Matrix product operator algebras

arXiv:2204.05940

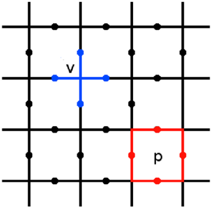
Andras Molnar, José Garre Rubio, Alberto Ruiz-de-Alarcón,
Norbert Schuch, David Pérez-García, Ignacio Cirac

(Intrinsic) Topological order

- Gapped phases
- Degenerate ground space, number of GS $\sim$ topology
- Different GS are locally indistinguishable $\Rightarrow$ no local order parameter for PT
- Excitations: point-like, free, anyons

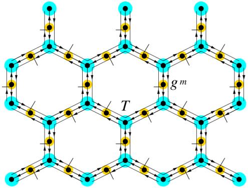
Toy models for spin systems

Kitaev models



- Groups
- C^*-Hopf algebras
- C^*-Weak Hopf algebras

String-net models



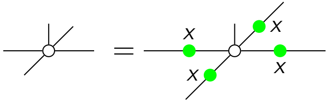
- Unitary fusion categories

Hamiltonian: commuting projectors
Anyons described by "doubled" object
PEPS description

[Kitaev:quant-ph/9707021, Levin:cond-mat/0404617, Gu:0809.2821, Buerschaper:1007.5283, Jia:2302.08131]

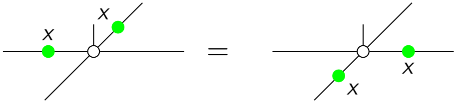
Topological PEPS

- PEPS allows perturbation from RFP, phase transitions
- Origin of topological properties: symmetry of the PEPS tensor
- Purely virtual
- "Size-independent"
- Matrix product operators



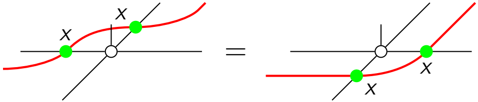
Topological PEPS

- PEPS allows perturbation from RFP, phase transitions
- Origin of topological properties: symmetry of the PEPS tensor
- Purely virtual
- "Size-independent"
- Matrix product operators



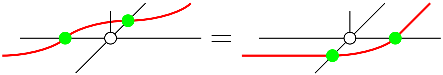
Topological PEPS

- PEPS allows perturbation from RFP, phase transitions
- Origin of topological properties: symmetry of the PEPS tensor
- Purely virtual
- “Size-independent”
- Matrix product operators



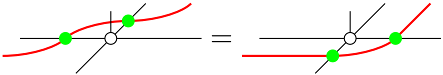
Size-independence of the symmetry

MPO symmetry of the PEPS tensor:

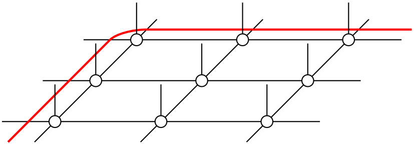


Size-independence of the symmetry

MPO symmetry of the PEPS tensor:

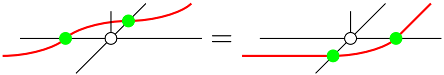


Symmetry of large area:

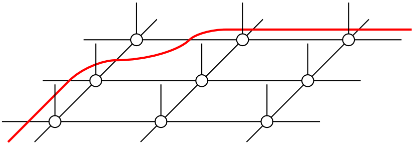


Size-independence of the symmetry

MPO symmetry of the PEPS tensor:

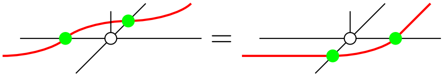


Symmetry of large area:

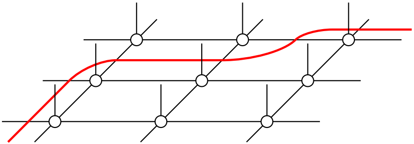


Size-independence of the symmetry

MPO symmetry of the PEPS tensor:

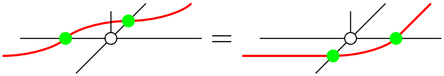


Symmetry of large area:

