# Lieb-Schultz-Mattis anomalies and web of dualities induced by gauging in quantum spin chains

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## Definition

Consider the quantum spin-1/2 *XYZ* chain

$$\begin{aligned}\widehat{H}_{b=0} := J_1 \sum_{j\in\Lambda} \left(\Delta_x \, \hat{\sigma}^x_j \, \hat{\sigma}^x_{j+1} + \Delta_y \, \hat{\sigma}^y_j \, \hat{\sigma}^y_{j+1} + \Delta_z \, \hat{\sigma}^z_j \, \hat{\sigma}^z_{j+1}\right) \\ + J_2 \sum_{j\in\Lambda} \left(\Delta_x \, \hat{\sigma}^x_j \, \hat{\sigma}^x_{j+2} + \Delta_y \, \hat{\sigma}^y_j \, \hat{\sigma}^y_{j+2} + \Delta_z \, \hat{\sigma}^z_j \, \hat{\sigma}^z_{j+2}\right)\end{aligned} \tag{1}$$

with $\mathcal{H}_{b=0}$ as domain of definition. Here,

- periodic ($b = 0$) boundary conditions are imposed,
- the lattice $\Lambda$ is made of $2N \equiv |\Lambda|$ sites with $N$ even,
- the dimensionfull nearest-neighbor antiferromagnetic exchange coupling $J_1 \geq 0$,
- the dimensionfull next-nearest-neighbor antiferromagnetic exchange coupling $J_2 \geq 0$,
- the dimensionless anisotropies $\Delta_x, \Delta_y, \Delta_z \geq 0$ in internal spin-1/2 space are positive,
- for every site $j \in \Lambda$, the triplet of Pauli operators $\hat{\boldsymbol{\sigma}}_j = \hat{\boldsymbol{\sigma}}_{j+2N}$ obeys the Pauli algebra, while these operator commute on different sites.

## Relevant symmetries

There are four symmetries of interest:

- two of which are crystalline;
  - translation generated by $j \mapsto t(j) = j + 1 \bmod 2N$ for any $j \in \Lambda$,
  - reflection $j \mapsto r(j) = 2N - j \bmod 2N$ for any $j \in \Lambda$ that leaves $j = N$ and $j = 2N$ unchanged,
- two of which are internal (on-site);
  - global rotation $r^{x}_{\pi}$ by $\pi$ about the $x$ axis in spin-1/2,
  - global rotation $r^{y}_{\pi}$ by $\pi$ about the $y$ axis in spin-1/2.

The space group is

$$G_{\mathrm{spa}} := \mathbb{Z}^{t}_{2N} \rtimes \mathbb{Z}^{r}_{2}, \tag{2a}$$

$$\mathbb{Z}^{t}_{2N} \equiv \left\{ t,\ t^{2},\ \cdots,\ t^{2N-1},\ t^{2N} \equiv e \right\}, \quad \mathbb{Z}^{r}_{2} \equiv \left\{ r,\ r^{2} \equiv e \right\}, \tag{2b}$$

$$r\, t\, r^{-1} = t^{2N-1} \qquad (\mathbb{Z}^{t}_{2N} \text{ is a normal subgroup of } G_{\mathrm{spa}}). \tag{2c}$$

The global internal symmetry group is

$$G_{\mathrm{int}} \equiv \mathbb{Z}^{x}_{2} \times \mathbb{Z}^{y}_{2}, \tag{3a}$$

$$\mathbb{Z}^{x}_{2} \equiv \left\{ r^{x}_{\pi},\ (r^{x}_{\pi})^{2} \equiv e \right\}, \qquad \mathbb{Z}^{y}_{2} \equiv \left\{ r^{y}_{\pi},\ (r^{y}_{\pi})^{2} \equiv e \right\}. \tag{3b}$$

The quantum mechanical representation of the symmetry group

$$\mathrm{G}_{\mathrm{tot}} \equiv \mathrm{G}_{\mathrm{spa}} \times \mathrm{G}_{\mathrm{int}}, \tag{4}$$

on the $2^{2N}$-dimensional Hilbert space $\mathcal{H}_{b=0}$ is the group isomorphism defined by

$$\widehat{U}_t := \prod_{j=1}^{2N-1} \frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}_{b=0}} + \hat{\boldsymbol{\sigma}}_j \cdot \hat{\boldsymbol{\sigma}}_{t(j)}\right), \qquad \widehat{U}_r := \prod_{j=1}^{N-1} \frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}_{b=0}} + \hat{\boldsymbol{\sigma}}_j \cdot \hat{\boldsymbol{\sigma}}_{r(j)}\right), \tag{5a}$$

$$\widehat{U}_{r^x_\pi} := \prod_{j=1}^{2N} \hat{\sigma}^x_j, \qquad \widehat{U}_{r^y_\pi} := \prod_{j=1}^{2N} \hat{\sigma}^y_j, \qquad \widehat{U}_{r^z_\pi} := (-1)^N\, \widehat{U}_{r^x_\pi}\, \widehat{U}_{r^y_\pi} = \prod_{j=1}^{2N} \hat{\sigma}^z_j. \tag{5b}$$

While the global representation of the $\mathbb{Z}^x_2 \times \mathbb{Z}^y_2$ group in Eq. (5b) is a group isomorphism, it is locally projective due to the Pauli algebra

$$\hat{\sigma}^x_j\, \hat{\sigma}^y_j = -\hat{\sigma}^y_j\, \hat{\sigma}^x_j, \qquad \hat{\sigma}^y_j\, \hat{\sigma}^z_j = -\hat{\sigma}^z_j\, \hat{\sigma}^y_j, \qquad \hat{\sigma}^z_j\, \hat{\sigma}^x_j = -\hat{\sigma}^x_j\, \hat{\sigma}^z_j, \tag{6}$$

for any $j \in \Lambda$.

# Translation LSM Theorem is applicable

## Theorem (Translation LSM)

Consider a one-dimensional lattice Hamiltonian with the symmetry group $\mathrm{G}_{\mathrm{tot}} \equiv \mathrm{Z}^{t}_{|\Lambda|} \times \mathbb{Z}^{x}_{2} \times \mathbb{Z}^{y}_{2}$, where the subgroup $\mathrm{Z}^{t}_{|\Lambda|}$ generates lattice translations and the subgroup $\mathbb{Z}^{x}_{2} \times \mathbb{Z}^{y}_{2}$ generates internal discrete spin-rotation symmetry. If the unit cell with respect to the translation symmetry $\mathrm{Z}^{t}_{|\Lambda|}$ hosts a half-integer spin representation of $\mathbb{Z}^{x}_{2} \times \mathbb{Z}^{y}_{2}$, then the ground states cannot be simultaneously gapped, non-degenerate, and $\mathrm{G}_{\mathrm{tot}}$-symmetric.

## Definition (Translation LSM anomaly)

When Theorem 1 holds, we say that there is a translation LSM anomaly.

## Reflection LSM Theorem is applicable

### Theorem (Reflection LSM)

Consider a one-dimensional lattice Hamiltonian with the symmetry group $\mathrm{G}_{\mathrm{tot}} \equiv \mathrm{Z}_2^r \times \mathbb{Z}_2^x \times \mathbb{Z}_2^y$, where the subgroup $\mathrm{Z}_2^r$ generates site-centered reflection and the subgroup $\mathbb{Z}_2^x \times \mathbb{Z}_2^y$ generates internal discrete spin-rotation symmetry. If each reflection center hosts a half-integer spin representation of $\mathbb{Z}_2^x \times \mathbb{Z}_2^y$, then the ground states cannot be simultaneously gapped, non-degenerate, and $\mathrm{G}_{\mathrm{tot}}$-symmetric.

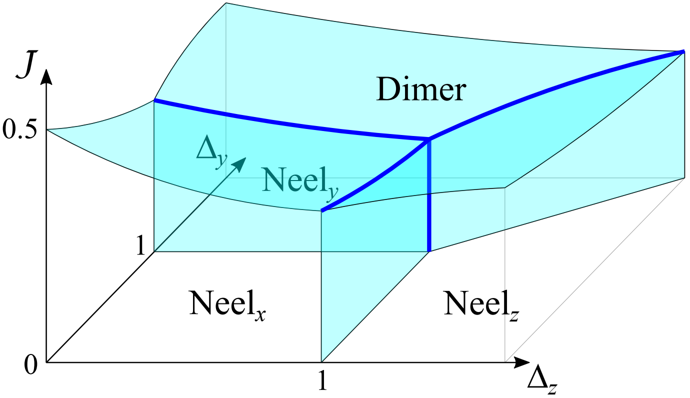
### Definition (Reflection LSM anomaly)

When the reflection LSM Theorem 2 holds, we say that there is a reflection LSM anomaly.

