Classification and construction for crystalline topological superconductors and topological insulators in interacting fermion systems

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JH Zhang, QR Wang, S Yang, Y Qi, and ZC GU, PRB, 100501(R) (2020)
Y Ouyang, QR Wang, ZC Gu, Y Qi, Chinese Physics Letters 38 (12), 127101 (2021)
JH Zhang, S Yang, Y Qi, ZC Gu, Phys. Rev. Research 4, 033081 (2022)
Jian-Hao Zhang, Yang Qi, Zheng-Cheng Gu, arXiv:2204.13558 (2022)

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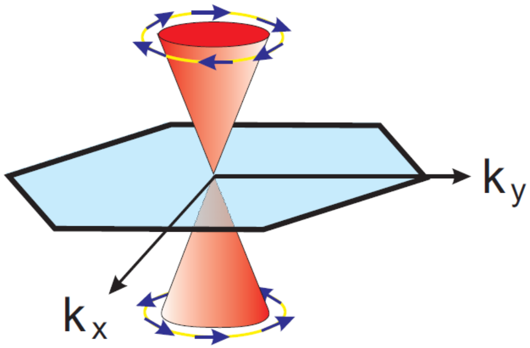
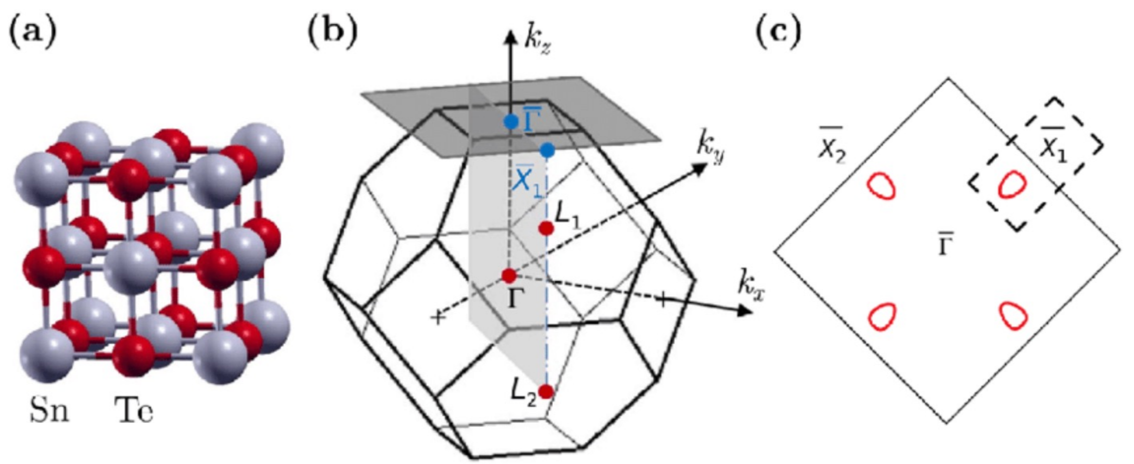
Example of symmetry protected topological (SPT) phases in free fermion systems

Topological insulator(with time reversal symmetry)



C L Kane, *et al*, 2005
B A Bernevig, et al 2006
W Molenkamp's group 2007
JE Moore, *et al*, 2007
M Zahid Hasan, *et al*, 2008

Crystalline topological insulator(with space group symmetry)



Liang Fu, 2011
TH Hsieh, *etal.* 2012

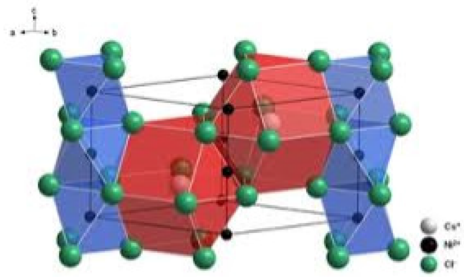
Topological insulator and superconductor in free fermion systems are classified by K theory(Kitaev 2008, A. P. Schnyder, *etal.* 2008, D. Freed, *etal.* 2012)

Crystalline SPT phases in interacting systems

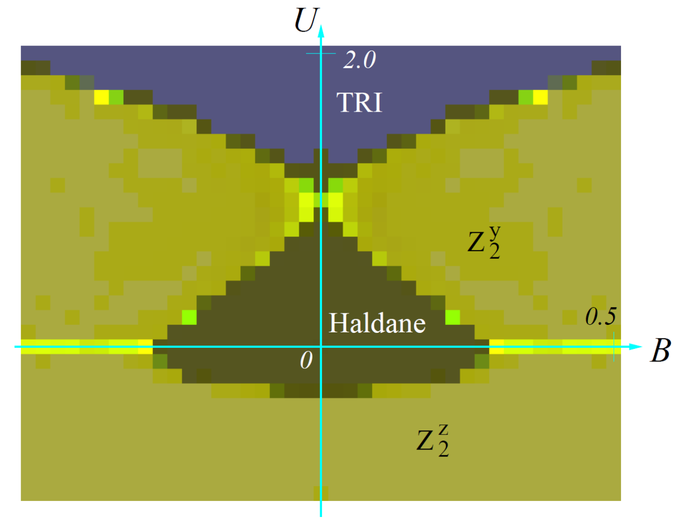
Spin-1 Haldane chain

$$H=\sum_i \left(\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1}+U(S_i^z)^2+BS_i^x\right)$$

Uc~1(B=0)

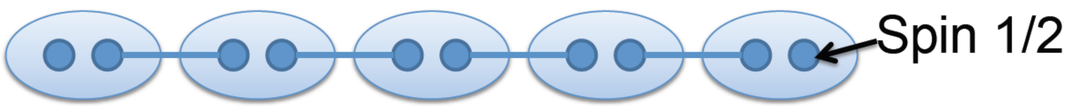


CsNiCl3(U~B~0)



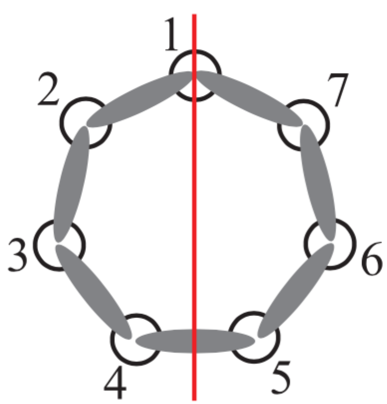
adiabatical connect to AKLT state

(Z C Gu, X G Wen 2009)



(Ian Affleck *etal.*, 1988)

SPT phases protected by many different symmetries, e.g., time reversal, spin rotation, bond inversion, etc.



- AKLT state carries mirror eigenvalue -1!

$$\mathcal{I}\left|\Psi_{S=1}^{\mathrm{AKLT}}\right\rangle_{L=7}=-\left|\Psi_{S=1}^{\mathrm{AKLT}}\right\rangle_{L=7}$$

(Frank Pollmann *etal.*, 2010)

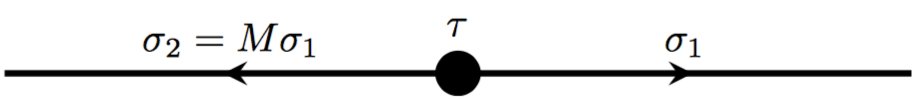
SPT phases with space group symmetry

Crystalline equivalence principle

- To compute the classification of space group SPT phases, we can just regard the space group as an internal symmetry, as long as mirror reflection symmetry is mapped into time reversal symmetry. (Ryan Thorngren and Dominic V. Else, PRX 8, 011040 (2018))

- All space group SPT phases can be constructed via the block state decoration scheme and realized as topological crystal! (Hao Song, Sheng-Jie Huang, Liang Fu, and Michael Hermele, Phys. Rev. X **7**, 011020 (2017), Zhida Song, Chen Fang, Yang Qi, Nature Communications 11, 4197 (2020))

- For example, SPT phases protected by time reversal symmetry and mirror reflection symmetry have the same classification.



For SPT protected by mirror reflection symmetry in 1D, one can just decorate a Z_2 charge on reflection point! The two different Z_2 eigenvalues gives rise to the correct classification!

Symmetry group G^f for fermion systems

- A generic G^f is defined by the short exact sequence:

$$1 \to \mathbb{Z}_2^f \to G_f \to G_b \to 1$$

- It is called central extension and specified by a $\mathbf{Z_2}$ coefficient cocycle:

$$\omega_2 \in H^2(G_b, \mathbb{Z}_2 = \{0, 1\})$$

- For a given group element in the total group G_f:

$$g_f = (P_f^{n(g)}, g_b) \in \mathbb{Z}_2^f \times G_b,$$

$$g_f \cdot h_f = \left(P_f^{n(g)}, g_b\right) \cdot \left(P_f^{n(h)}, h_b\right) := \left(P_f^{n(g)+n(h)+\omega_2(g_b,h_b)}, g_b h_b\right)$$

$$P_f^{n(g)+n(h)+\omega_2(g_b,h_b)} \in \mathbb{Z}_2^f \text{ and } g_b h_b \in G_b.$$

- The associativity of group multiplication holds naturally due to the fact that:

$$\omega_2(h,k) + \omega_2(gh,k) + \omega_2(g,hk) + \omega_2(g,h) = 0$$

Construction and classification of crystalline FSPT phases in 1D

- For spinless fermion systems, we have $M^2=1$, and the corresponding onsite symmetry on the 0D block(mirror reflection point) is $Z_2 \times Z_2^f$.

