

Matrix product operator algebras

arXiv:2204.05940

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Norbert Schuch, David Pérez-García, Ignacio Cirac



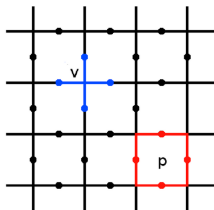
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(Intrinsic) Topological order

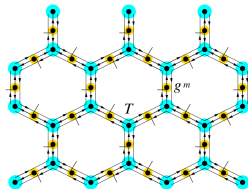
- ▶ Gapped phases
- ▶ Degenerate ground space, number of GS \sim topology
- ▶ Different GS are locally indistinguishable \Rightarrow no local order parameter for PT
- ▶ Excitations: point-like, free, anyons

Toy models for spin systems

Kitaev models



String-net models



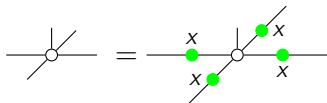
- ▶ Groups
- ▶ C^* -Hopf algebras
- ▶ C^* -Weak Hopf algebras

- ▶ Unitary fusion categories

Hamiltonian: commuting projectors
Anyons described by “doubled” object
PEPS description

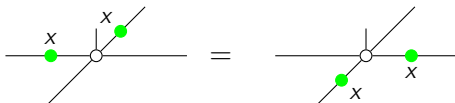
Topological PEPS

- ▶ PEPS allows perturbation from RFP, phase transitions
- ▶ Origin of topological properties: symmetry of the PEPS tensor
- ▶ Purely virtual
- ▶ “Size-independent”
- ▶ Matrix product operators



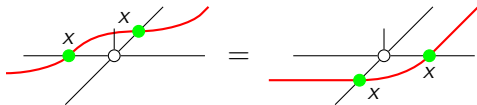
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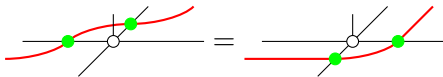
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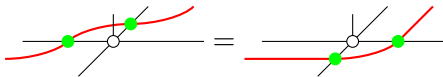
Size-independence of the symmetry

MPO symmetry of the PEPS tensor:

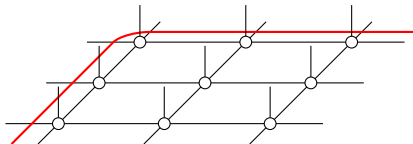


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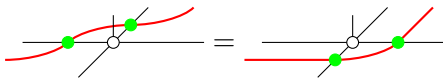


Symmetry of large area:

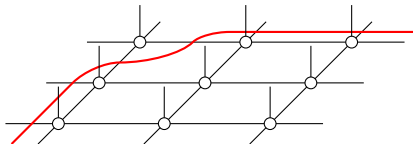


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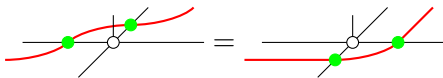


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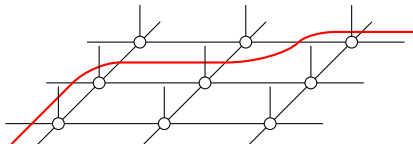


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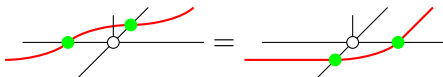


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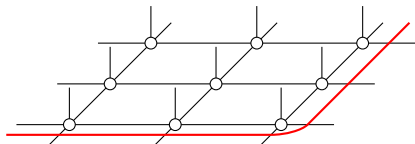


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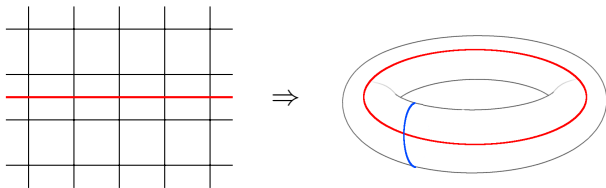
Topological ground space degeneracy

Hamiltonian:

$$|\Psi\rangle \approx \sum_i \left[\text{Diagram} \right] \otimes |\Psi_{rest}^i\rangle$$

The diagram in the equation shows a 3D perspective of a 2D grid of sites. The grid is composed of horizontal and diagonal lines. Four sites are highlighted with white circles. Vertical lines extend upwards from each of these sites, and a label b_i is placed above the top-right site, indicating a symmetry operator acting on these sites.