## Phase diagram ($J \equiv J_2/J_1$ and $\Delta_x \equiv 1$)

The constraints from the translation and reflection LSM theorems are fully consistent with the phase diagram (Haldane 1981, Mudry et al. 2019).

Indeed, all gapped ground states in the phase diagram




break spontaneously $\mathrm{G}_{\mathrm{tot}}$ if finitely degenerate.

## Remark (LSM anomaly versus mixed 't Hooft anomaly)

The translation LSM and reflection LSM Theorems (anomalies) have been interpreted as the presence of a mixed 't Hooft anomaly between crystalline symmetry groups, either $\mathbb{Z}^{t}_{2N}$ or $\mathbb{Z}^{r}_{2}$, and internal symmetry group $\mathbb{Z}^{x}_{2} \times \mathbb{Z}^{y}_{2}$ (Cheng et al. 2016, Cho et al. 2017, Jian et al. 2018, Else et al. 2020, and Cheng et al 2023). Accordingly, one cannot gauge the full internal symmetry group $\mathrm{G}_{\mathrm{int}}$, while maintaining the space group $\mathrm{G}_{\mathrm{spa}}$. However, a non-anomalous subgroup $\mathrm{H}_{\mathrm{int}} \subset \mathrm{G}_{\mathrm{int}}$ can still be consistently gauged.

**Question: What happens if we gauge the symmetry group $\mathbb{Z}^{z}_{2}$, which is the diagonal subroup of $\mathbb{Z}^{x}_{2} \times \mathbb{Z}^{y}_{2}$?**

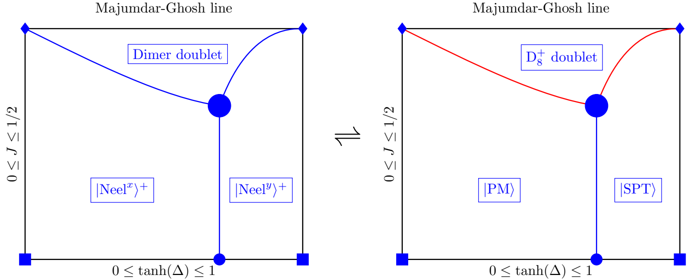
It turns ou that the there are two ways to gauge the symmetry group $\mathbb{Z}^{z}_{2}$:

- Kramers-Wannier (KW) gauging,
- Jordan-Wigner (JW) gauging.

# Main result for Kramers-Wannier (KW) gauging: KW duality

Set

$$\Delta \equiv \Delta_y/\Delta_x, \qquad \Delta_z = 0, \qquad 0 \leq J \equiv J_2/J_1 \leq 1/2.$$



# Main result for Jordan-Wigner (JW) gauging: JW duality

Set

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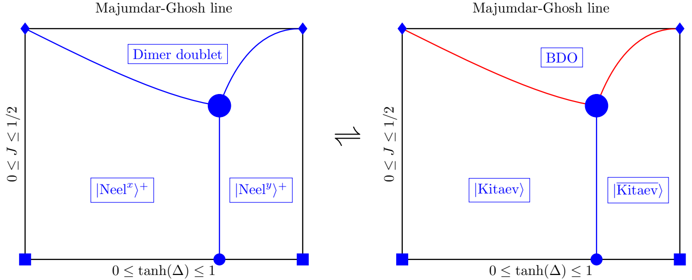


# Outline

Starting from $\mathbb{Z}_2$-symmetric quantum spin-$1/2$ $XYZ$ chains defined on the lattice

$$\Lambda := \left\{ j \,\middle|\, j = 1, \cdots, 2N \right\}, \tag{7}$$

we are going to gauge the global $\mathbb{Z}_2$ symmetry in two ways.

- The first way delivers a bosonic bond algebra with global $\mathbb{Z}_2$-symmetry that is supported on the dual lattice

$$\Lambda^\star := \left\{ j^\star \equiv j + \frac{1}{2} \,\middle|\, j \in \Lambda \right\}, \tag{8}$$

i.e., the links of the lattice $\Lambda$.

- The second way delivers a fermionic bond algebra with global $\mathbb{Z}_2$ fermion parity symmetry that is supported on the lattice $\Lambda$.

We will then establish a triality between all three bond algebras, i.e., any pair of the three bond algebras form dual pairs provided appropriate consistency conditions are imposed.

## The bond algebra $\mathfrak{B}_b$ with $b = 0, 1$

The Hilbert space is

$$\mathcal{H}_b := \mathrm{span}\left\{ \bigotimes_{j\in\Lambda} \left(\frac{\hat{\sigma}^x_j - \mathrm{i}\hat{\sigma}^y_j}{2}\right)^{n_j} |\uparrow\rangle_j \,\middle|\, n_j = 0, 1, \qquad \hat{\sigma}^z_j\,|\uparrow\rangle_j = |\uparrow\rangle_j \right\} \cong \mathbb{C}^{2^{2N}}, \tag{9a}$$

where

$$\hat{\sigma}^\alpha_j\,\hat{\sigma}^\beta_j = \delta^{\alpha\beta}\,\widehat{\mathbb{1}}_{\mathcal{H}_b} + \mathrm{i}\epsilon^{\alpha\beta\gamma}\,\hat{\sigma}^\gamma_j, \qquad \left[\hat{\sigma}^\alpha_i, \hat{\sigma}^\beta_j\right] = 0, \qquad i < j \in \Lambda, \tag{9b}$$

with the symmetry twisted boundary conditions

$$\hat{\sigma}^x_{j+2N} = (-1)^b\,\hat{\sigma}^x_j, \qquad \hat{\sigma}^y_{j+2N} = (-1)^b\,\hat{\sigma}^y_j, \qquad \hat{\sigma}^z_{j+2N} = \hat{\sigma}^z_j. \tag{9c}$$

The bond algebra

$$\mathfrak{B}_b \equiv \left\langle \hat{\sigma}^z_j, \quad \hat{\sigma}^x_j\,\hat{\sigma}^x_{j+1} \,\middle|\, j \in \Lambda \right\rangle, \tag{10}$$

is spanned by all complex-valued linear combinations of products of the generators $\hat{\sigma}^z_i$ and $\hat{\sigma}^x_j\,\hat{\sigma}^x_{j+1}$ for any $i, j \in \Lambda$.

It is invariant under the **global** internal symmetry group $\mathbb{Z}^z_2$ generated by $\widehat{U}_{r^z_\pi}$.

## The extended bond algebra $\mathfrak{B}_{b,b'}$ with $b, b' = 0, 1$

The extended Hilbert space is the tensor product

$$\mathcal{H}_{b,b'} \equiv \mathcal{H}_{b} \otimes \mathcal{H}_{b'} \tag{11a}$$

with

$$\mathcal{H}_{b'} := \mathrm{span}\left\{ \bigotimes_{j^{\star}\in\Lambda^{\star}} \left(\frac{\hat{\tau}^{x}_{j^{\star}} - \mathrm{i}\hat{\tau}^{y}_{j^{\star}}}{2}\right)^{n_{j^{\star}}} |\uparrow\rangle_{j^{\star}} \;\middle|\; n_{j^{\star}} = 0, 1, \quad \hat{\tau}^{z}_{j^{\star}} |\uparrow\rangle_{j^{\star}} = |\uparrow\rangle_{j^{\star}} \right\} \cong \mathbb{C}^{2^{2N}}, \tag{11b}$$

where

$$\hat{\tau}^{\alpha}_{j^{\star}}\, \hat{\tau}^{\beta}_{j^{\star}} = \delta^{\alpha\beta}\, \widehat{\mathbb{1}}_{\mathcal{H}_{b'}} + \mathrm{i}\epsilon^{\alpha\beta\gamma}\, \hat{\tau}^{\gamma}_{j^{\star}}, \qquad \left[\hat{\tau}^{\alpha}_{i^{\star}}, \hat{\tau}^{\beta}_{j^{\star}}\right] = 0, \qquad i^{\star} < j^{\star} \in \Lambda^{\star}, \tag{11c}$$

obey the symmetry twisted boundary conditions

$$\hat{\tau}^{x}_{j^{\star}+2N} = (-1)^{b'}\, \hat{\tau}^{x}_{j^{\star}}, \quad \hat{\tau}^{y}_{j^{\star}+2N} = (-1)^{b'}\, \hat{\tau}^{y}_{j^{\star}}, \quad \hat{\tau}^{z}_{j^{\star}+2N} = \hat{\tau}^{z}_{j^{\star}}, \qquad b' = 0, 1. \tag{11d}$$