Two possible 0D block-states:

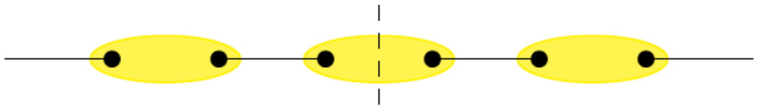
1. The fermion parity of the 0D block is odd.

2. The inversion eigenvalue of the 0D block is −1. (Trivialized by atomic state $c_1^\dagger c_2^\dagger|0\rangle$)

One possible 1D block-state (half infinite chain on the left/right side):

Kitaev's Majorana chain (There is no additional symmetry)

Nevertheless, a single Majorana chain is not comparable with a mirror reflection symmetry in spinless fermion systems.



$$M' : \ i\gamma_1\gamma_2 \longmapsto i\gamma_2\gamma_1 = -i\gamma_1\gamma_2$$

- The final classification results is Z_2 which is the same as FSPT protected by the time reversal symmetry $T^2=-1$(More precisely, $T^2=P^f$).

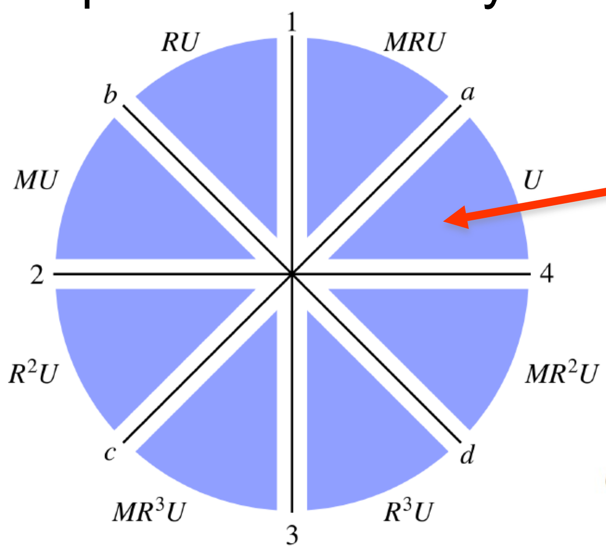
Real space construction for point group FSPT phases in 2D

Can be realized by free fermion systems

G_b \ spin	spinless	spin-1/2
C_{2m-1}	$\mathbb{Z}_{2m-1}$	$\mathbb{Z}_{2m-1}$
C_{2m}	$\mathbb{Z}_m$	$\begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_{4m}, & m \in \text{even} \\ \mathbb{Z}_{8m}, & m \in \text{odd} \end{cases}$
D_{2m-1}	$\mathbb{Z}_2$	$\mathbb{Z}_1$
D_{2m}	$\begin{cases} \mathbb{Z}_2 & \text{if } m \in \text{odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } m \in \text{even} \end{cases}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

C Wang, CH Lin, ZC Gu, PRB 95, 195147 (2017)
Meng Cheng, Chenjie Wang PRB 105, 195154 (2022)
Jian-Hao Zhang, *etal.*, PRB, 100501(R) (2020)

- An example of intrinsic interacting crystalline FSPT state with D_4 symmetry for spinless fermion systems



Extended trivialization:

$$O_U^{\text{loc}}|\psi\rangle = |T_U\rangle \otimes |\psi_{\bar{U}}\rangle$$

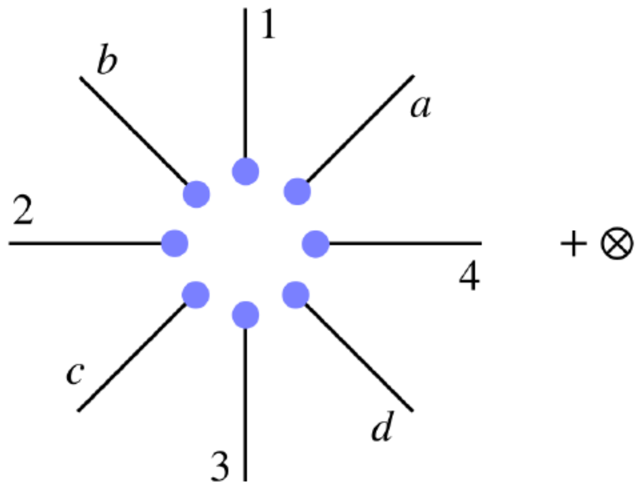
- T_U is a product state in region U

Symmetric trivialization:

$$O_R^{\text{loc}} = \bigotimes_{g\in D_4} V_g O_U^{\text{loc}} V_g^{-1} \Rightarrow |\psi'\rangle = O_R^{\text{loc}}|\psi\rangle = \bigotimes_{g\in D_4} |T_{gU}\rangle \otimes \bigotimes_{i=1,a}^{4,d} |\psi_i\rangle \otimes |\psi_{0\text{D}}\rangle$$

1D Block state decoration

- Divide 8 semi-infinite chains into two categories: category-I, (1234) and category-II, (abcd). These two categories are independent under the space-group symmetry actions



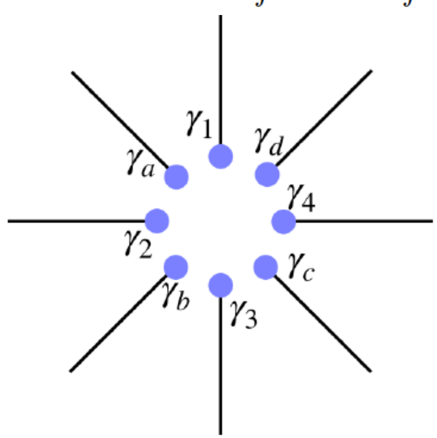
Two possible 1D block-states:

1. Majorana chain(not allowed)

2. 1D FSPT state with a total symmetry $\mathbf{Z_2 \times Z_2^f}$.

- Decoration Majorana chains on category-I or category-II only: $\boldsymbol{R}P_f = -P_f\boldsymbol{R}$

$$P_f = i^2\prod_{j=1}^{4}\gamma_j = -\gamma_1\gamma_2\gamma_3\gamma_4,\ \boldsymbol{R}:\ (\gamma_1,\gamma_2,\gamma_3,\gamma_4)\mapsto(\gamma_2,\gamma_3,\gamma_4,\gamma_1)$$

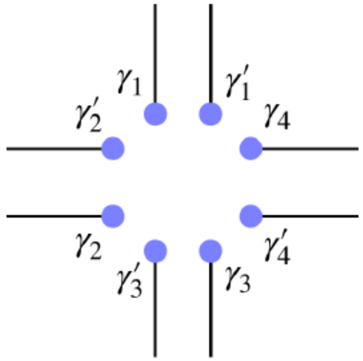


- Decoration Majorana chains on both categories, total fermion parity still anticommutes with M

$$P_f = i^4\prod_{i=1}^{4}\prod_{j=a}^{d}\gamma_i\gamma_j,\ \boldsymbol{M}:\ P_f\mapsto -P_f$$

• Decoration 1D FSPT state with a total symmetry $Z_2 \times Z_2^f$ on category-I or category-II only:

$$\boldsymbol{R}: \ \gamma_i \longmapsto \gamma_{i+1}, \quad \gamma_i' \longmapsto \gamma_{i+1}', \quad i \in \mathbb{Z}(\text{mod } 4)$$
$$\boldsymbol{M}: \ \gamma_i \longmapsto -\gamma_{6-i}, \quad \gamma_i' \longmapsto \gamma_{6-i}', \quad i \in \mathbb{Z}(\text{mod } 4)$$



• The minus sign for M is required as M is realized as a Z_2 internal symmetry on reflection axis.