Hamiltonian does not detect the symmetry operator:

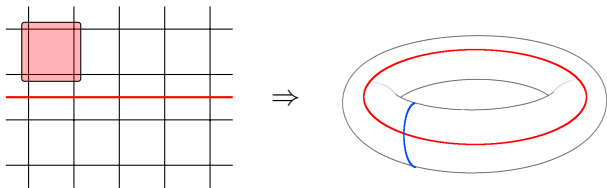


Topological ground space degeneracy

Hamiltonian:

$$|\Psi\rangle \approx \sum_i \text{[Diagram of a 2D lattice with four sites and vertical bonds labeled } b_i] \otimes |\Psi_{rest}^i\rangle$$

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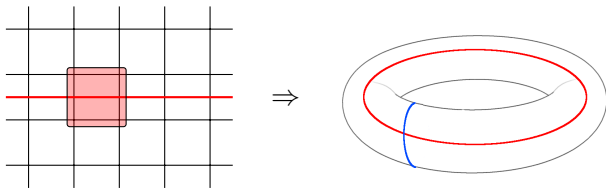


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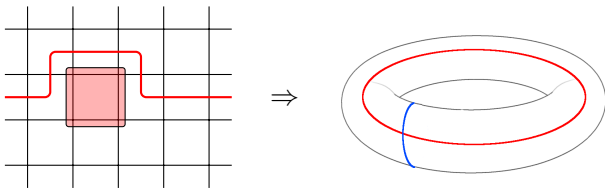


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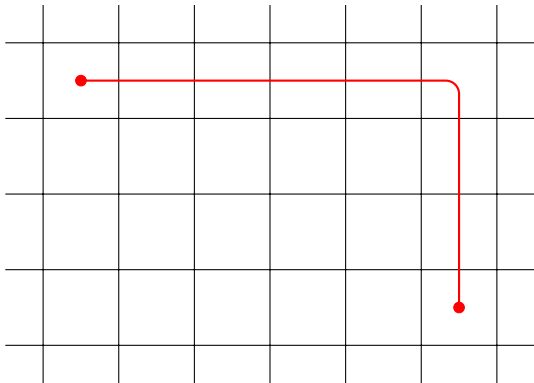
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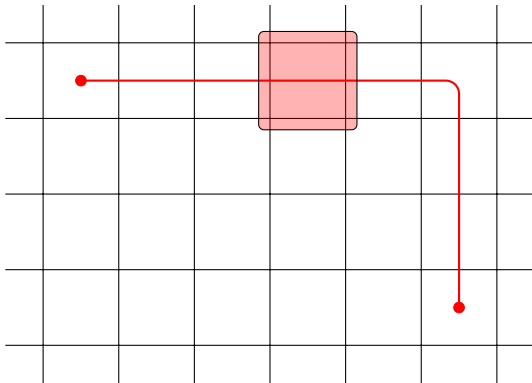


Excitations – example



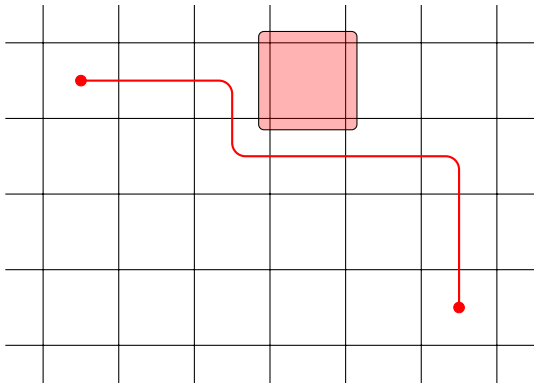
[Bultinck:1511.08090]