The extended bond algebra

$$\mathfrak{B}_{b,b'} \equiv \left\langle \hat{\sigma}^z_j, \quad \hat{\sigma}^x_j\, \hat{\tau}^z_{j^\star}\, \hat{\sigma}^x_{j+1} \quad \middle| \quad j^\star := j + \frac{1}{2}, \qquad j \in \Lambda \right\rangle , \tag{12}$$

is spanned by all complex-valued linear combinations of products of the generators $\hat{\sigma}^z_i$ and $\hat{\sigma}^x_j\, \hat{\tau}^z_{j^\star}\, \hat{\sigma}^x_{j+1}$ for any $i, j \in \Lambda$.

It is invariant under the **local** internal symmetry group generated by the **local Gauss operators**

$$\widehat{G}_{b,b';j} := \hat{\tau}^x_{j^\star-1}\, \hat{\sigma}^z_j\, \hat{\tau}^x_{j^\star} = \widehat{G}_{b,b';j+2N}, \qquad j \in \Lambda. \tag{13}$$

## Projection to the gauge-invariant states spanning $\mathcal{H}^{\vee}_{b'}$

Physical states are defined to be those states in $\mathcal{H}_{b,b'}$ that are eigenstates of all local Gauss operators with the eigenvalue one. There exists a unitary transformation $\widehat{U}_{b,b'}$ such that

$$\widehat{U}_{b,b'}\,\hat{\sigma}^{x}_{j}\,\left(\widehat{U}_{b,b'}\right)^{\dagger}=\hat{\sigma}^{x}_{j},\qquad \widehat{U}_{b,b'}\,\hat{\sigma}^{z}_{j}\,\left(\widehat{U}_{b,b'}\right)^{\dagger}=\hat{\tau}^{x}_{j^{\star}-1}\,\hat{\sigma}^{z}_{j}\,\hat{\tau}^{x}_{j^{\star}},\tag{14a}$$

$$\widehat{U}_{b,b'}\,\hat{\tau}^{x}_{j^{\star}}\,\left(\widehat{U}_{b.b'}\right)^{\dagger}=\hat{\tau}^{x}_{j^{\star}},\qquad \widehat{U}_{b,b'}\,\hat{\tau}^{z}_{j^{\star}}\,\left(\widehat{U}_{b,b'}\right)^{\dagger}=\hat{\sigma}^{x}_{j}\,\hat{\tau}^{z}_{j^{\star}}\,\hat{\sigma}^{x}_{j+1},\tag{14b}$$

and

$$\widehat{U}_{b,b'}\,\widehat{G}_{b,b';j}\,\left(\widehat{U}_{b,b'}\right)^{\dagger}=\hat{\sigma}^{z}_{j},\qquad j\in\Lambda.\tag{14c}$$

After this unitary transformation, the physical subspace is the $2^{2N}$-dimensional gauge-invariant subspace

$$\mathcal{H}^{\vee}_{b'}:=\widehat{P}_{b,b';\mathrm{G}}\,\mathcal{H}_{b,b'}\subset\mathcal{H}_{b,b'},\tag{15a}$$

where

$$\widehat{P}_{b,b';\mathrm{G}}:=\prod_{j\in\Lambda}\frac{1}{2}\left[\widehat{\mathbb{1}}_{\mathcal{H}_{b,b'}}+\widehat{U}_{b,b'}\,\widehat{G}_{b,b';j}\,\left(\widehat{U}_{b,b'}\right)^{\dagger}\right].\tag{15b}$$

The KW dual bond algebra $\mathfrak{B}_{b'}$

The $2N$ triplets of projected operators

$$\begin{aligned}
\hat{\tau}^{x\,\vee}_{j^\star} &:= \widehat{P}_{b,b';\mathrm{G}} \left[ \widehat{U}_{b,b'}\, \hat{\tau}^{x}_{j^\star} \, \left(\widehat{U}_{b,b'}\right)^{\dagger} \right] \widehat{P}_{b,b';\mathrm{G}} = \widehat{P}_{b,b';\mathrm{G}}\, \hat{\tau}^{x}_{j^\star}\, \widehat{P}_{b,b';\mathrm{G}}\,, \\
\hat{\tau}^{y\,\vee}_{j^\star} &:= \widehat{P}_{b,b';\mathrm{G}} \left[ \widehat{U}_{b,b'}\, \hat{\sigma}^{x}_{j}\, \hat{\tau}^{y}_{j^\star}\, \hat{\sigma}^{x}_{j+1} \, \left(\widehat{U}_{b,b'}\right)^{\dagger} \right] \widehat{P}_{b,b';\mathrm{G}} = \widehat{P}_{b,b';\mathrm{G}}\, \hat{\tau}^{y}_{j^\star}\, \widehat{P}_{b,b';\mathrm{G}}\,, \\
\hat{\tau}^{z\,\vee}_{j^\star} &:= \widehat{P}_{b,b';\mathrm{G}} \left[ \widehat{U}_{b,b'}\, \hat{\sigma}^{x}_{j}\, \hat{\tau}^{z}_{j^\star}\, \hat{\sigma}^{x}_{j+1} \, \left(\widehat{U}_{b,b'}\right)^{\dagger} \right] \widehat{P}_{b,b';\mathrm{G}} = \widehat{P}_{b,b';\mathrm{G}}\, \hat{\tau}^{z}_{j^\star}\, \widehat{P}_{b,b';\mathrm{G}}\,,
\end{aligned} \tag{16a}$$

realize a Pauli algebra on the Hilbert space $\mathcal{H}^{\vee}_{b'}$ that is isomorphic to the Pauli algebra (11c) on the Hilbert space $\mathcal{H}_{b'}$. This implies that the projection to $\mathcal{H}^{\vee}_{b'}$ of the bond algebra $\mathfrak{B}_{b,b'}$ delivers the KW dual bond algebra

$$\mathfrak{B}_{b'} := \widehat{P}_{b,b';\mathrm{G}} \left[ \widehat{U}_{b,b'}\, \mathfrak{B}_{b,b'} \, \left(\widehat{U}_{b,b'}\right)^{\dagger} \right] \widehat{P}_{b,b';\mathrm{G}} = \left\langle \hat{\tau}^{x\,\vee}_{j^\star-1}\, \hat{\tau}^{x\,\vee}_{j^\star}, \quad \hat{\tau}^{z\,\vee}_{j^\star} \;\middle|\; j^\star \in \Lambda^\star \right\rangle . \tag{16b}$$

This is the bond algebra of operators that are symmetric under the dual $\mathbb{Z}_2^{z\,\vee}$ symmetry with the generator

$$\widehat{U}^{\vee}_{r^z_\pi} := \prod_{j^\star \in \Lambda^\star} \hat{\tau}^{z\,\vee}_{j^\star} \tag{16c}$$

of the global rotation by $\pi$ about the $z$ axis in internal spin-1/2 space attached to the dual lattice $\Lambda^\star$.