• The 8 dangling Majorana modes can be gapped out without breaking both R and M.

$$\mathcal{H} = \mathcal{H}_U + \mathcal{H}_J$$

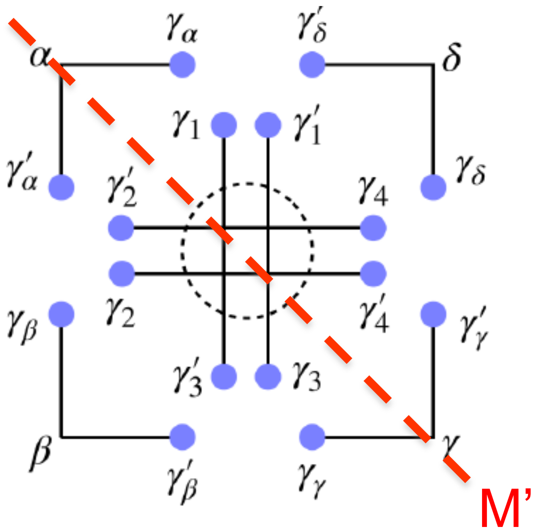
$$\mathcal{H}_U = U\left[\gamma_1\gamma_1'\gamma_3\gamma_3' + \gamma_2\gamma_2'\gamma_4\gamma_4'\right], \quad U > 0$$

Large U limit: $\gamma_1\gamma_1'\gamma_3\gamma_3' = \gamma_2\gamma_2'\gamma_4\gamma_4' = -1$

Within the above subspace, the following term will split the degeneracy:

$$\mathcal{H}_J = J(\gamma_1\gamma_2\gamma_1'\gamma_2' + \gamma_1\gamma_2\gamma_3'\gamma_4'), \quad J > 0$$

Trivialization



- Whether the 1D FSPT state decoration on category-I or category-II can be trivialized or not?

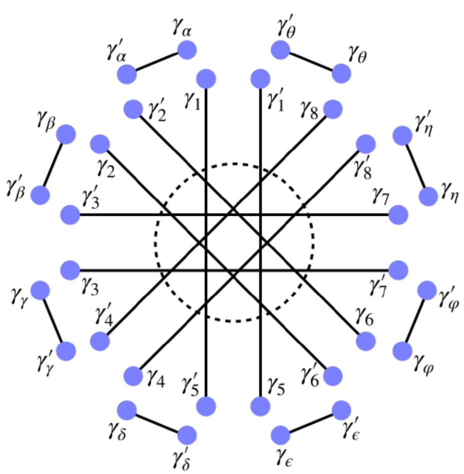
Attaching four Majorana chains on the edge

- For a local patch made by four Majorana zero modes, M is comparable with fermion parity.

$$M: \ (\gamma_1, \gamma_1', \gamma_\alpha, \gamma_\delta') \mapsto (-\gamma_1, \gamma_1', \gamma_\delta', \gamma_\alpha)$$

Nevertheless, a single Majorana chain is not comparable with a mirror reflection symmetry in spinless fermion systems.

$$M' = MR^3 \in D_4 \qquad (M')^2 = 1$$



- 1D FSPT state decoration on category-I or category-II can not be trivialized, however, decorations on both category-I and category-II can be trivialized.
- 1D FSP decoration is not allowed for D_2 or D_6

0D Block state decoration

- The full data of 0D block-state can be calculated by:

$$\mathcal{H}^0(\mathbb{Z}_4 \rtimes \mathbb{Z}_2, \mathbb{Z}_2) \times \mathcal{H}^1[\mathbb{Z}_4 \rtimes \mathbb{Z}_2, U(1)] = \mathbb{Z}_2 \times \mathbb{Z}_2^2$$

Fermion parity   Bosonic SPT states in 0D

Trivialization

- The space group action at the 0D block as **$D_4 \times Z_2^f$** on-site symmetry. The complex fermion decoration which changes the fermion parity cannot be trivialized. However, all bosonic SPT states can be trivialized. Consider an atomic insulating state with 4 complex fermions:

$$|\psi\rangle_{0D} = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger |0\rangle, \quad \begin{aligned} \boldsymbol{R} &: \ (c_1, c_2, c_3, c_4) \mapsto (c_2, c_3, c_4, c_1) \\ \boldsymbol{M} &: \ (c_1, c_2, c_3, c_4) \mapsto (c_1, c_4, c_3, c_2) \end{aligned}$$

$$\boldsymbol{R}|\psi\rangle_{0D} = \boldsymbol{M}|\psi\rangle_{0D} = -|\psi\rangle_{0D}$$

- For both reflection and rotation symmetries, eigenvalue -1 corresponds to a trivial atomic insulator.
- The final classification results is **$Z_2 \times Z_2$** for D_4 symmetry, and **Z_2** for D_2/D_6.

- For spin-1/2 systems, there is no 1D block state because the total symmetry is $\mathbf{Z_4^f}$. No nontrivial 1D FSPT phase for this case.

- The full data of 0D block-state can still be calculated by:

$$\mathcal{H}^0(\mathbb{Z}_4 \rtimes \mathbb{Z}_2, \mathbb{Z}_2) \times \mathcal{H}^1[\mathbb{Z}_4 \rtimes \mathbb{Z}_2, U(1)] = \mathbb{Z}_2 \times \mathbb{Z}_2^2$$

Subject to the following obstruction $\mathrm{d}\nu_1 = (-1)^{\omega_2 \smile n_0}$

- The first $\mathbf{Z_2}$ is killed and only two bosonic $\mathbf{Z_2}$ root states are allowed.

- For spin-1/2 systems all bosonic SPT states cannot be trivialized due to symmetry requirement R^4=-1 and M^2=-1 on fermion operator. Consider the same atomic insulating state with 4 complex fermions:

$$|\psi\rangle_{0\mathrm{D}} = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger |0\rangle, \quad \begin{aligned} \boldsymbol{R}: &\ (c_1, c_2, c_3, c_4) \mapsto (c_2, c_3, c_4, -c_1) \\ \boldsymbol{M}: &\ (c_1, c_2, c_3, c_4) \mapsto (c_1, c_4, c_3, -c_2) \end{aligned}$$

- The final classification results is $\mathbf{Z_2 \times Z_2}$ for D_2, D_4 and D_6.

Real space construction for space group FSPT phases in 2D

Three major steps:

- Cell decomposition

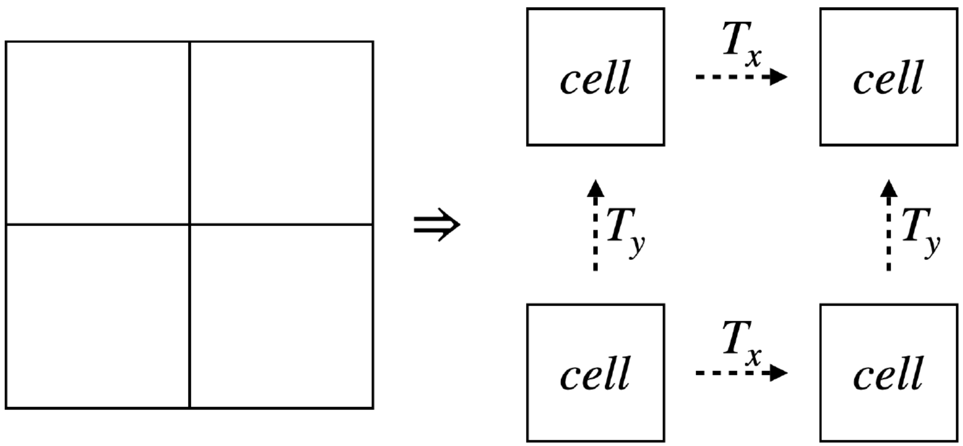
- 1D and 0D Block-state decoration(with or without symmetries): Possible obstructions for block-states; all obstruction-free block-states form a group {OFBS}

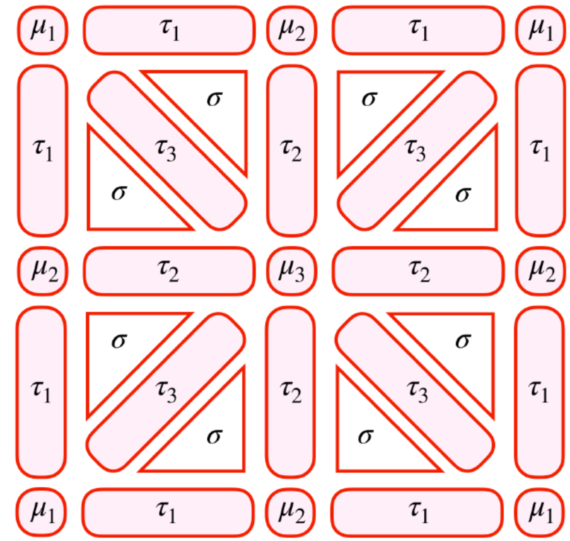
- Bubble equivalence: Some block-states might be equivalent via bubble equivalences; all trivial block-states form a group {TBS}

Topological distinct phases can be labeled by different group elements of the quotient group G= {OFBS}/{TBS}

p4m space group with spinless fermion

- Space group is composed by point group and translational symmetry.