Excitations – example



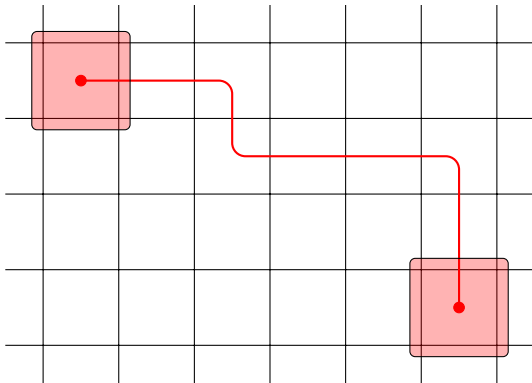
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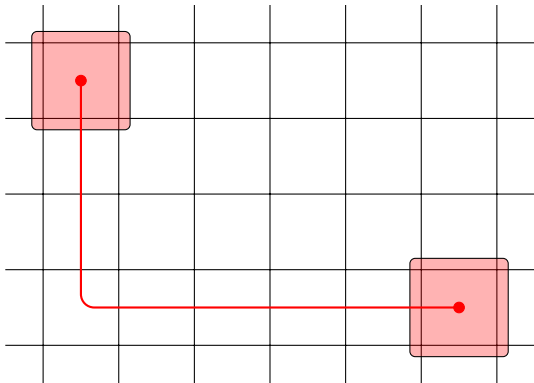
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Excitations – example



[Bultinck:1511.08090]

The MPO symmetries

Toric code: 2 MPOs,

$$\begin{array}{c} | \\ | \\ \text{---} \\ | \\ | \end{array} \begin{array}{c} | \\ | \\ \text{---} \\ | \\ | \end{array} \cdots \begin{array}{c} | \\ | \\ \text{---} \\ | \\ | \end{array} \equiv \begin{cases} 1 \otimes 1 \otimes \cdots \otimes 1 \\ X \otimes X \otimes \cdots \otimes X \end{cases}$$

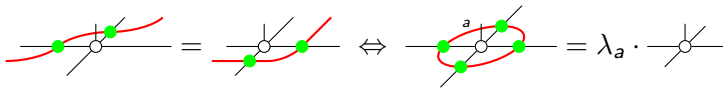
G-injective PEPS: MPOs elements of G

$$\begin{array}{c} | \\ | \\ \text{---} \\ | \\ | \end{array} \begin{array}{c} | \\ | \\ \text{---} \\ | \\ | \end{array} \cdots \begin{array}{c} | \\ | \\ \text{---} \\ | \\ | \end{array} \equiv g \otimes g \otimes \cdots \otimes g, \quad g \in G$$

String-net models: MPO = elements of fusion category \mathcal{C}

$$\begin{array}{c} | \\ | \\ \text{---} \\ | \\ | \end{array} \begin{array}{c} | \\ | \\ \text{---} \\ | \\ | \end{array} \cdots \begin{array}{c} | \\ | \\ \text{---} \\ | \\ | \end{array} \equiv a \in \text{Obj}(\mathcal{C})$$

Algebra of symmetries

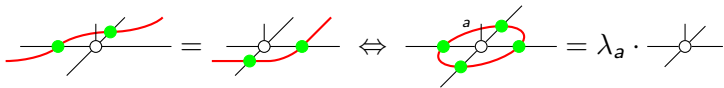


The diagram illustrates an equation in the algebra of symmetries. It consists of four terms connected by mathematical symbols: an equals sign, a double-headed arrow, and another equals sign.

- Term 1:** A central white circle with four lines extending from it (up, down, left, right). A red curve passes through the circle, with two green dots on it. The curve is concave up on the left and concave down on the right.
- Term 2:** A central white circle with four lines. A red curve passes through the circle, with two green dots on it. The curve is concave down on the left and concave up on the right.
- Term 3:** A central white circle with four lines. A red ellipse encloses the circle and two green dots on the horizontal line. The label 'a' is placed above the top-right green dot.
- Term 4:** A central white circle with four lines.

The equation is: $\text{Term 1} = \text{Term 2} \Leftrightarrow \text{Term 3} = \lambda_a \cdot \text{Term 4}$

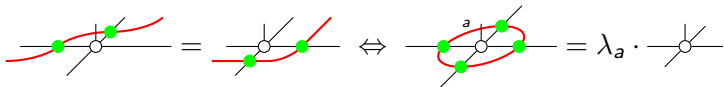
Algebra of symmetries



Linear combination and product of MPOs is symmetry:

$$(O_a + O_b)|T\rangle = (\lambda_a + \lambda_b)|T\rangle \quad \text{and} \quad O_a \cdot O_b|T\rangle = \lambda_a \cdot \lambda_b|T\rangle$$