## Consistency conditions for the KW duality

The pair of operators

$$\left(\hat{\sigma}^{z}_{j}\,, \qquad \hat{\tau}^{x\,\vee}_{j^\star-1}\,\hat{\tau}^{x\,\vee}_{j^\star}\right) \tag{17a}$$

and the pair of operators

$$\left(\hat{\sigma}^{x}_{j}\,\hat{\sigma}^{x}_{j+1}\,, \qquad \hat{\tau}^{z\,\vee}_{j^\star}\right) \tag{17b}$$

each form a dual pair if and only if the pair of operators

$$\left(\prod_{j=1}^{2N}\hat{\sigma}^{z}_{j}=\widehat{U}_{r^{z}_{\pi}}\,, \qquad \prod_{j=1}^{2N}\left(\hat{\tau}^{x\,\vee}_{j^\star-1}\,\hat{\tau}^{x\,\vee}_{j^\star}\right)=(-1)^{b'}\,\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b'}}\right) \tag{18a}$$

and the pair of operators

$$\left(\prod_{j=1}^{2N}\left(\hat{\sigma}^{x}_{j}\,\hat{\sigma}^{x}_{j+1}\right)=(-1)^{b}\,\widehat{\mathbb{1}}_{\mathcal{H}_{b}}, \qquad \prod_{j=1}^{2N}\hat{\tau}^{z\,\vee}_{j^\star}=:\widehat{U}^{\vee}_{r^{z}_{\pi}}\right) \tag{18b}$$

each form a dual pair, respectively. The duality between the bond algebras (10) and (16b) holds only on the $2^{2N-1}$-dimensional subspaces

$$\mathcal{H}_{b;\,(-1)^{b'}}=\frac{1}{2}\left[\widehat{\mathbb{1}}_{\mathcal{H}_{b}}+(-1)^{b'}\,\widehat{U}_{r^{z}_{\pi}}\right]\mathcal{H}_{b}, \qquad \mathcal{H}^{\vee}_{b';\,(-1)^{b}}=\frac{1}{2}\left[\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b'}}+(-1)^{b}\,\widehat{U}^{\vee}_{r^{z}_{\pi}}\right]\mathcal{H}^{\vee}_{b'}, \tag{19}$$

of $2^{2N}$-dimensional Hilbert spaces $\mathcal{H}_{b}$ and $\mathcal{H}^{\vee}_{b'}$, respectively. This duality between the bond algebras (10) and (16b) acting on Hilbert spaces $\mathcal{H}_{b;\,(-1)^{b'}}$ and $\mathcal{H}^{\vee}_{b';\,(-1)^{b}}$ respectively, is nothing but the Kramers-Wannier (KW) duality with symmetry twisted boundary conditions.

## The extended bond algebra $\mathfrak{B}_{b,f}$ with $b, f = 0, 1$

The extended Hilbert space is the tensor product

$$\mathcal{H}_{b,f} \equiv \mathcal{H}_b \otimes \mathcal{H}_f \tag{20a}$$

with

$$\mathcal{H}_f := \mathrm{span}\left\{ \left[ \prod_{j^\star \in \Lambda^\star} \left( \frac{\hat{\beta}_{j^\star} - \mathrm{i}\hat{\alpha}_{j^\star}}{2} \right)^{n_{j^\star}} \right] |0\rangle \,\middle|\, n_{j^\star} = 0, 1, \qquad \frac{\hat{\beta}_{j^\star} + \mathrm{i}\hat{\alpha}_{j^\star}}{2} |0\rangle = 0 \right\} \cong \mathbb{C}^{2^{2N}}, \tag{20b}$$

where

$$\left\{ \hat{\alpha}_{i^\star}, \hat{\alpha}_{j^\star} \right\} = \left\{ \hat{\beta}_{i^\star}, \hat{\beta}_{j^\star} \right\} = 2\delta_{i^\star, j^\star} \, \widehat{\mathbb{1}}_{\mathcal{H}_f}, \qquad \left\{ \hat{\alpha}_{i^\star}, \hat{\beta}_{j^\star} \right\} = 0, \quad i^\star, j^\star \in \Lambda^\star, \tag{20c}$$

obey the symmetry twisted boundary conditions

$$\hat{\alpha}_{j^\star + 2N} = \left( \widehat{P}_{\mathrm{F}} \right)^f \hat{\alpha}_{j^\star} \left( \widehat{P}_{\mathrm{F}}^\dagger \right)^f = (-1)^f \, \hat{\alpha}_{j^\star}, \qquad \hat{\beta}_{j^\star + 2N} = \left( \widehat{P}_{\mathrm{F}} \right)^f \hat{\beta}_{j^\star} \left( \widehat{P}_{\mathrm{F}}^\dagger \right)^f = (-1)^f \, \hat{\beta}_{j^\star}. \tag{20d}$$

Here, we have introduced the **global fermion parity operator**

$$\widehat{P}_{\mathrm{F}} := \prod_{j^\star \in \Lambda^\star} \left( \mathrm{i}\hat{\beta}_{j^\star} \, \hat{\alpha}_{j^\star} \right) \tag{21a}$$

that generates a $2^{2N}$-dimensional representation of the cyclic group

$$\mathbb{Z}_2^{\mathrm{F}} = \left\{ p_{\mathrm{F}}, \ (p_{\mathrm{F}})^2 \equiv e \right\}. \tag{21b}$$

The extended bond algebra

$$\mathfrak{B}_{b,f} \equiv \left\langle \hat{\sigma}^{z}_{j}, \quad \hat{\sigma}^{x}_{j} \left( \mathrm{i}\hat{\beta}_{j^\star}\, \hat{\alpha}_{j^\star} \right) \hat{\sigma}^{x}_{j+1} \quad \middle| \quad j^\star := j + \frac{1}{2}, \qquad j \in \Lambda \right\rangle , \tag{22}$$

is spanned by all complex-valued linear combinations of products of the generators $\hat{\sigma}^{z}_{j}$ and $\hat{\sigma}^{x}_{j} \left( \mathrm{i}\hat{\beta}_{j^\star}\, \hat{\alpha}_{j^\star} \right) \hat{\sigma}^{x}_{j+1}$ for any $i, j \in \Lambda$.

It is invariant under the **local** internal symmetry group generated by the **local Gauss operators**

$$\widehat{G}_{b,f;j} := \mathrm{i}\hat{\beta}_{j^\star-1}\, \hat{\sigma}^{z}_{j}\, \hat{\alpha}_{j^\star} = \widehat{G}_{b,f;j+2N}, \qquad j \in \Lambda. \tag{23}$$

## Projection to the gauge-invariant states spanning $\mathcal{H}_f^{\vee}$

Physical states are defined to be those states in $\mathcal{H}_{b,f}$ that are eigenstates of all local Gauss operators with the eigenvalue one. There exists a unitary transformation $\widehat{U}_{b,f}$ such that

$$\widehat{U}_{b,f}\,\hat{\sigma}^x_j\,\left(\widehat{U}_{b,f}\right)^\dagger=\hat{\sigma}^x_j,\qquad \widehat{U}_{b,f}\,\hat{\sigma}^z_j\,\left(\widehat{U}_{b,f}\right)^\dagger=\mathrm{i}\hat{\beta}_{j^\star-1}\,\hat{\sigma}^z_j\,\hat{\alpha}_{j^\star},$$
$$\widehat{U}_{b,f}\,\hat{\beta}_{j^\star}\,\left(\widehat{U}_{b,f}\right)^\dagger=\hat{\beta}_{j^\star}\,\hat{\sigma}^x_{j+1},\qquad \widehat{U}_{b,f}\,\hat{\alpha}_{j^\star}\,\left(\widehat{U}_{b,f}\right)^\dagger=\hat{\sigma}^x_j\,\hat{\alpha}_{j^\star}\,, \tag{24a}$$

and

$$\widehat{U}_{b,f}\,\widehat{G}_{b,f;j}\,\left(\widehat{U}_{b,f}\right)^\dagger=\hat{\sigma}^z_j,\qquad j\in\Lambda. \tag{24b}$$

After this unitary transformation, the physical subspace is the $2^{2N}$-dimensional gauge-invariant subspace

$$\mathcal{H}_f^{\vee}:=\widehat{P}_{b,f;\mathrm{G}}\,\mathcal{H}_{b,f}\subset\mathcal{H}_{b,f}, \tag{25a}$$

where

$$\widehat{P}_{b,f;\mathrm{G}}:=\prod_{j\in\Lambda}\frac{1}{2}\left[\widehat{\mathbb{1}}_{\mathcal{H}_{b,b'}}+\widehat{U}_{b,f}\,\widehat{G}_{b,f;j}\,\left(\widehat{U}_{b,f}\right)^\dagger\right]. \tag{25b}$$