- For wallpaper group p4m, the corresponding point group is D_4. Cell decomposition of p4m leads to an assembly of following lower dimensional blocks with corresponding on-site symmetries:

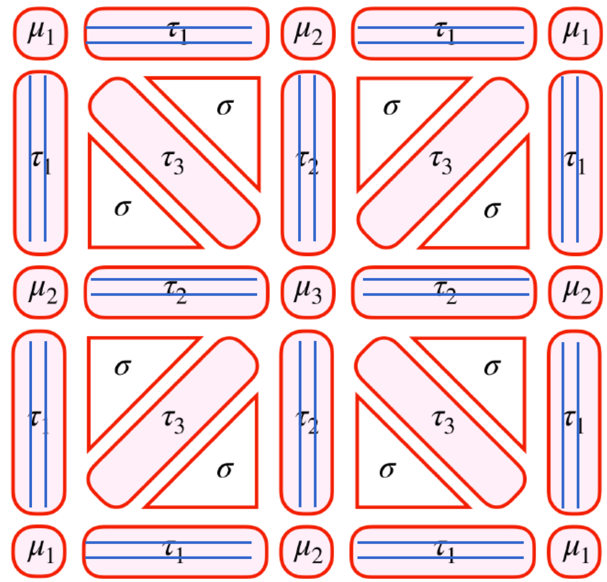
  2D blocks: no on-site symmetry;

  1D blocks: Z_2 on-site symmetry;

  0D blocks 1 and 3: D_4 on-site symmetry; 2: D_2 on-site symmetry.

Block state decoration

- Similar as the D_4 point group case, Majorana chain decoration is not allowed. Only possible for 1D FSPT state decoration with a total symmetry $Z_2 \times Z_2^f$.(double Majorana chain)



There are two independent 1D blocks comparable with 1D FSPT decoration

$$\{\mathrm{OFBS}\}_{p4m,0}^{\mathrm{1D}} = \mathbb{Z}_2^2$$

- Similar as the D_4/D_2 point group case, 0D block-states decoration is classified by:

$$\mathcal{H}^1\left[\mathbb{Z}_2^f \times (\mathbb{Z}_n \rtimes \mathbb{Z}_2), U(1)\right] = \mathbb{Z}_2^3$$

There are three independent 0D blocks, each can be labelled as fermion parity and two mirror eigenvalues

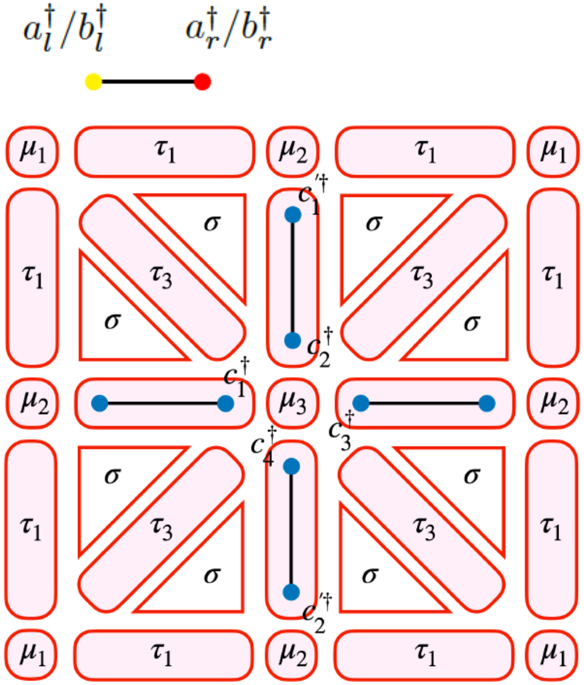
$$\{\mathrm{OFBS}\}_{p4m,0}^{\mathrm{0D}} = \mathbb{Z}_2^9 \qquad [(\pm,\pm,\pm),(\pm,\pm,\pm),(\pm,\pm,\pm)]$$

Final classification of obstruction free block states:

$$\{\mathrm{OFBS}\}_{p4m,0} = \{\mathrm{OFBS}\}_{p4m,0}^{\mathrm{1D}} \times \{\mathrm{OFBS}\}_{p4m,0}^{\mathrm{0D}} = \mathbb{Z}_2^2 \times \mathbb{Z}_2^9 = \mathbb{Z}_2^{11}$$

Bubble equivalence

1D bubble construction can be realized as creating a pair of complex fermion on all 1D blocks.:



$$|\psi\rangle_{p4m}^{\mu_3} = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger|0\rangle, \ \ |\psi\rangle_{p4m}^{\mu_2} = c_1'^\dagger c_2'^\dagger|0\rangle$$

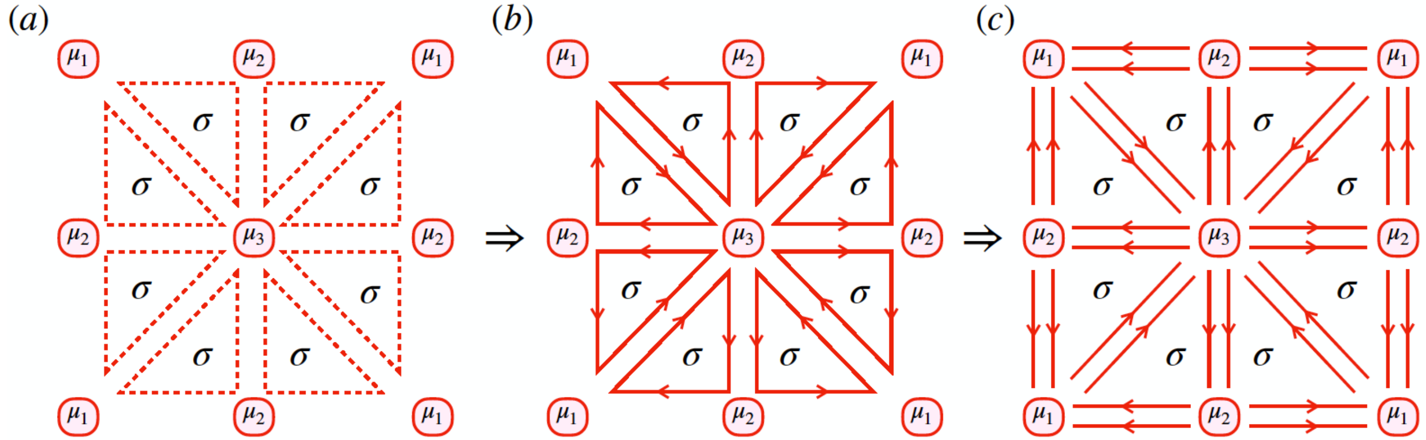
$$\boldsymbol{M}_{\tau_3}|\psi\rangle_{p4m}^{\mu_3} = c_2^\dagger c_1^\dagger c_4^\dagger c_3^\dagger|0\rangle = |\psi\rangle_{p4m}^{\mu_3}$$

$$\boldsymbol{M}_{\tau_2}|\psi\rangle_{p4m}^{\mu_3} = c_1^\dagger c_4^\dagger c_3^\dagger c_2^\dagger|0\rangle = -|\psi\rangle_{p4m}^{\mu_3}$$

$$\boldsymbol{M}_{\tau_1}|\psi\rangle_{p4m}^{\mu_2} = c_2'^\dagger c_1'^\dagger|0\rangle = -|\psi\rangle_{p4m}^{\mu_2}$$

$$\boldsymbol{M}_{\tau_2}|\psi\rangle_{p4m}^{\mu_2} = c_1'^\dagger c_2'^\dagger|0\rangle = |\psi\rangle_{p4m}^{\mu_2}$$

2D bubble construction can be realized as creating a Majorana chain on the boundaries of all 2D blocks.



Classification results

- 1D bubble constructions can be characterized by three integers(mod 2), describing the changing of two mirror eigenvalues for each 0D block

$$[(+,(-1)^{n_1},(-1)^{n_3}),(+,(-1)^{n_2},(-1)^{n_1}),(+,(-1)^{n_2},(-1)^{n_3})]$$

$$\{\mathrm{TBS}\}_{p4m,0}^{\mathrm{0D}}=\mathbb{Z}_2^3$$

$$G_{p4m,0}^{\mathrm{0D}}=\{\mathrm{OFBS}\}_{p4m,0}^{\mathrm{0D}}/\{\mathrm{TBS}\}_{p4m,0}^{\mathrm{0D}}=\mathbb{Z}_2^9/\mathbb{Z}_2^3=\mathbb{Z}_2^3\times\mathbb{Z}_2^3$$

- 2D bubble constructions will trivialize one of the 1D FSPT decoration

$$\{\mathrm{TBS}\}_{p4m,0}^{\mathrm{1D}}=\mathbb{Z}_2$$

$$G_{p4m,0}^{\mathrm{1D}}=\{\mathrm{OFBS}\}_{p4m,0}^{\mathrm{1D}}/\{\mathrm{TBS}\}_{p4m,0}^{\mathrm{1D}}=\mathbb{Z}_2^2/\mathbb{Z}_2=\mathbb{Z}_2$$

$$\mathcal{G}_{p4m,0}=\{\mathrm{OFBS}\}_{p4m,0}/\{\mathrm{TBS}\}_{p4m,0}=\mathbb{Z}_2^{11}/\mathbb{Z}_2^4=\mathbb{Z}_2^4\times\mathbb{Z}_2^3$$

- For spin-1/2 systems, there would be no nontrivial 1D decorations, and 0D block decorations on both D_2 and D_4 centers give rise to a $\mathbf{Z_2\times Z_2}$ classification. There is also no trivialization, hence the final classification results is $\mathbf{Z_2^6}$.