Algebra of symmetries



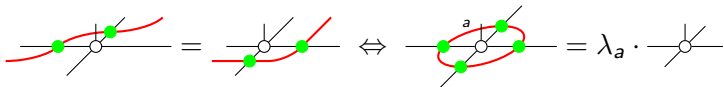
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Algebra of symmetries:

$$\mathcal{A}_{PBC} \equiv \left\{ \sum_a \lambda_a \cdot \boxed{\text{---} \overset{a}{\bullet} \text{---} \bullet \text{---} \cdots \text{---} \bullet \text{---}} \mid \lambda_a \in \mathbb{C} \right\}$$

Algebra of symmetries



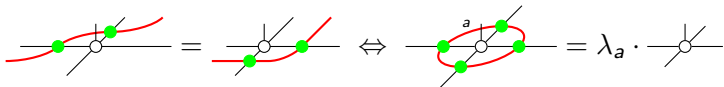
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Algebra of symmetries:

$$\mathcal{A}_{PBC} \equiv \left\{ \left[\overset{\lambda}{\bullet} \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \right] \mid \lambda = \bigoplus_a \lambda_a \cdot \mathbb{1}_{D_a} \right\}$$

Algebra of symmetries



Linear combination and product of MPOs is symmetry:

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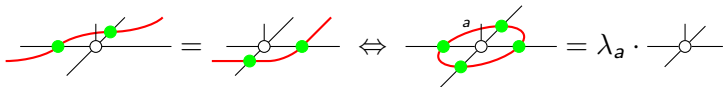
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The product for **groups**:

$$O_g \cdot O_h = O_{gh}$$

Algebra of symmetries



Linear combination and product of MPOs is symmetry:

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The product for **fusion categories**:

$$O_a \cdot O_b = \sum_c N_{ab}^c O_c, \quad N_{ab}^c \in \mathbb{Z}^+$$

PEPS from MPO symmetries

Special element $O \in \mathcal{A}_{PBC}$:

$$O_a \cdot O = d_a O \quad \text{or} \quad \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} = d_a \cdot \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array}$$

PEPS from MPO symmetries

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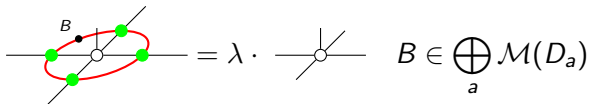
Then a symmetric PEPS tensor is:

$$\begin{array}{c} \diagup \\ \bigcirc \\ \diagdown \end{array} \equiv \begin{array}{c} \diagup \bullet \diagdown \\ \text{---} \bullet \text{---} \bullet \text{---} \\ \diagdown \bullet \diagup \end{array}$$

[Bultinck:1511.08090, Lootens:2008.11187]

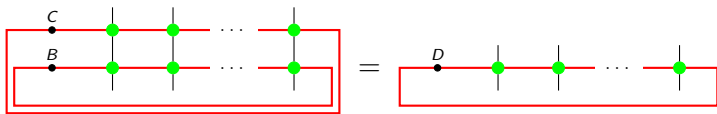
Our perspective

Open boundary MPOs are also symmetries:



$$B \in \bigoplus_a \mathcal{M}(D_a)$$

Open boundary MPOs form an algebra: $\forall B, C \exists D$ s.t.



The algebra of symmetries:

$$\mathcal{A}_{OBC} \equiv \left\{ \begin{array}{c} B \\ \text{MPO diagram} \end{array} \middle| B \in \bigoplus_a \mathcal{M}(D_a) \right\}$$

Weak Hopf algebra

► Injective tensors:

$$\mathcal{A} \equiv \left\{ \begin{array}{c} \text{---} B \text{---} \bullet \text{---} \bullet \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \right| B \in \bigoplus_a \mathcal{M}(D_a) \right\}$$

Weak Hopf algebra

- ▶ Injective tensors:

$$\mathcal{A} \equiv \left\{ \begin{array}{c} \text{Diagram: a red rectangle with a black dot labeled } B \text{ and a green dot on the top edge, and a vertical line passing through the green dot.} \\ \left| B \in \bigoplus_a \mathcal{M}(D_a) \right. \end{array} \right\}$$

