} = (N_{\hat{1}}, \cdots, N_{\hat{d}}) \in \Lambda$, and $\boldsymbol{j} = (n_{\hat{1}}, \cdots, n_{\hat{d}})$ with $n_{\hat{\mu}} = 1, \cdots, N_{\hat{\mu}}$.

One verifies the symmetry twisted algebra

$$\widehat{U}(h)^{-1}\,\widehat{T}_{\hat{1}}(g)\,\widehat{U}(h) = e^{\mathrm{i}\chi(g,h)}\,\widehat{T}_{\hat{1}}(g), \tag{22a}$$

where

$$\chi(g,h) := \phi(g,h) - \phi(h,g) + (|\Lambda| - 1)\pi\,\rho(h)[\rho(g) + 1], \tag{22b}$$

Step 4 proof Theorem II revisited

Instead of extracting spectral properties of Hamiltonian $\widehat{H}_{\mathrm{pbc}}$ directly, we shall do so with the family of Hamiltonians indexed by $g \in \mathrm{G}_f$ and given by

$$\widehat{H}^{\mathrm{tilt}}_{\mathrm{twis}}(g) := \sum_{a=1}^{|\Lambda|} \left(\widehat{T}_{\hat{1}}(g)\right)^{a} \, \hat{h}^{\mathrm{tilt}}_{\boldsymbol{l}} \, \left(\widehat{T}^{-1}_{\hat{1}}(g)\right)^{a}, \tag{23}$$

where $\hat{h}^{\mathrm{tilt}}_{1}$ is a G_f-symmetric and local Hermitian operator.

Step 5 proof Theorem II revisited

In terms of the Majorana spinors $\hat{\chi}_{\boldsymbol{j}}$, the total fermion parity operator $\widehat{P}$ has the representation

$$\widehat{P} := \mathrm{i}^{|\Lambda|/2} \prod_{\boldsymbol{j}\in\Lambda} \prod_{l=1}^{2n+1} \hat{\chi}_{\boldsymbol{j},l}. \tag{24}$$

Conjugation of the fermion parity operator $\widehat{P}$ by the tilted translation operator $\widehat{T}_{\hat{1}}$ delivers

$$\widehat{T}_{\hat{1}}\,\widehat{P}\,\widehat{T}_{\hat{1}}^{-1} = (-1)^{|\Lambda|-1}\widehat{P} = -\widehat{P}, \tag{25}$$

where we arrived at the last equality by noting that $|\Lambda|$ is an even integer. The factor $(-1)^{|\Lambda|-1}$ arises since each spinor $\hat{\chi}_{\boldsymbol{j}}$ consists of an odd number of Majorana operators.

The nontrivial algebra (25) implies that the ground state of any Hamiltonian that commutes with $\widehat{P}$, $\widehat{T}_{\hat{1}}$, and the generators of G_f is either degenerate or spontaneously breaks translation or G_f symmetry.

Quantum field theories in $(d+1)$ spacetime

Let space be $d = 0, 1, 2, \cdots$ dimensional.

Spacetime is then $(d+1) = 1, 2, 3, \cdots$ dimensional.

A quantum field theory in $(d+1)$-dimensional space time is **defined** by the unitary time evolution

$$\mathcal{U}(t_2, t_1) := \int \mathcal{D}[\varphi, \partial_\mu \varphi, \cdots]\, e^{+\frac{\mathrm{i}}{\hbar}\int\limits_{t_1}^{t_2} \mathrm{d}t\, L[\varphi, \partial_\mu \varphi, \cdots]} \tag{26}$$

for given initial and final fields at time t_1 and t_2, respectively.

Anomalies in quantum $(d+1)$ space quantum field theory

If a quantum field theory is invariant under a **global internal symmetry**, while it is **impossible to gauge consistently** this global internal symmetry by the addition of **local** terms to the Lagrangian density, then the (many-body) eigenstates are constrained by a **'t Hooft anomaly**.

The **'t Hooft anomaly matching condition** implies that the ground state of the quantum field theory must be compatible with the **'t Hooft anomaly**.

Any trivially gapped and non-degenerate ground state is free from a **'t Hooft anomaly**.

The **'t Hooft anomaly matching condition** thus requires that any trivially gapped ground state must necessarily break the global internal symmetry down to a subgroup that trivializes its t'Hooft anomaly.

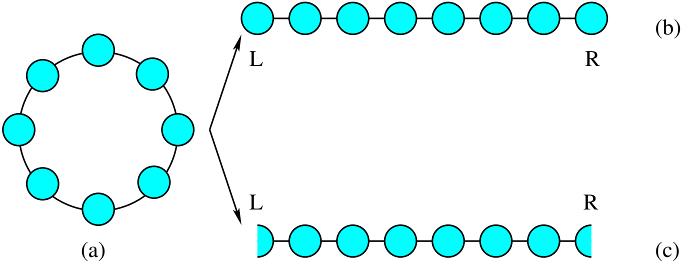
LSM anomalies

The LSM Theorem I is the lattice counterpart to the **’t Hooft anomaly** in quantum-field theory.

We thus reinterpret the LSM Theorem I as an LSM anomaly.

LSM anomalies and crystalline invertible topological phases of matter

The boundaries of **crystalline** invertible topological phases of matters hosting topologically protected zero modes realize LSM anomalies.



In one-dimensional space, fermionic invertible topological phase of matter are labeled by three indices ν, ρ, and μ associated to the second cohomology group

$$([(\nu,\rho)],[\mu]) \in \mathrm{H}^2(\mathrm{G}_f, \mathrm{U}(1)_{\mathfrak{c}}), \qquad \mathrm{G}_f \text{ a central extension of } \mathrm{G} \text{ by } \mathbb{Z}_2^{\mathrm{F}}, \text{ i.e., } \mathrm{G}_f/\mathbb{Z}_2^{\mathrm{F}} = \mathrm{G}.$$