Classification of TSC in 2D spin-1/2 and spinless fermion systems

G_b	E_0^{1D}	E_0^{0D}	$\mathcal{G}_0$
$p1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
$p2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$
pm	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^5 \times \mathbb{Z}_2$
pg	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
cm	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$
pmm	$\mathbb{Z}_1$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^4$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^4$
pmg	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$
pgg	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$
cmm	$\mathbb{Z}_1$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$
$p4$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$p4m$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^3$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^3$
$p4g$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$
$p3$	$\mathbb{Z}_1$	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$
$p3m1$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$
$p31m$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$
$p6$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$
$p6m$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$

G_b	$E_{1/2}^{1D}$	$E_{1/2}^{0D}$	$\mathcal{G}_{1/2}$
$p1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
$p2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_4^4$	$\mathbb{Z}_4 \times \mathbb{Z}_8^3$
pm	$\mathbb{Z}_2$	$\mathbb{Z}_4^2$	$\mathbb{Z}_4 \times \mathbb{Z}_8$
pg	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
cm	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
pmm	$\mathbb{Z}_1$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^8$
pmg	$\mathbb{Z}_2^2$	$\mathbb{Z}_4^3$	$\mathbb{Z}_4 \times \mathbb{Z}_8^2$
pgg	$\mathbb{Z}_2^2$	$\mathbb{Z}_4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
cmm	$\mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2^4$	$\mathbb{Z}_8 \times \mathbb{Z}_2^4$
$p4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8^2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_8^3$
$p4m$	$\mathbb{Z}_1$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^6$
$p4g$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_2^2$
$p3$	$\mathbb{Z}_1$	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$
$p3m1$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_4$
$p31m$	$\mathbb{Z}_1$	$\mathbb{Z}_4 \times \mathbb{Z}_3$	$\mathbb{Z}_4 \times \mathbb{Z}_3$
$p6$	$\mathbb{Z}_2$	$\mathbb{Z}_{12} \times \mathbb{Z}_4 \times \mathbb{Z}_3$	$\mathbb{Z}_{12} \times \mathbb{Z}_8 \times \mathbb{Z}_3$
$p6m$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$

- Fermionic/bosonic root phases are denoted with red/blue color.
- The 1D Majorana chain decoration, 0D complex fermion decoration and bosonic SPT phases has a one to one correspondence with internal symmetry.

JH Zhang, S Yang, Y Qi, ZC Gu, Phys. Rev. Research 4, 033081 (2022)

Generalize into topological insulators(TI) in 2D interacting fermion systems

G_b	spinless	spin-1/2
$p1$	$\mathbb{Z}$	$\mathbb{Z}$
$p2$	$\mathbb{Z}\times\mathbb{Z}_4^3\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4^3\times\mathbb{Z}_2$
pm	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$
pg	$\mathbb{Z}$	$\mathbb{Z}$
cm	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_2$
pmm	$\mathbb{Z}\times\mathbb{Z}_4^3\times\mathbb{Z}_2^4$	$2\mathbb{Z}\times\mathbb{Z}_2^8$
pmg	$\mathbb{Z}\times\mathbb{Z}_4^2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4^2\times\mathbb{Z}_2$
pgg	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$
cmm	$\mathbb{Z}\times\mathbb{Z}_4^2\times\mathbb{Z}_2^2$	$2\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2^4$
$p4$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2$
$p4m$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2^3$	$2\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_2^6$
$p4g$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_2^2$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_2^2$
$p3$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_3^3$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_3^3$
$p3m1$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_2$
$p31m$	$\mathbb{Z}\times\mathbb{Z}_3\times\mathbb{Z}_6$	$\mathbb{Z}\times\mathbb{Z}_3\times\mathbb{Z}_6$
$p6$	$\mathbb{Z}\times\mathbb{Z}_{12}\times\mathbb{Z}_6\times\mathbb{Z}_3$	$\mathbb{Z}\times\mathbb{Z}_{12}\times\mathbb{Z}_6\times\mathbb{Z}_3$
$p6m$	$\mathbb{Z}\times\mathbb{Z}_{12}\times\mathbb{Z}_2^2$	$2\mathbb{Z}\times\mathbb{Z}_6\times\mathbb{Z}_2^3$

- Fermionic/bosonic root phases are denoted with red/blue color.
- No 1D block state decoration: no 1D topological insulator without symmetry or with a Z_2 onsite symmetry.

Classifying fermionic SPT(FSPT) phases protected by internal symmetry

- 1D fermionic systems can be mapped to bosonic systems with an additional unbroken fermion parity symmetry. (Xie Chen, Z C Gu, X G Wen, PRB 84, 235128 (2011))

- The braiding/three loop braiding statistics of the gauge flux/flux line is a good way to understand the 2D/3D classification. (Z.-C. Gu, M. Levin, PRB 89, 201113(R) (2014) M. Cheng, Z. Bi, Y. Z. You, Z. C. Gu, PRB 97, 205109, (2018), C Wang, CH Lin, ZC Gu, PRB 95, 195147(2017), J. R. Zhou, Q. R. Wang, C. Wang, Z. C. Gu, Nature communications 12, 1, (2021))

- Decoration of Kitaev's Majorana chains on intersection lines of symmetry domains and decoration of complex fermion on intersection points of symmetry domain walls lead to a general group super-cohomolgy theory(equivalent to Atiyah–Hirzebruch spectral sequence) for fermionic SPT phases! (Z.-C. Gu, X.-G. Wen, Phys. Rev. B 90, 115141 (2014), Q. R. Wang, ZC Gu, PRX, 8, 011055 (2018), Q. R. Wang, Z. C. Gu, PRX,10, 031055 (2020))

Classification of general FSPT phases in 2D

$$\begin{cases} n_1 \in H^1(G_b, \mathbb{Z}_2), \\ n_2 \in C^2(G_b, \mathbb{Z}_2)/B^2(G_b, \mathbb{Z}_2)/\Gamma^2, \\ \nu_3 \in C^3(G_b, U(1)_T)/B^3(G_b, U(1)_T)/\Gamma^3. \end{cases}$$

$$\begin{cases} n_1(gg_0, gg_1) = n_1(g_0, g_1) = n_1(g_0^{-1}g_1), \\ n_2(gg_0, gg_1, gg_2) = n_2(g_0, g_1, g_2) = n_2(g_0^{-1}g_1, g_1^{-1}g_2), \\ \nu_3(g, ga, gab, gabc) = {}^g\nu_3(a, b, c) = \nu_3(a, b, c)^{1-2s_1(g)} \cdot \mathcal{O}_4^{\text{symm}}(g, ga, gab, gabc) \end{cases}$$

$$\begin{cases} \mathrm{d}n_1 = 0, \\ \mathrm{d}n_2 = \omega_2 \smile n_1 + s_1 \smile n_1 \smile n_1, \\ \mathrm{d}\nu_3 = \mathcal{O}_4[n_2] \end{cases}$$

$$\begin{cases} \Gamma^2 = \{\omega_2 \in H^2(G_b, \mathbb{Z}_2)\}, \\ \Gamma^3 = \{(-1)^{\omega_2 \smile n_1} \in H^3(G_b, U(1)_T) | n_1 \in H^1(G_b, \mathbb{Z}_2)\}. \end{cases}$$

$$\begin{aligned} \mathcal{O}_4^{\text{symm}}(g_0, g_1, g_2, g_3) &= (-1)^{\omega_2(g_0, g_0^{-1}g_1) n_2(123) + [s_1(g_0) + \omega_2(g_0, g_0^{-1}g_2)] \mathrm{d}n_2(0123)} \\ &= (-1)^{(\omega_2 \smile n_2 + s_1 \smile \mathrm{d}n_2)(g_0, g_0^{-1}g_1, g_1^{-1}g_2, g_2^{-1}g_3) + \omega_2(g_0, g_0^{-1}g_2) \mathrm{d}n_2(g_0^{-1}g_1, g_1^{-1}g_2, g_2^{-1}g_3)} \end{aligned}$$

$$\mathcal{O}_4[n_2](01234) = (-1)^{(\omega_2 \smile n_2 + n_2 \smile n_2 + n_2 \smile_1 \mathrm{d}n_2)(01234) + \omega_2(013)\mathrm{d}n_2(1234) + \mathrm{d}n_2(0124)\mathrm{d}n_2(0234)} (-i)^{\mathrm{d}n_2(0123)[1-\mathrm{d}n_2(0124)]}$$

$s_1 \in H^1(G_b, \mathbb{Z}_2)$ is related to time reversal symmetry.