## The JW dual bond algebra $\mathfrak{B}_f$

The $2N$ doublets of projected operators

$$\begin{aligned} \hat{\beta}^{\vee}_{j+1} &:= \widehat{P}_{b,f;\mathrm{G}} \left[ \widehat{U}_{b,f} \left( \hat{\beta}_{j^\star}\, \hat{\sigma}^{x}_{j+1} \right) \left( \widehat{U}_{b,f} \right)^{\dagger} \right] \widehat{P}_{b,f;\mathrm{G}} = \widehat{P}_{b,f;\mathrm{G}}\, \hat{\beta}_{j^\star}\, \widehat{P}_{b,f;\mathrm{G}} , \\ \hat{\alpha}^{\vee}_{j} &:= \widehat{P}_{b,f;\mathrm{G}} \left[ \widehat{U}_{b,f} \left( \hat{\sigma}^{x}_{j}\, \hat{\alpha}_{j^\star} \right) \left( \widehat{U}_{b,f} \right)^{\dagger} \right] \widehat{P}_{b,f;\mathrm{G}} = \widehat{P}_{b,f;\mathrm{G}}\, \hat{\alpha}_{j^\star}\, \widehat{P}_{b,f;\mathrm{G}}, \end{aligned} \tag{26}$$

realize a Majorana algebra on the Hilbert space $\mathcal{H}^{\vee}_{f}$ that is isomorphic to the Majorana algebra (20c) on the Hilbert space $\mathcal{H}_f$.

We also find that the projection to $\mathcal{H}^{\vee}_{f}$ of the bond algebra $\mathfrak{B}_{b,f}$ is the JW dual bond algebra

$$\mathfrak{B}_f := \left\langle \mathrm{i}\hat{\beta}^{\vee}_{j}\, \hat{\alpha}^{\vee}_{j}, \qquad \mathrm{i}\hat{\beta}^{\vee}_{j+1}\, \hat{\alpha}^{\vee}_{j} \;\middle|\; j \in \Lambda \right\rangle , \tag{27a}$$

which is the algebra of operators invariant under conjugation by the generator

$$\widehat{P}^{\vee}_{\mathrm{F}} := \prod_{j\in\Lambda} \left( \mathrm{i}\hat{\beta}^{\vee}_{j}\, \hat{\alpha}^{\vee}_{j} \right) . \tag{27b}$$

of a global fermion-parity symmetry $\mathbb{Z}^{\mathrm{F}}_{2}$.

## Consistency conditions for the JW duality

The pairs of operators

$$\left(\hat{\sigma}^{z}_{j}, \qquad \mathrm{i}\hat{\beta}^{\vee}_{j}\,\hat{\alpha}^{\vee}_{j}\right) \tag{28a}$$

and

$$\left(\hat{\sigma}^{x}_{j}\,\hat{\sigma}^{x}_{j+1}, \qquad \mathrm{i}\hat{\beta}^{\vee}_{j+1}\,\hat{\alpha}^{\vee}_{j}\right) \tag{28b}$$

each form a dual pair if and only if the pairs of operators

$$\left(\prod_{j=1}^{2N}\hat{\sigma}^{z}_{j} = \widehat{U}_{r^{z}_{\pi}}, \qquad \prod_{j=1}^{2N}\left(\mathrm{i}\hat{\beta}^{\vee}_{j}\,\hat{\alpha}^{\vee}_{j}\right) = \widehat{P}^{\vee}_{\mathrm{F}}\right) \tag{28c}$$

and

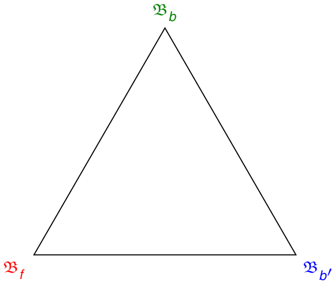
$$\left(\prod_{j=1}^{2N}\left(\hat{\sigma}^{x}_{j}\,\hat{\sigma}^{x}_{j+1}\right) = (-1)^{b}\,\widehat{\mathbb{1}}_{\mathcal{H}_{b}}, \qquad \prod_{j=1}^{2N}\left(\mathrm{i}\hat{\beta}^{\vee}_{j+1}\,\hat{\alpha}^{\vee}_{j}\right) = (-1)^{f+1}\,\widehat{P}^{\vee}_{\mathrm{F}}\right) \tag{28d}$$

each form a dual pair, respectively. The duality between the bond algebras (10) and (27) holds only on the $2^{2N-1}$-dimensional subspaces

$$\mathcal{H}_{b;\,(-1)^{b+f+1}} = \frac{1}{2}\left[\widehat{\mathbb{1}}_{\mathcal{H}_{b}} + (-1)^{b+f+1}\,\widehat{U}_{r^{z}_{\pi}}\right]\mathcal{H}_{b}, \qquad \mathcal{H}^{\vee}_{f;\,(-1)^{b+f+1}} = \frac{1}{2}\left[\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{f}} + (-1)^{b+f+1}\,\widehat{P}^{\vee}_{\mathrm{F}}\right]\mathcal{H}^{\vee}_{f}, \tag{29a}$$

of $2^{2N}$-dimensional Hilbert spaces $\mathcal{H}_{b}$ and $\mathcal{H}^{\vee}_{f}$, respectively. This duality between the bond algebras (10) and (27a) acting on Hilbert spaces $\mathcal{H}_{b;\,(-1)^{b+f+1}}$ and $\mathcal{H}^{\vee}_{f;\,(-1)^{b+f+1}}$ respectively, is nothing but the Jordan-Wigner (JW) duality with symmetry twisted boundary conditions.

# Triality of the bond algebras $\mathfrak{B}_{b}$, $\mathfrak{B}_{b'}$, $\mathfrak{B}_{f}$




| $(f,\ b',\ b)$ | $\mathcal{H}_{f;\,(-1)^{b+f+1}}$ | $\mathcal{H}^{\vee}_{b';\,(-1)^{b}}$ | $\mathcal{H}^{\vee}_{b;\,(-1)^{b'}}$ |
|---|---|---|---|
| $(0,1,0)$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}_f}-\widehat{P}_{\mathrm{F}}\right)\mathcal{H}_f$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b'}}+\widehat{U}^{\vee}_{r^{z}_{\pi}}\right)\mathcal{H}^{\vee}_{b'}$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b}}-\widehat{U}^{\vee}_{r^{z}_{\pi}}\right)\mathcal{H}^{\vee}_{b}$ |
| $(0,0,1)$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}_f}+\widehat{P}_{\mathrm{F}}\right)\mathcal{H}_f$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b'}}-\widehat{U}^{\vee}_{r^{z}_{\pi}}\right)\mathcal{H}^{\vee}_{b'}$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b}}+\widehat{U}^{\vee}_{r^{z}_{\pi}}\right)\mathcal{H}^{\vee}_{b}$ |
| $(1,0,0)$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}_f}+\widehat{P}_{\mathrm{F}}\right)\mathcal{H}_f$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b'}}+\widehat{U}^{\vee}_{r^{z}_{\pi}}\right)\mathcal{H}^{\vee}_{b'}$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b}}+\widehat{U}^{\vee}_{r^{z}_{\pi}}\right)\mathcal{H}^{\vee}_{b}$ |
| $(1,1,1)$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}_f}-\widehat{P}_{\mathrm{F}}\right)\mathcal{H}_f$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b'}}-\widehat{U}^{\vee}_{r^{z}_{\pi}}\right)\mathcal{H}^{\vee}_{b'}$ | $\frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b}}-\widehat{U}^{\vee}_{r^{z}_{\pi}}\right)\mathcal{H}^{\vee}_{b}$ |

# KW dual of the LSM anomaly

To study the fate of the translation and reflection LSM Theorems under KW duality, we choose $b = b' = 0$ so that periodic boundary conditions are imposed.