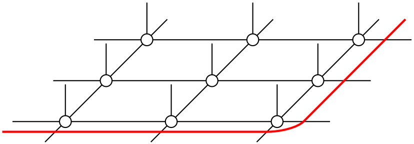


Size-independence of the symmetry

MPO symmetry of the PEPS tensor:

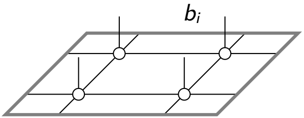


Symmetry of large area:



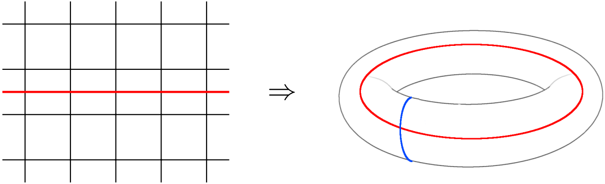
Topological ground space degeneracy

Hamiltonian:

$$|\Psi\rangle \approx \sum_i$$ 

$$\otimes |\Psi^{i}_{rest}\rangle$$

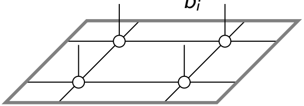
Hamiltonian does not detect the symmetry operator:



Topological ground space degeneracy

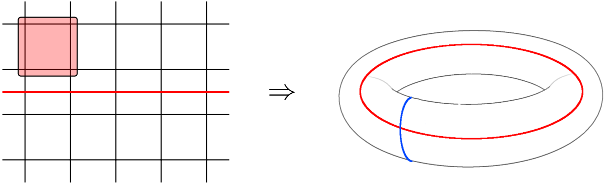
Hamiltonian:

$$|\Psi\rangle \approx \sum_i$$



$$\otimes |\Psi^i_{rest}\rangle$$

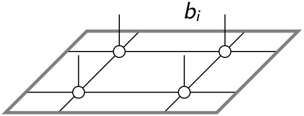
Hamiltonian does not detect the symmetry operator:



Topological ground space degeneracy

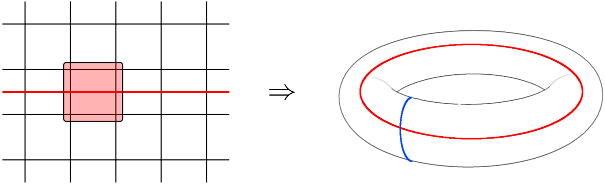
Hamiltonian:

$$|\Psi\rangle \approx \sum_i$$



$$\otimes\, |\Psi^i_{rest}\rangle$$

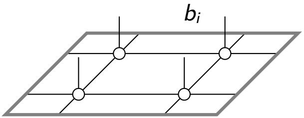
Hamiltonian does not detect the symmetry operator:



Topological ground space degeneracy

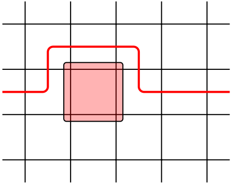
Hamiltonian:

$$|\Psi\rangle \approx \sum_i$$

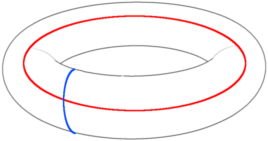


$$\otimes\, |\Psi^i_{rest}\rangle$$

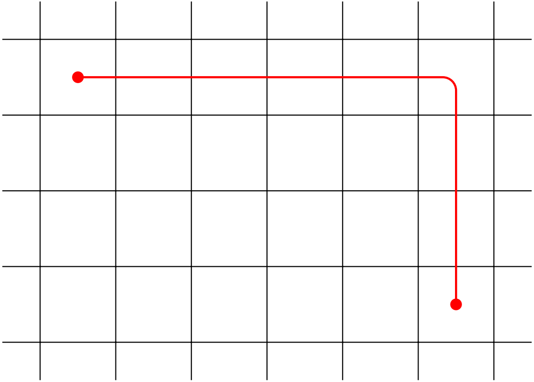
Hamiltonian does not detect the symmetry operator:



$\Rightarrow$

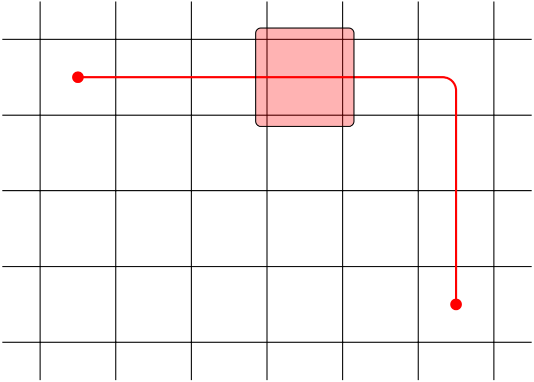


Excitations – example



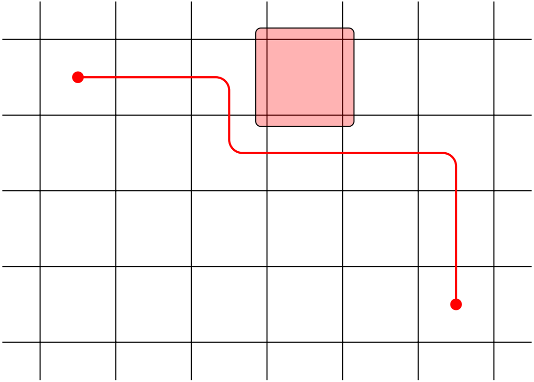
[Bultinck:1511.08090]

Excitations – example



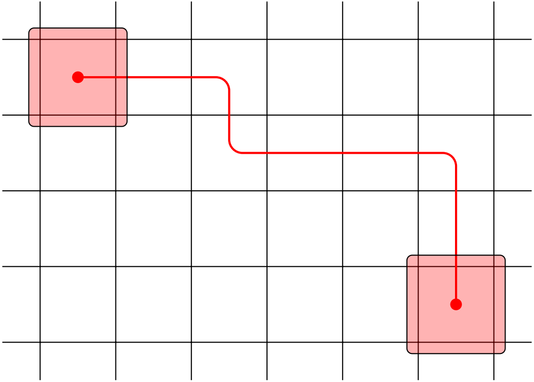
[Bultinck:1511.08090]

Excitations – example



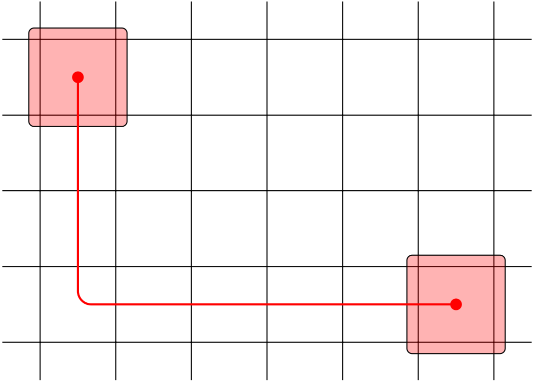
[Bultinck:1511.08090]

Excitations – example



[Bultinck:1511.08090]

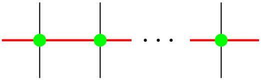
Excitations – example



[Bultinck:1511.08090]

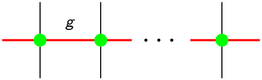
The MPO symmetries

Toric code: 2 MPOs,



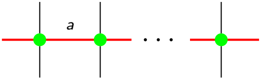
$$\equiv \begin{cases} 1 \otimes 1 \otimes \cdots \otimes 1 \\ X \otimes X \otimes \cdots \otimes X \end{cases}$$

G-injective PEPS: MPOs elements of G



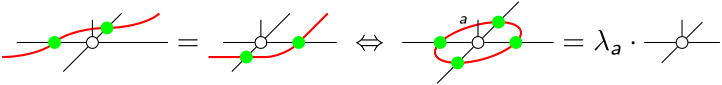
$$\equiv g \otimes g \otimes \cdots \otimes g, \quad g \in G$$