2-cocycle $\omega_2 \in H^2(G_b, \mathbb{Z}_2)$ which tells us how G_b is extended

Q. R. Wang, Z. C. Gu, PRX,10, 031055 (2020)

Classification of space group SPT phases for 2D interacting fermion systems

Fermionic crystalline equivalence principle

- A mirror reflection symmetry action should still be mapped onto a time reversal symmetry action. In addition, spinless(spin-1/2) fermion systems should be mapped onto spin-1/2(spinless) fermion system.

Bosonic Crystalline SPT phases

SG	MC	CF	B	Total
p1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
p2	$3\mathbb{Z}_2$	$4\mathbb{Z}_2$	$4\mathbb{Z}_2$	2048
p1m1	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	32
p1g1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
c1m1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	8
p2mm	0	0	$8\mathbb{Z}_2$	256
p2mg	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$3\mathbb{Z}_2$	256
p2gg	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	64
c2mm	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$5\mathbb{Z}_2$	128
p4	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus 2\mathbb{Z}_4$	1024
p4mm	0	0	$6\mathbb{Z}_2$	64
p4gm	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_4$	64
p3	0	$\mathbb{Z}_2$	$3\mathbb{Z}_3$	54
p3m1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	4
p31m	0	$\mathbb{Z}_2$	$\mathbb{Z}_6$	12
p6	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_6$	288
p6mm	0	0	$4\mathbb{Z}_2$	16

Spinless (map to internal symmetry)

SG	MC	CF	B	Total
p1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
p2	0	$3\mathbb{Z}_2$	$\mathbb{Z}_2$	16
p1m1	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2$	64
p1g1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
c1m1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	16
p2mm	0	$4\mathbb{Z}_2$	$4\mathbb{Z}_2$	256
p2mg	$\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2$	32
p2gg	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	8
c2mm	0	$3\mathbb{Z}_2$	$2\mathbb{Z}_2$	32
p4	0	$2\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	32
p4mm	0	$4\mathbb{Z}_2$	$3\mathbb{Z}_2$	128
p4gm	0	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	16
p3	0	$\mathbb{Z}_2$	$3\mathbb{Z}_3$	54
p3m1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	8
p31m	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	24
p6	0	$\mathbb{Z}_2$	$\mathbb{Z}_3 \oplus \mathbb{Z}_6$	36
p6mm	0	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	16

Spin-1/2 (map to internal symmetry)

Y Ouyang, QR Wang, ZC Gu, Y Qi, Chinese Physics Letters 38 (12), 127101 (2021)

Real space construction for point group FSPT phases in 3D

For 3D case, we need to consider 2D block states(with or without on-site symmetries) decoration.

- Obstruction free condition requires no open edges: all 1D gapless edge modes of 2D block states must be gapped out without breaking the symmetry.
- We can use 2n components non-chiral Luttinger liquids to describe the gapless edge:

$$\mathcal{L}_{1\mathrm{D}} = \frac{K_{IJ}}{4\pi}\left(\partial_x \phi^I\right)\left(\partial_t \phi^J\right) + \frac{V_{IJ}}{8\pi}\left(\partial_x \phi^I\right)\left(\partial_x \phi^J\right)$$

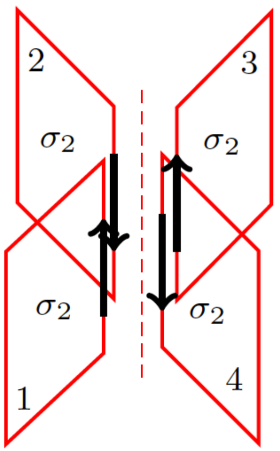
- With proper back scattering term under certain symmetry transformation:

$$U = \sum_k U(\Lambda_k) = \sum_k U(x)\cos\left[\Lambda_k^T K\phi - \alpha(x)\right]$$

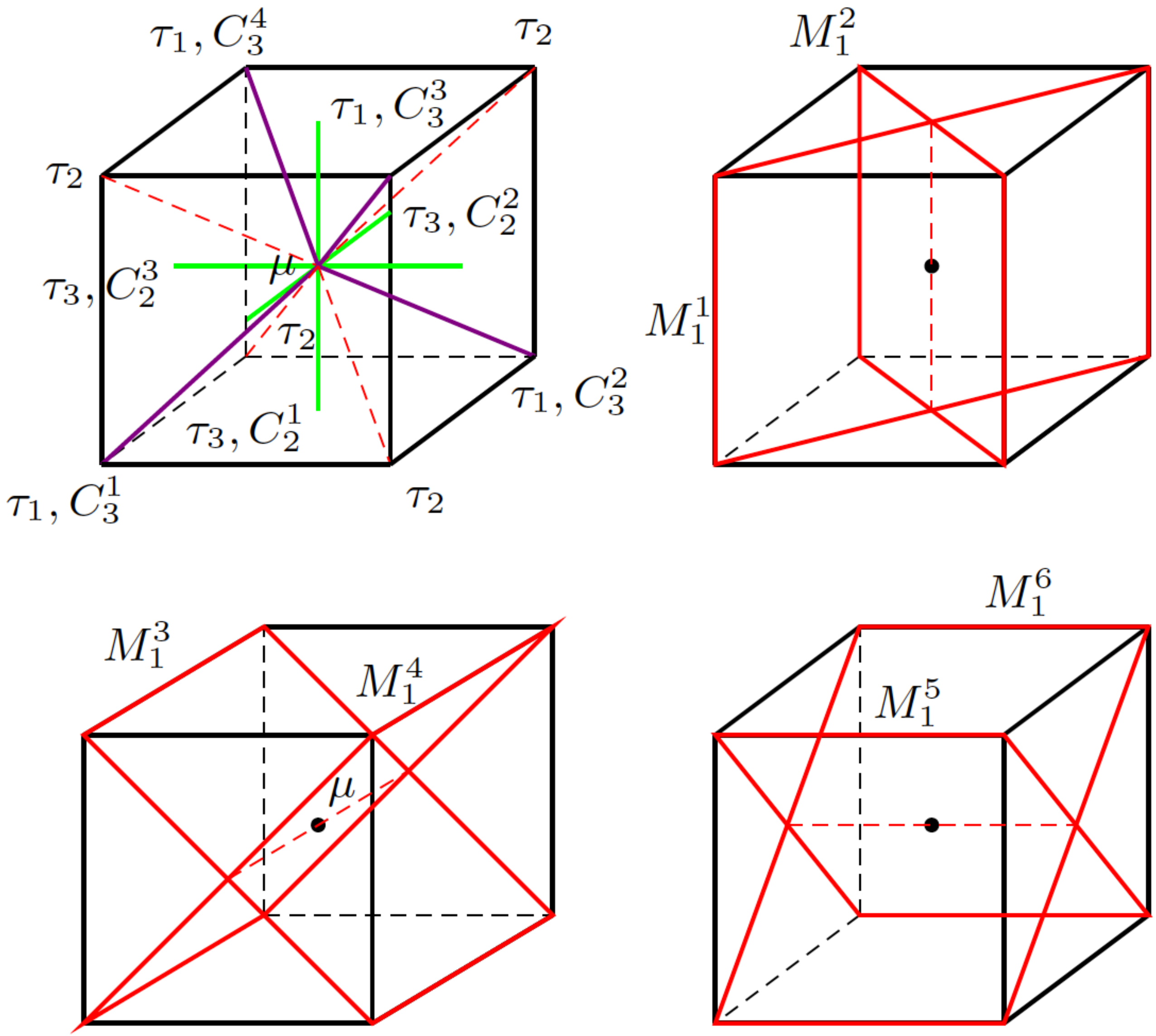
$$S : \phi \mapsto W^S \phi + \delta\phi^S$$