Space group and global symmetries need to be dualized. One finds

$$\widehat{U}_t^{\vee} := \prod_{j=1}^{2N-1} \frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b'=0}} + \hat{\boldsymbol{\tau}}^{\vee}_{j^\star} \cdot \hat{\boldsymbol{\tau}}^{\vee}_{t(j^\star)}\right), \qquad \widehat{U}_r^{\vee} := \prod_{j=1}^{N} \frac{1}{2}\left(\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_{b'=0}} + \hat{\boldsymbol{\tau}}^{\vee}_{j^\star} \cdot \hat{\boldsymbol{\tau}}^{\vee}_{r(j^\star)}\right), \tag{30}$$

for the space group and

$$\widehat{U}_{\mathrm{o}}^{\vee} = \prod_{j=1}^{N} \hat{\tau}^{z\,\vee}_{2j-1+\frac{1}{2}}, \qquad \widehat{U}_{\mathrm{e}}^{\vee} = \prod_{j=1}^{N} \hat{\tau}^{z\,\vee}_{2j+\frac{1}{2}}, \qquad \widehat{U}_{r^z_\pi}^{\vee} = \widehat{U}_{\mathrm{o}}^{\vee}\,\widehat{U}_{\mathrm{e}}^{\vee} = \prod_{j=1}^{2N} \hat{\tau}^{z\,\vee}_{j+\frac{1}{2}}, \tag{31}$$

for the global internal symmetry group.

The action of $\mathrm{G}^{\vee}_{\mathrm{spa}}$ on $\mathrm{G}^{\vee}_{\mathrm{int}}$ is now non-trivial as it is given by the composition rules

$$\widehat{U}^{\vee}_{t}\,\widehat{U}^{\vee}_{\mathrm{o}}\left(\widehat{U}^{\vee}_{t}\right)^{\dagger}=\widehat{U}^{\vee}_{\mathrm{e}},\qquad \widehat{U}^{\vee}_{t}\,\widehat{U}^{\vee}_{\mathrm{e}}\left(\widehat{U}^{\vee}_{t}\right)^{\dagger}=\widehat{U}^{\vee}_{\mathrm{o}}, \tag{32a}$$

$$\widehat{U}^{\vee}_{r}\,\widehat{U}^{\vee}_{\mathrm{o}}\left(\widehat{U}^{\vee}_{r}\right)^{\dagger}=\widehat{U}^{\vee}_{\mathrm{e}},\qquad \widehat{U}^{\vee}_{r}\,\widehat{U}^{\vee}_{\mathrm{e}}\left(\widehat{U}^{\vee}_{r}\right)^{\dagger}=\widehat{U}^{\vee}_{\mathrm{o}}. \tag{32b}$$

In other words, the dual symmetry group

$$\mathrm{G}^{\vee}_{\mathrm{tot}}\equiv\mathrm{G}^{\vee}_{\mathrm{spa}}\ltimes\mathrm{G}^{\vee}_{\mathrm{int}} \tag{33}$$

of the Hamiltonian $\widehat{H}^{\vee}_{b'=0}$ in the bond algebra (16b) with $b'=0$ that is dual to the Hamiltonian $\widehat{H}_{b=0}$ in the bond algebra (10) with $b=0$ is a **semi-direct product** of crystalline symmetries $\mathrm{G}^{\vee}_{\mathrm{spa}}$ and internal symmetries $\mathrm{G}^{\vee}_{\mathrm{int}}$.

One observes that the two LSM Theorems 1 and 2 do not apply to the dual symmetry group $\mathrm{G}^{\vee}_{\mathrm{tot}}$.

**Proof.**

This is because the local representation of $\mathrm{G}^{\vee}_{\mathrm{int}}$ is not projective, unlike that of $\mathrm{G}_{\mathrm{int}}$. □

We further note that while being isomorphic to $\mathrm{G}_{\mathrm{spa}}$ the dual crystalline symmetry group $\mathrm{G}^{\vee}_{\mathrm{spa}}$ is such that

1. the "natural" unit cell on which the internal symmetry group $\mathrm{G}^{\vee}_{\mathrm{int}}$ acts on-site is associated with the generator $\left(\widehat{U}^{\vee}_{t}\right)^{2}$ of translations, i.e., it is twice that of the unit cell associated with the generator $\widehat{U}^{\vee}_{t}$ of translations,
2. the operator $\widehat{U}^{\vee}_{r}$ acts as a link-centered reflection on lattice $\Lambda^{\star}$ such that there are no invariant unit cells.

Both properties can be interpreted as a trivialization of mixed anomalies between internal and spatial symmetries under the gauging of a subgroup of the internal symmetries.

In anticipation of the discussion of the phase diagram of the quantum spin-1/2 *XYZ* chain, we close this discussion by focusing on the reflection symmetry subgroup $\mathbb{Z}_2^r$ of $\mathrm{G}_{\mathrm{spa}}^{\vee}$.

As a consequence of the underlying LSM anomaly, the Abelian group

$$\mathbb{Z}_2^r \times \mathbb{Z}_2^x \times \mathbb{Z}_2^y \tag{34a}$$

formed by the subgroup of reflection symmetry $\mathbb{Z}_2^r$ together with the group of internal symmetries $G_{\mathrm{int}} \equiv \mathbb{Z}_2^x \times \mathbb{Z}_2^y$ is mapped to the non-Abelian dihedral group of order eight

$$\begin{aligned}\mathrm{D}_8 :=& \left\{r,\ r^2 \equiv e\right\} \ltimes \left\{e,\ r_{\mathrm{o}},\ r_{\mathrm{e}},\ r_{\mathrm{o}}\, r_{\mathrm{e}} \;\middle|\; (r_{\mathrm{o}})^2 \equiv (r_{\mathrm{e}})^2 \equiv e, \qquad r_{\mathrm{o}}\, r_{\mathrm{e}} = r_{\mathrm{e}}\, r_{\mathrm{o}}\right\}\\ =& \left\{e,\ a,\ a^2,\ a^3,\ r,\ r\,a,\ r\,a^2,\ r\,a^3 \;\middle|\; a \equiv r\,r_{\mathrm{o}},\ a^4 \equiv r^2 \equiv e,\ r\,a\,r = r\,a^3\right\},\end{aligned} \tag{34b}$$

after gauging the diagonal subgroup $\mathbb{Z}_2^z \subset \mathbb{Z}_2^x \times \mathbb{Z}_2^y$ by KW duality.

## JW dual of the LSM anomaly

Having set $b = b' = 0$ for the KW duality, triality of the bond algebras **enforces** $f = 1$. However, we will consider the generic case of $f = 0, 1$.

The extension of the crystalline symmetries on the Hilbert space $\mathcal{H}_{b,f}$ are obtained by demanding the **covariance** of the **local Gauss operators** under translation and reflection. We thus define the unitary operators

$$\widehat{U}^{\vee}_{t,f} := \left(\mathrm{i}\hat{\beta}_1\,\hat{\alpha}_1\right)^f \prod_{j=1}^{2N-1} \frac{\mathrm{i}}{2}\left[\left(\hat{\beta}^{\vee}_j - \hat{\beta}^{\vee}_{t(j)}\right)\left(\hat{\alpha}^{\vee}_j - \hat{\alpha}^{\vee}_{t(j)}\right)\right], \tag{35a}$$

$$\widehat{U}^{\vee}_{r,f} := \left(\mathrm{i}\hat{\beta}_{2N}\,\hat{\alpha}_{2N}\right)^f \prod_{j=1}^{2N} \frac{1}{\sqrt{2}}\left(\widehat{\mathbb{1}}_{\mathcal{H}^{\vee}_f} + \hat{\beta}^{\vee}_{r(j)}\,\hat{\alpha}^{\vee}_j\right), \tag{35b}$$

where the global fermion parity $\widehat{P}^{\vee}_{\mathrm{F}}$ takes the form (27b). For any $j \in \Lambda$, conjugation of $\hat{\alpha}^{\vee}_j$ and $\hat{\beta}^{\vee}_j$ by $\widehat{U}^{\vee}_{t,f}$ and $\widehat{U}^{\vee}_{r,f}$ implement the maps

$$\hat{\alpha}^{\vee}_j \mapsto (-1)^{f\,\delta_{j,2N}}\,\hat{\alpha}^{\vee}_{t(j)}, \qquad \hat{\beta}^{\vee}_j \mapsto (-1)^{f\,\delta_{j,2N}}\,\hat{\beta}^{\vee}_{t(j)}, \tag{36a}$$

$$\hat{\alpha}^{\vee}_j \mapsto +(-1)^{f\,\delta_{j,2N}}\,\hat{\beta}^{\vee}_{r(j)}, \qquad \hat{\beta}^{\vee}_j \mapsto -(-1)^{f\,\delta_{j,2N}}\,\hat{\alpha}^{\vee}_{r(j)}, \tag{36b}$$

respectively.