String-net models: MPO = elements of fusion category $\mathcal{C}$

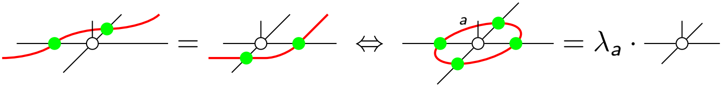


$$\equiv a \in Obj(\mathcal{C})$$

Algebra of symmetries



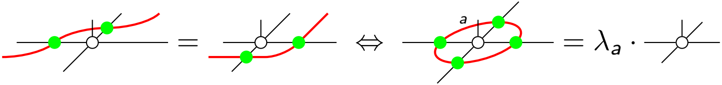
Algebra of symmetries



Linear combination and product of MPOs is symmetry:

$$(O_a + O_b)|T\rangle = (\lambda_a + \lambda_b)|T\rangle \quad \text{and} \quad O_a \cdot O_b|T\rangle = \lambda_a \cdot \lambda_b|T\rangle$$

Algebra of symmetries

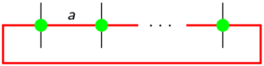


Linear combination and product of MPOs is symmetry:

$$(O_a + O_b)|T\rangle = (\lambda_a + \lambda_b)|T\rangle \quad \text{and} \quad O_a \cdot O_b|T\rangle = \lambda_a \cdot \lambda_b|T\rangle$$

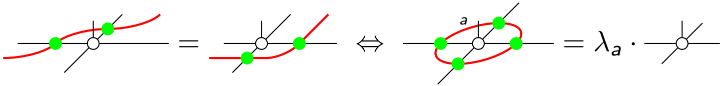
Algebra of symmetries:

$$\mathcal{A}_{PBC} \equiv \left\{ \sum_a \lambda_a \cdot \right.$$



$$\left. \middle| \lambda_a \in \mathbb{C} \right\}$$

Algebra of symmetries

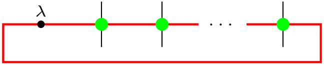


Linear combination and product of MPOs is symmetry:

$$(O_a + O_b)|T\rangle = (\lambda_a + \lambda_b)|T\rangle \quad \text{and} \quad O_a \cdot O_b|T\rangle = \lambda_a \cdot \lambda_b|T\rangle$$

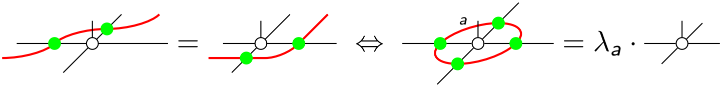
Algebra of symmetries:

$$\mathcal{A}_{PBC} \equiv \Big\{$$



$$\Big|\, \lambda = \bigoplus_a \lambda_a \cdot \mathbb{1}_{D_a} \Big\}$$

Algebra of symmetries

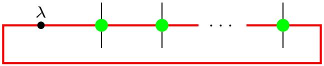


Linear combination and product of MPOs is symmetry:

$$(O_a + O_b)|T\rangle = (\lambda_a + \lambda_b)|T\rangle \quad \text{and} \quad O_a \cdot O_b|T\rangle = \lambda_a \cdot \lambda_b|T\rangle$$

Algebra of symmetries:

$$\mathcal{A}_{PBC} \equiv \Bigg\{$$

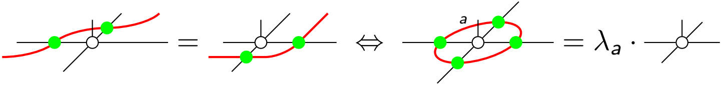


$$\Bigg| \lambda = \bigoplus_a \lambda_a \cdot \mathbb{1}_{D_a} \Bigg\}$$

The product for groups:

$$O_g \cdot O_h = O_{gh}$$

Algebra of symmetries

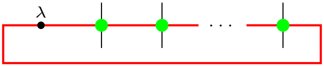


Linear combination and product of MPOs is symmetry:

$$(O_a + O_b)|T\rangle = (\lambda_a + \lambda_b)|T\rangle \quad \text{and} \quad O_a \cdot O_b|T\rangle = \lambda_a \cdot \lambda_b|T\rangle$$