- With n primitive "null vectors" satisfying:

$$\Lambda_i^T K \Lambda_j = 0$$



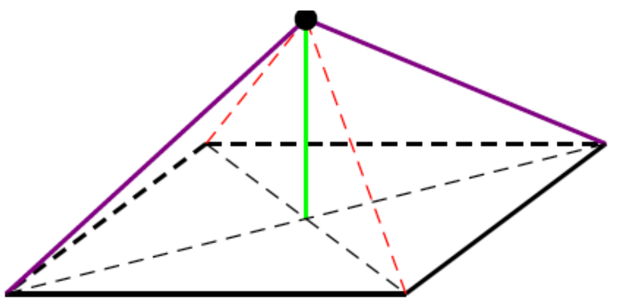
Example: T_d point group symmetry

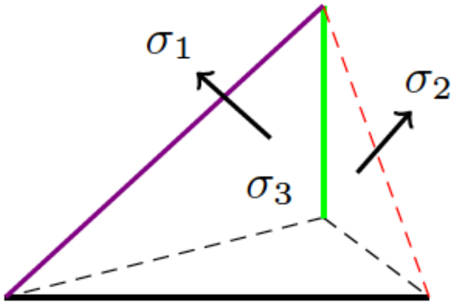


Cell decomposition and block states (spinless fermion):

Block state construction for the wavefunction:

$$|\Psi\rangle = \bigotimes_{g\in T_d} |T_{g\lambda}\rangle \otimes \sum_{k=1}^{3} |\gamma_{g\sigma_k}\rangle \otimes \sum_{l=1}^{3} |\beta_{g\tau_l}\rangle \otimes |\alpha_\mu\rangle$$





- Trivial decoration on 3D block.
- $Z_2 \times Z_2^f$ symmetry on all 2D blocks.
- $D_3 \times Z_2^f$ symmetry on all pink and red(dash) 1D blocks.
- $D_2 \times Z_2^f$ symmetry on all greeen1D blocks.
- $S_4 \times Z_2^f$ symmetry on 0D block.

2D block states(spinless fermion)

- p+ip TSC with a **Z** classification. Monolayer of p+ip TSC is obstructed (a rigorous proof can be achieved based on anomaly indicator, (Chris Heinrich and Michael Levin, Phys. Rev. B 98, 035101 (2018)) while double layer of p+ip TSC is trivial.(Similar to the double layer E_8 state which can always be trivialized.)

- Z_2 fermionic SPT state with a $\mathbf{Z_8}$ classification. Monolayer layer Z_2 fermionic SPT states are all obstructed, double layer and four layers of Z_2 fermionic SPT states(equivalent to the bosonic Levin-Gu state) is obstruction free!

$$W_2^{M_1^1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}, \; \delta\phi_2^{M_1^1} = \pi\chi_1$$

$$W_2^{M_1^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_{2\times 2}, \; \delta\phi_2^{M_1^2} = \pi\chi_2$$

$$K = (\sigma^z)^{\oplus 4}$$

$$\chi_1 = (1,0,1,0,0,1,0,1)^T$$

$$\chi_2 = (0,1,0,1,1,0,1,0).$$

(Yuan-Ming Lu and Ashvin Vishwanath, Phys. Rev. B 86, 125119 (2012))

We can find 4 independent backscattering terms to open the bulk gap without breaking any symmetry.

$$\Lambda_1 = (1,0,0,1,1,0,0,1)^T$$

$$\Lambda_2 = (0,1,1,0,0,1,1,0)^T$$

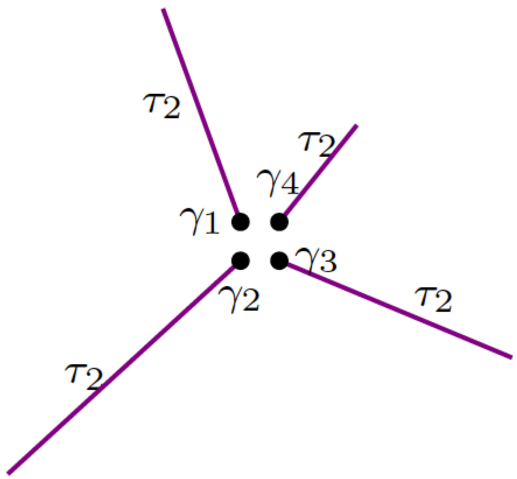
$$\Lambda_3 = (1,0,0,1,-1,0,0,-1)^T$$

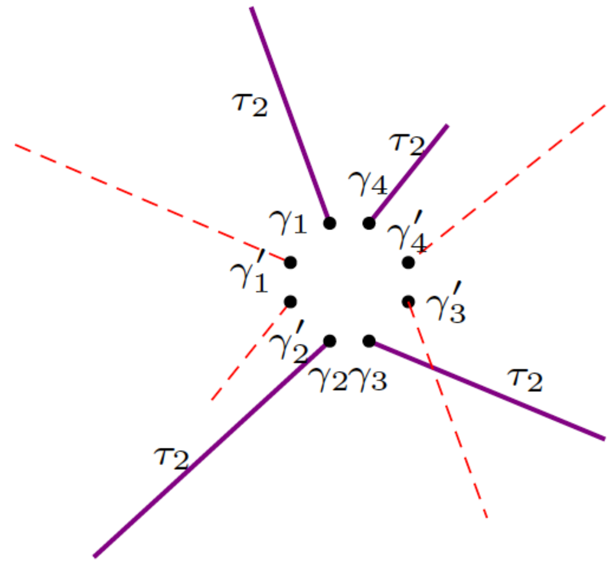
$$\Lambda_4 = (0,1,1,0,0,-1,-1,0)^T$$

1D block states(spinless fermion)

Using the same method as in 2D, we derive all obstruction free 1D block states:

- Majorana chain decoration on both pink and red(dash) 1D blocks;





No way to gap out four Majorana modes in a symmetric way!

$$H = i\gamma_1\gamma_3' + i\gamma_2\gamma_4' + i\gamma_3\gamma_1' + i\gamma_4\gamma_2'$$

- $Z_2 \times Z_2^f$ FSPT(double Majorana chains) decoration on pink or red(dash) 1D blocks;

- Haldane chain decoration on green 1D blocks.

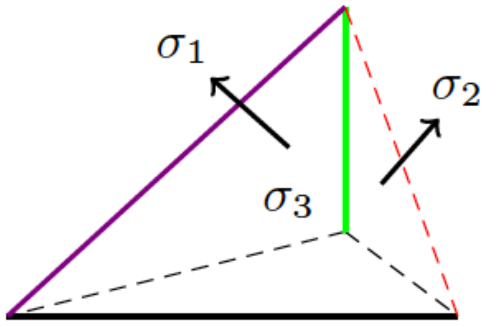
2D bubble constructions:

- Majorana bubble on type-3 2D blocks: leaves three Majorana chains on each pink and red(dash)1D blocks. Majorana chain decoration on both pink and red(dash)1D blocks is trivialized;

- Double Majorana bubble on type-3 2D blocks : leaves three 1D Z_2-fSPT phases on each pink and red(dash)1D blocks. Double Majorana chains decoration on pink and red(dash)1D blocks is trivialized;

- Double Majorana bubble on type-1 2D blocks : leaves three 1D Z_2-fSPT phases on each pink 1D blocks and two 1D Z_2-fSPT phases(equivalent to a Haldane chain.) on each green 1D blocks.

All trivial 1D blocks: $\{\mathrm{TBS}\}_{T_d}^{\mathrm{1D}} = \mathbb{Z}_2^3.$



Remaining non-trivial 1D block-state: $\mathcal{G}_{T_d}^{\mathrm{1D}} = \{\mathrm{OFBS}\}_{T_d}^{\mathrm{1D}} / \{\mathrm{TBS}\}_{T_d}^{\mathrm{1D}} = \mathbb{Z}_2.$

0D block states(spinless fermion)

- Effective on-cite symmetry is S_4

$$n_0 \in \mathcal{H}^0(S_4, \mathbb{Z}_2) = \mathbb{Z}_2, \qquad \nu_1 \in \mathcal{H}^1\left[S_4, U(1)\right] = \mathbb{Z}_2,$$

Fermion parity

Bosonic SPT states in 0D characterizing by the reflection eigenvalues.