In the fermionic case, reflection is not an order two operation. Instead, one verifies that

$$\left(\widehat{U}_{r,f}^{\vee}\right)^2 = -\widehat{P}_{\mathrm{F}}^{\vee}. \tag{37a}$$

Similarly, translation is not an order $2N$ operator if $f = 1$, instead

$$\left(\widehat{U}_{t,f}^{\vee}\right)^{2N} = \left(\widehat{P}_{\mathrm{F}}^{\vee}\right)^{f}. \tag{37b}$$

This leads to a mixing of crystalline symmetries with the fermion parity. We denote the crystalline group obtained after JW duality as $\mathrm{G}_{\mathrm{spa}}^{\vee,\mathrm{F}}$.

What is the relation between the JW dual $\mathrm{G}_{\mathrm{spa}}^{\vee,\mathrm{F}}$ and the KW dual $\mathrm{G}_{\mathrm{spa}}^{\vee}$?

The JW dual $\mathrm{G}_{\mathrm{spa}}^{\vee,\mathrm{F}}$ is obtained by the central extension of the KW dual $\mathrm{G}_{\mathrm{spa}}^{\vee}$ by fermion parity $\mathbb{Z}_2^{\mathrm{F}}$ specified by the short exact sequence

$$0 \to \mathbb{Z}_2^{\mathrm{F}} \to \mathrm{G}_{\mathrm{spa}}^{\vee,\mathrm{F}} \to \mathrm{G}_{\mathrm{spa}}^{\vee} \to 0, \tag{38}$$

with the extension class $[\gamma_f] \in H^2(\mathrm{G}_{\mathrm{spa}}^{\vee}, \mathbb{Z}_2^{\mathrm{F}})$ and the extension map

$$\gamma_f(r,r) := p_{\mathrm{F}}, \qquad \gamma_f(t^a, t^b) = (p_{\mathrm{F}})^{f\,\lfloor (a+b)/2N \rfloor}, \qquad \gamma_f(r,t) = (p_{\mathrm{F}})^{f}, \tag{39}$$

where $p_{\mathrm{F}}$ was defined in Eq. (21b) and $\lfloor \cdot \rfloor$ is the lower floor function. All other maps can be derived using these relations and the cocycle condition for $\gamma_f$.

Having defined the crystalline symmetries, we now turn to the internal symmetries.

Under the JW duality,

$$\widehat{U}_{r_\pi^x} = \prod_{j=1}^{N} \hat{\sigma}^x_{2j-1}\,\hat{\sigma}^x_{2j} \longmapsto \widehat{U}^{\vee}_{\mathrm{o}} := \prod_{j=1}^{N} \left(\mathrm{i}\hat{\alpha}^{\vee}_{2j-1}\,\hat{\beta}^{\vee}_{2j}\right), \tag{40a}$$

$$\widehat{U}_{r_\pi^y} = \prod_{j=1}^{N} \hat{\sigma}^y_{2j-1}\,\hat{\sigma}^y_{2j} \longmapsto \widehat{U}^{\vee}_{\mathrm{e}} := \prod_{j=1}^{N} \left(\mathrm{i}\hat{\beta}^{\vee}_{2j-1}\,\hat{\alpha}^{\vee}_{2j}\right). \tag{40b}$$

The pair $\widehat{U}^{\vee}_{\mathrm{o}}$ and $\widehat{U}^{\vee}_{\mathrm{e}}$ of dual internal symmetry operators compose to the fermion parity operator,

$$\widehat{U}^{\vee}_{\mathrm{o}}\,\widehat{U}^{\vee}_{\mathrm{e}} = \widehat{P}^{\vee}_{\mathrm{F}}. \tag{40c}$$

The pair of operators $\widehat{U}^{\vee}_{\mathrm{o}}$ and $\widehat{U}^{\vee}_{\mathrm{e}}$ generates a $2^{2N}$-dimensional representation of the internal symmetry group

$$\mathrm{G}^{\vee,\mathrm{F}}_{\mathrm{int}} \equiv \mathbb{Z}^{\mathrm{o}}_2 \times \mathbb{Z}^{\mathrm{e}}_2, \tag{41a}$$

with

$$\mathbb{Z}^{\mathrm{o}}_2 \equiv \left\{r_{\mathrm{o}},\ (r_{\mathrm{o}})^2 \equiv e\right\}, \qquad \mathbb{Z}^{\mathrm{e}}_2 \equiv \left\{r_{\mathrm{e}},\ (r_{\mathrm{e}})^2 \equiv e\right\}. \tag{41b}$$

The generators (35) of the dual crystalline symmetries act on the operators $\widehat{U}_{\mathrm{o}}^{\vee}$ and $\widehat{U}_{\mathrm{e}}^{\vee}$ according to the composition rules

$$
\begin{aligned}
&\widehat{U}_{t}^{\vee}\,\widehat{U}_{\mathrm{o}}^{\vee}\left(\widehat{U}_{t}^{\vee}\right)^{\dagger}=(-1)^{f+1}\,\widehat{U}_{\mathrm{e}}^{\vee}, &\qquad &\widehat{U}_{r}^{\vee}\,\widehat{U}_{\mathrm{o}}^{\vee}\left(\widehat{U}_{r}^{\vee}\right)^{\dagger}=(-1)^{f+1}\,\widehat{U}_{\mathrm{e}}^{\vee}\,,\\
&\widehat{U}_{t}^{\vee}\,\widehat{U}_{\mathrm{e}}^{\vee}\left(\widehat{U}_{t}^{\vee}\right)^{\dagger}=(-1)^{f+1}\,\widehat{U}_{\mathrm{o}}^{\vee}\,, &\qquad &\widehat{U}_{r}^{\vee}\,\widehat{U}_{\mathrm{e}}^{\vee}\left(\widehat{U}_{r}^{\vee}\right)^{\dagger}=(-1)^{f+1}\,\widehat{U}_{\mathrm{o}}^{\vee}\,,\\
&\widehat{U}_{t}^{\vee}\,\widehat{P}_{\mathrm{F}}^{\vee}\left(\widehat{U}_{t}^{\vee}\right)^{\dagger}=\widehat{P}_{\mathrm{F}}^{\vee}\,, &\qquad &\widehat{U}_{r}^{\vee}\,\widehat{P}_{\mathrm{F}}^{\vee}\left(\widehat{U}_{r}^{\vee}\right)^{\dagger}=\widehat{P}_{\mathrm{F}}^{\vee}\,.
\end{aligned}
\tag{42}
$$

The total symmetry group $\mathrm{G}_{\mathrm{tot}}^{\vee,\mathrm{F}}$ is obtained by taking the semi-direct product of $\mathrm{G}_{\mathrm{spa}}^{\vee,\mathrm{F}}$ and $\mathrm{G}_{\mathrm{int}}^{\vee,\mathrm{F}}$ together with coseting by the fermion parity group $\mathbb{Z}_2^{\mathrm{F}}$ defined in Eq. (21b), i.e.,

$$
\mathrm{G}_{\mathrm{tot}}^{\vee,\mathrm{F}}=\left(\mathrm{G}_{\mathrm{spa}}^{\vee,\mathrm{F}}\ltimes\mathrm{G}_{\mathrm{int}}^{\vee,\mathrm{F}}\right)\Big/\,\mathbb{Z}_2^{\mathrm{F}}. \tag{43a}
$$

Here, the semi-direct product $\mathrm{G}_{\mathrm{spa}}^{\vee,\mathrm{F}}\ltimes\mathrm{G}_{\mathrm{int}}^{\vee,\mathrm{F}}$ is specified by the action

$$
\begin{aligned}
t\,r_{\mathrm{o}}t^{-1}&=r_{\mathrm{e}}\,, &\qquad r\,r_{\mathrm{o}}r^{-1}&=r_{\mathrm{e}}\,,\\
t\,r_{\mathrm{e}}t^{-1}&=r_{\mathrm{o}}\,, &\qquad r\,r_{\mathrm{e}}r^{-1}&=r_{0}\,,\\
t\,p_{\mathrm{F}}t^{-1}&=p_{\mathrm{F}}\,, &\qquad r\,p_{\mathrm{F}}r^{-1}&=p_{\mathrm{F}}.
\end{aligned}
\tag{43b}
$$

of dual crystalline symmetry group $\mathrm{G}_{\mathrm{spa}}^{\vee,\mathrm{F}}$ on the dual internal symmetry group $\mathrm{G}_{\mathrm{int}}^{\vee,\mathrm{F}}$.