Algebra of symmetries:

$$\mathcal{A}_{PBC} \equiv \Bigg\{$$



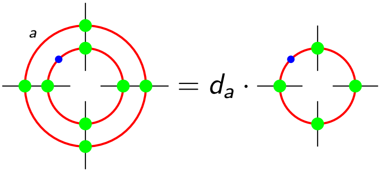
$$\Bigg| \lambda = \bigoplus_a \lambda_a \cdot \mathbb{1}_{D_a} \Bigg\}$$

The product for fusion categories:

$$O_a \cdot O_b = \sum_c N_{ab}^c O_c, \quad N_{ab}^c \in \mathbb{Z}^+$$

PEPS from MPO symmetries

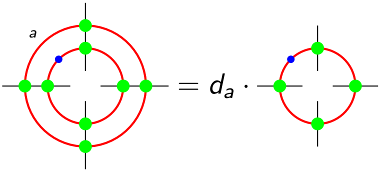
Special element $O \in \mathcal{A}_{PBC}$:

$O_a \cdot O = d_a O$ or 

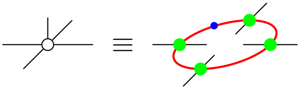
PEPS from MPO symmetries

Special element $O \in \mathcal{A}_{PBC}$:

$O_a \cdot O = d_a O$ or



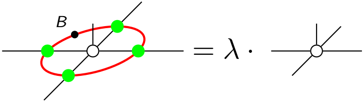
Then a symmetric PEPS tensor is:



[Bultinck:1511.08090, Lootens:2008.11187]

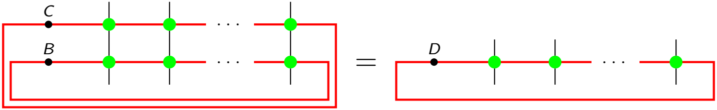
Our perspective

Open boundary MPOs are also symmetries:

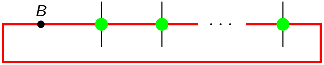


$B \in \bigoplus_a \mathcal{M}(D_a)$

Open boundary MPOs form an algebra: $\forall B, C\ \exists D$ s.t.



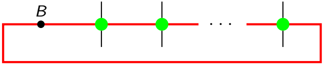
The algebra of symmetries:

$\mathcal{A}_{OBC} \equiv \Bigg\{$  $\Bigg| B \in \bigoplus_a \mathcal{M}(D_a) \Bigg\}$

Weak Hopf algebra

- Injective tensors:

$$\mathcal{A} \equiv \Bigg\{$$

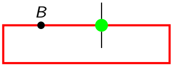


$$\Bigg| B \in \bigoplus_a \mathcal{M}(D_a) \Bigg\}$$

Weak Hopf algebra

- Injective tensors:

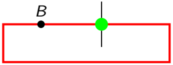
$$\mathcal{A} \equiv \Bigg\{$$



$$\Bigg| B \in \bigoplus_a \mathcal{M}(D_a) \Bigg\}$$

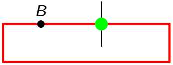
Weak Hopf algebra

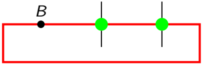
- Injective tensors:

$\mathcal{A} \equiv \Bigg\{$ 

$\Bigg| B \in \bigoplus_a \mathcal{M}(D_a) \Bigg\}$

- Growing of MPO: $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ "coproduct":

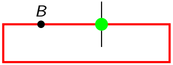
$\Delta :$ 

$\mapsto$ 

$\in \mathcal{A} \otimes \mathcal{A}$

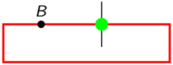
Weak Hopf algebra

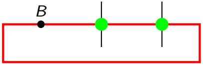
- Injective tensors:

$\mathcal{A} \equiv \Big\{$ 

$\Big| B \in \bigoplus_a \mathcal{M}(D_a) \Big\}$

- Growing of MPO: $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ "coproduct":