- However, the mirror eigenvalue -1 can be trivialized by the following atomic insulator

$$\left(c_1^\dagger, c_2^\dagger, c_3^\dagger, c_4^\dagger\right) \mapsto \left(c_1^\dagger, c_3^\dagger, c_2^\dagger, c_4^\dagger\right) \qquad c_1^\dagger c_3^\dagger c_2^\dagger c_4^\dagger |0\rangle = -|\psi_\mu\rangle$$

The final classification result is $\mathbf{Z_4 \times Z_2^2}$, with the following block-states:

- 2D fermionic Levin-Gu state on each 2D block;
- Haldane chain decoration on each green 1D block;
- Complex fermion decoration on 0D block ;

Classification of TSC protected by point group symmetry in 3D

G_b	E_0^{2D}	E_0^{1D}	E_0^{0D}	$\mathcal{G}_0$
C_1	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_i = S_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_2	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_s = C_{1h}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_{16}$
C_{2h}	$\mathbb{Z}_8$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8$
$D_2 = V$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{2v}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
$D_{2h} = V_h$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
C_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C_{4h}	$\mathbb{Z}_8$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
D_4	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
C_{4v}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$
$D_{2d} = V_d$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
D_{4h}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^6$
C_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
D_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3v}	$\mathbb{Z}_{16}$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_{16}$
D_{3d}	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
C_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3h}	$\mathbb{Z}_8$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8$
C_{6h}	$\mathbb{Z}_8$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8$
D_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{6v}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
D_{3h}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$
D_{6h}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
T	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
T_h	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
T_d	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$
O	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
O_h	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$

G_b	$E_{1/2}^{2D}$	$E_{1/2}^{1D}$	$E_{1/2}^{0D}$	$\mathcal{G}_{1/2}$
C_1	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_i = S_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_2	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_s = C_{1h}$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{2h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
$D_2 = V$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_{2v}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$D_{2h} = V_h$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
C_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C_{4h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
D_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_{4v}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$D_{2d} = V_d$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
D_{4h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
C_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
D_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3v}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
D_{3d}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_{3h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{6h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
D_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_{6v}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
D_{3h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
D_{6h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
T	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
T_h	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
T_d	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
O	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
O_h	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$

For usual fermion systems with spin-1/2, all 2D and 1D block states decorations are all trivial, there would be no interesting crystalline TSC.

Although spinless fermion systems are not quite natural, SC with co-planar magnetic order or applying strong magnetic field may still possible to realize them in experiments.

Block-states for topological insulator

With the additional $U^f(1)$ symmetry, we have the following 2D block-states:

1. Integer quantum Hall insulator(Chern insulator) with **Z** classification;

2. Kitaev E_8 state with **Z** classification;

3. 2D $U^f(1)$ Z_2 fermionic SPT state with **Z_4** classification.

With the additional $U^f(1)$ symmetry, we have only nontrivial root state – the Haldane chain, on green 1D block-states for spinless fermion systems and there is no nontrivial one 1D block state for spin-1/2 fermions.

With the additional $U^f(1)$ symmetry, the 0D block states can be classified by the following data:

$$n_0 \in \mathcal{H}^0(S_4, \mathbb{Z}) = \mathbb{Z}, \quad \nu_1 \in \mathcal{H}^1\left[S_4, U(1)\right] = \mathbb{Z}_2$$

Subject to the following obstruction $\mathrm{d}\nu_1 = (-1)^{\omega_2 \smile n_0}$

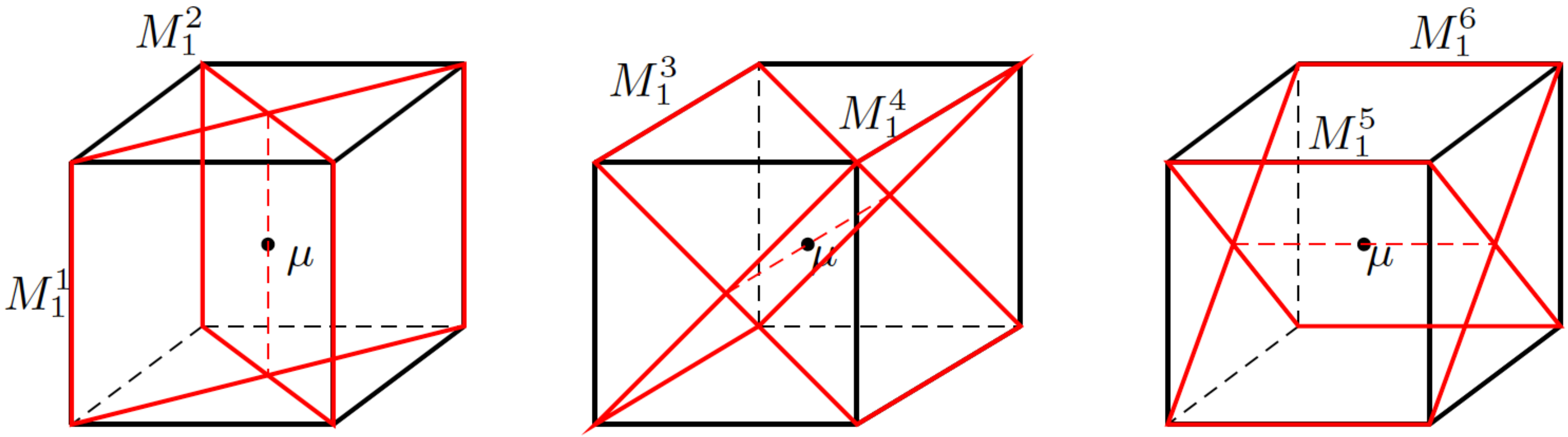
- Consider additional trivialization from atomic insulators, we end up with **Z_2** (parity of $U^f(1)$ charge) for spinless fermions and $U^f(1)$ charge n=2 (mod 4) for spin-1/2 fermon systems.

Classification of TI protected by point group symmetry in 3D

G_b	$E^{2D}_{0,U(1)}$	$E^{1D}_{0,U(1)}$	$E^{0D}_{0,U(1)}$	$\mathcal{G}_{0,U(1)}$
C_1	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_i = S_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_2	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_s = C_{1h}$	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_{2h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$D_2 = V$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{2v}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$
$D_{2h} = V_h$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^3$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^7$
C_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_4	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
C_{4h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^2$
D_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{4v}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$
$D_{2d} = V_d$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^3$
D_{4h}	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^3$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^7$
C_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_6	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$
D_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3v}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
D_{3d}	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$
C_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_{6h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
D_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{6v}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$
D_{3h}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
D_{6h}	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^3$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^7$
T	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
T_h	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^3$
T_d	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$
O	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
O_h	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^4$

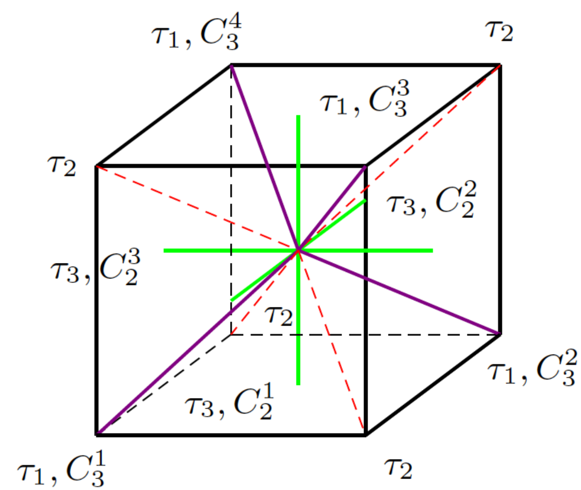
G_b	$E^{2D}_{1/2,U(1)}$	$E^{1D}_{1/2,U(1)}$	$E^{0D}_{1/2,U(1)}$	$\mathcal{G}_{1/2,U(1)}$
C_1	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_i = S_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_2	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_s = C_{1h}$	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_{2h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^3$
$D_2 = V$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
C_{2v}	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$D_{2h} = V_h$	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^5$
C_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_4	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
C_{4h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^3$
D_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_{4v}	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$D_{2d} = V_d$	$\mathbb{Z}_8 \times \mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_8 \times \mathbb{Z}_2^5$
D_{4h}	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^5$
C_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_6	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$
D_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3v}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
D_{3d}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^3$
C_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_{6h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^3$
D_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
C_{6v}	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
D_{3h}	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^3$
D_{6h}	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^5$
T	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
T_h	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^3$
T_d	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^2$
O	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
O_h	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^4$

Higher order TSC and TI



By decoration of p+ip TSC, Z_2 SPT states, Chern insulator or E8 states onto the 2D blocks, we can derive various second order topological TSC/TI with hinge gapless modes for interacting fermion systems.

Depending the decoration of Kitaev's Majarona chain or Z_2 SPT states onto the 1D blocks, we can derive various third order topological TSC/TI with corner zero modes for interacting fermion systems.



Summary and future works

- We classify and construct all crystalline topological superconductors and topological insulators in interacting fermion systems.

- The #11 p4m space group is the only one that support intrinsic 2D interacting topological superconductor for spinless fermion systems. All other 2D crystalline topological superconductors and topological insulators can be realized in free fermion systems!

- In 3D, almost all point group SPT states with 2D block-states decorations are intriguing interacting topological phases.

- Experimental realization for interacting crystalline topological insulators and superconductors.

- Crystalline equivalence principle suggest the space-time symmetry can be regarded as internal symmetry for topological quantum field theory, which might have important implications for quantum gravity! (Tianyao Fang and Z C Gu, to appear 2023)