- ▶ Growing of MPO: $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ "coproduct":

$$\Delta : \begin{array}{c} \text{Diagram: a red rectangle with a black dot labeled } B \text{ and a green dot on the top edge, and a vertical line passing through the green dot.} \\ \hline \text{Diagram: a red rectangle with a black dot labeled } B \text{ and two green dots on the top edge, and two vertical lines passing through the green dots.} \end{array} \in \mathcal{A} \otimes \mathcal{A}$$

Weak Hopf algebra

- ▶ Injective tensors:

$$\mathcal{A} \equiv \left\{ \begin{array}{c} \text{---} \overset{B}{\bullet} \text{---} \color{green}\bullet \text{---} \\ \color{red}\boxed{\phantom{\text{---} \overset{B}{\bullet} \text{---} \color{green}\bullet \text{---}}} \\ \color{red}\phantom{\text{---} \overset{B}{\bullet} \text{---} \color{green}\bullet \text{---}} \end{array} \middle| B \in \bigoplus_a \mathcal{M}(D_a) \right\}$$

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- ▶ (\mathcal{A}, Δ) + additional properties = (weak) Hopf algebra

[Bohm:math/9805116, Montgomery: Rep Theory of Semisimple Hopf]

Weak Hopf algebra

- ▶ Injective tensors:

$$\mathcal{A} \equiv \left\{ \begin{array}{c} \text{Diagram: a red rectangle with a black dot labeled } B \text{ on the top wire and a green dot on the bottom wire.} \\ \left| B \in \bigoplus_a \mathcal{M}(D_a) \right. \end{array} \right\}$$

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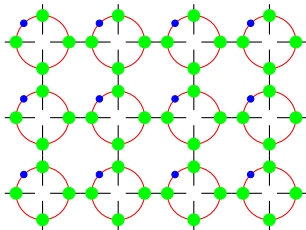
- ▶ (\mathcal{A}, Δ) + additional properties = (weak) Hopf algebra
[Bohm:math/9805116, Montgomery: Rep Theory of Semisimple Hopf]
- ▶ Fusion category \equiv Weak Hopf algebras
[Etingof:math/0203060, Kitaev:1104.5047]

Use of algebraic formulation

- ▶ Transfer operator of topological PEPS: renormalization fixed point MPDO
[Ruiz-de-Alarcón:2204.06295]
- ▶ Phase classification of RFP MPDO
[Ruiz-de-Alarcón:2204.06295]
- ▶ Characterization of symmetries in topologically ordered PEPS
[Molnar: in preparation]
- ▶ Other possibly interesting states with topological properties

New models?

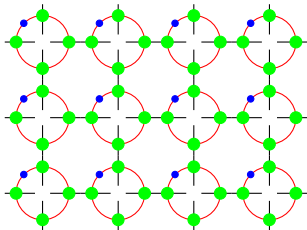
- ▶ Algebraic object
 - ▶ Unitary Fusion Category
 - ▶ C^* weak Hopf algebra
- ▶ MPO representations
- ▶ Special algebra element \rightarrow PEPS
 - ▶ Kitaev/String-net Hamiltonian
 - ▶ Parent Hamiltonian



New models?

- ▶ Algebraic object
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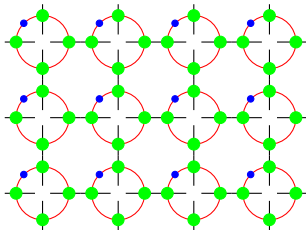
Taft Hopf algebra



New models?

- ▶ Algebraic object
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Taft Hopf algebra



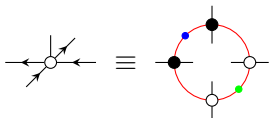
Taft-Hopf-injective PEPS

- ▶ The PEPS is non-zero on finite region, torus
- ▶ The PEPS is zero on a sphere
- ▶ The GS of the parent Hamiltonian is the expected one
- ▶ Rotation: local unitaries
- ▶ Not zero correlation length
- ▶ Nicer Hamiltonian? Excitations?
- ▶ Connection to non-semisimple TQFTs?

Conclusion

- ▶ Topological order: MPO symmetries
- ▶ Topological GS degeneracy, anyons
- ▶ MPO from fusion categories
- ▶ Alternative formulation: weak Hopf algebra

MPO injectivity of string-net models



- ▶ MPOs built from fusion categories
- ▶ 2 types of MPO tensors (incoming vs outgoing index)
- ▶ Blue and green tensors: quantum dimensions