$\Delta :$ 

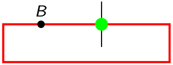
$\mapsto$ 

$\in \mathcal{A} \otimes \mathcal{A}$

- $(\mathcal{A}, \Delta)$ + additional properties = (weak) Hopf algebra
  [Bohm:math/9805116, Montgomery: Rep Theory of Semisimple Hopf]

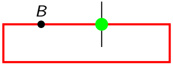
Weak Hopf algebra

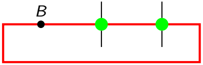
- Injective tensors:

$\mathcal{A} \equiv \Bigg\{$ 

$\Bigg| \, B \in \bigoplus_a \mathcal{M}(D_a) \Bigg\}$

- Growing of MPO: $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ "coproduct":

$\Delta :$ 

$\mapsto$ 

$\in \mathcal{A} \otimes \mathcal{A}$

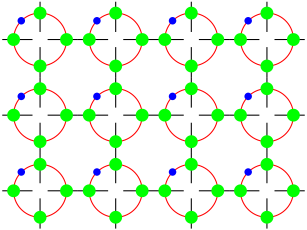
- $(\mathcal{A}, \Delta)$ + additional properties = (weak) Hopf algebra
[Bohm:math/9805116, Montgomery: Rep Theory of Semisimple Hopf]
- Fusion category $\equiv$ Weak Hopf algebras
[Etingof:math/0203060, Kitaev:1104.5047]

Use of algebraic formulation

- Transfer operator of topological PEPS: renormalization fixed point MPDO
  [Ruiz-de-Alarcón:2204.06295]
- Phase classification of RFP MPDO
  [Ruiz-de-Alarcón:2204.06295]
- Characterization of symmetries in topologically ordered PEPS
  [Molnar: in preparation]
- Other possibly interesting states with topological properties

New models?

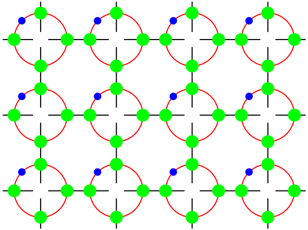
- Algebraic object
  - Unitary Fusion Category
  - C^* weak Hopf algebra
- MPO representations
- Special algebra element $\rightarrow$ PEPS
  - Kitaev/String-net Hamiltonian
  - Parent Hamiltonian



New models?

- Algebraic object
  - Unitary Fusion Category
  - C^* weak Hopf algebra
- MPO representations
- Special algebra element → PEPS
  - Kitaev/String-net Hamiltonian
  - Parent Hamiltonian

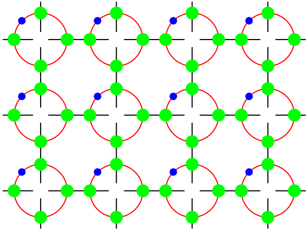
Taft Hopf algebra



New models?

- Algebraic object
  - ~~Unitary Fusion Category~~
  - ~~C^* weak Hopf algebra~~
- MPO representations
- Special algebra element $\rightarrow$ PEPS
  - ~~Kitaev/String-net Hamiltonian~~
  - Parent Hamiltonian

Taft Hopf algebra



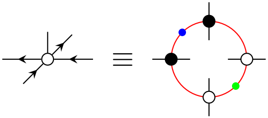
Taft-Hopf-injective PEPS

- The PEPS is non-zero on finite region, torus
- The PEPS is zero on a sphere
- The GS of the parent Hamiltonian is the expected one
- Rotation: local unitaries
- Not zero correlation length
- Nicer Hamiltonian? Excitations?
- Connection to non-semisimple TQFTs?

Conclusion

- Topological order: MPO symmetries
- Topological GS degeneracy, anyons
- MPO from fusion categories
- Alternative formulation: weak Hopf algebra

MPO injectivity of string-net models



- MPOs built from fusion categories
- 2 types of MPO tensors (incoming vs outgoing index)
- Blue and green tensors: quantum dimensions