We emphasize that the structure of $\mathrm{G}_{\mathrm{tot}}^{\vee,\mathrm{F}}$ is different from $\mathrm{G}_{\mathrm{tot}}^{\vee}$ in Eq. (33) obtained via the KW duality.

More precisely, under the JW duality the resulting dual total symmetry group $\mathrm{G}_{\mathrm{tot}}^{\vee,\mathrm{F}}$ is assembled from the crystalline $\mathrm{G}_{\mathrm{spa}}^{\vee,\mathrm{F}}$ and internal $\mathrm{G}_{\mathrm{int}}^{\vee,\mathrm{F}}$ symmetry groups using a nontrivial central extension in addition to the semi-direct product structure.

In contrast, the dual of $\mathrm{G}_{\mathrm{tot}}$ under the KW duality is a semi-direct product of the crystalline and internal symmetry groups.

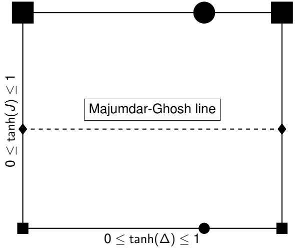
Finally, one observes that the two LSM Theorems 1 and 2 do not apply to the dual symmetry group $\mathrm{G}_{\mathrm{tot}}^{\vee,\mathrm{F}}$. This is because the local representation of $\mathrm{G}_{\mathrm{int}}^{\vee,\mathrm{F}}$ is not projective, unlike that of $\mathrm{G}_{\mathrm{int}}$. As was the case with the KW dual $\mathrm{G}_{\mathrm{int}}^{\vee}$, the trivialization of mixed anomalies between internal and spatial symmetries under the JW gauging of a subgroup of the internal symmetries can be attributed to a doubling of the natural unit cell for the dual internal symmetries.

# Phase diagram of the quantum spin-1/2 *XYX* chain

We start from

$$\widehat{H}_{b=0} := J_1 \sum_{j\in\Lambda} \left(\Delta_x\, \hat{\sigma}^x_j\, \hat{\sigma}^x_{j+1} + \Delta_y\, \hat{\sigma}^y_j\, \hat{\sigma}^y_{j+1} + \Delta_z\, \hat{\sigma}^z_j\, \hat{\sigma}^z_{j+1}\right) + J_2 \sum_{j\in\Lambda} \left(\Delta_x\, \hat{\sigma}^x_j\, \hat{\sigma}^x_{j+2} + \Delta_y\, \hat{\sigma}^y_j\, \hat{\sigma}^y_{j+2} + \Delta_z\, \hat{\sigma}^z_j\, \hat{\sigma}^z_{j+2}\right). \tag{44}$$

Its zero-temperature phase diagram has the exactly soluble points

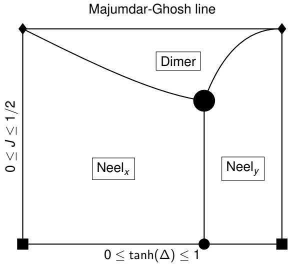


# Reduced coupling space

The reduced coupling space is defined by

$$0 \leq \Delta \equiv \Delta_y/\Delta_x, \qquad \Delta_z = 0, \qquad 0 \leq J \equiv J_2/J_1 \leq 1/2. \tag{45}$$

The corresponding zero-temperature phase diagram is




At $(\Delta, J) = (0, 0)$, the two degenerate ground states are

$$|\mathrm{Neel}^x_{\mathrm{o}}\rangle := |\rightarrow, \leftarrow, \rightarrow, \leftarrow, \cdots\rangle\,, \quad |\mathrm{Neel}^x_{\mathrm{e}}\rangle := |\leftarrow, \rightarrow, \leftarrow, \rightarrow, \cdots\rangle\,. \tag{46a}$$

At $(\Delta, J) = (\infty, 0)$, the two degenerate ground states are

$$|\mathrm{Neel}^y_{\mathrm{o}}\rangle := |\nearrow, \swarrow, \nearrow, \swarrow, \cdots\rangle\,, \quad |\mathrm{Neel}^y_{\mathrm{e}}\rangle := |\swarrow, \nearrow, \swarrow, \nearrow, \cdots\rangle\,. \tag{46b}$$

Along the MG line $(\Delta, J) = (\Delta, 1/2)$, the two degenerate ground states are

$$|\mathrm{Dimer}_{\mathrm{o}}\rangle := \bigotimes_{j=1}^{N} |[2j-1, 2j]\rangle\,, \quad |\mathrm{Dimer}_{\mathrm{e}}\rangle := \bigotimes_{j=1}^{N} |[2j, 2j+1]\rangle\,. \tag{46c}$$

## Correlation functions

These ground states are distinguished by the non-vanishing expectations values of the order parameters

$$\widehat{O}^{\mathrm{o}}_{\mathrm{Neel}^x} := \frac{1}{2N}\sum_{j=1}^{2N}(-1)^{j+1}\,\hat{\sigma}^x_j, \tag{47a}$$

$$\widehat{O}^{\mathrm{o}}_{\mathrm{Neel}^y} := \frac{1}{2N}\sum_{j=1}^{2N}(-1)^{j+1}\,\hat{\sigma}^y_j, \tag{47b}$$

$$\widehat{O}_{\mathrm{dimer}} := \frac{1}{N}\sum_{j=1}^{2N}(-1)^{j}\,\frac{1}{3}\,\hat{\boldsymbol{\sigma}}_j\cdot\hat{\boldsymbol{\sigma}}_{j+1}, \tag{47c}$$

respectively. The order parameters for the $\mathrm{Neel}_x$ and $\mathrm{Neel}_y$ phases are odd under $\widehat{U}_{r^z_\pi}$ symmetry, while the dimer order parameter is even. In other words, the order parameter for the two Neel phases do not belong to the bond algebra (10) and do not have an image in the dual bond algebras (16b) and (27). For this reason, it is more convenient to define the operators

$$\widehat{C}^x_{j,j+n} := \hat{\sigma}^x_j\,\hat{\sigma}^x_{j+n}, \tag{48a}$$

$$\widehat{C}^y_{j,j+n} := \hat{\sigma}^y_j\,\hat{\sigma}^y_{j+n}, \tag{48b}$$

$$\widehat{D}_j := \frac{1}{3}\,\hat{\boldsymbol{\sigma}}_j\cdot\hat{\boldsymbol{\sigma}}_{j+1}. \tag{48c}$$

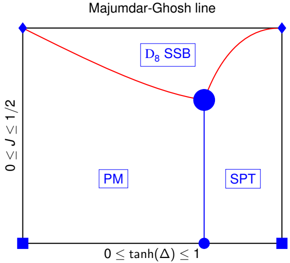
The KW dual to Eq. (44) is

$$\widehat{H}^{\vee}_{b'=0} := J_1 \sum_{j^\star\in\Lambda^\star} \left[\Delta_x\,\hat{\tau}^{z\,\vee}_{j^\star} - \Delta_y\left(\hat{\tau}^{x\,\vee}_{j^\star-1}\,\hat{\tau}^{z\,\vee}_{j^\star}\,\hat{\tau}^{x\,\vee}_{j^\star+1}\right) + \Delta_z\left(\hat{\tau}^{x\,\vee}_{j^\star-1}\,\hat{\tau}^{x\,\vee}_{j^\star+1}\right)\right]$$
$$+ J_2 \sum_{j^\star\in\Lambda^\star} \left[\Delta_x\,\hat{\tau}^{z\,\vee}_{j^\star}\,\hat{\tau}^{z\,\vee}_{j^\star+1} + \Delta_y\left(\hat{\tau}^{x\,\vee}_{j^\star-1}\,\hat{\tau}^{z\,\vee}_{j^\star}\,\hat{\tau}^{x\,\vee}_{j^\star+1}\right)\left(\hat{\tau}^{x\,\vee}_{j^\star}\,\hat{\tau}^{z\,\vee}_{j^\star+1}\,\hat{\tau}^{x\,\vee}_{j^\star+2}\right)\right.$$
$$\left. +\Delta_z\left(\hat{\tau}^{x\,\vee}_{j^\star-1}\,\hat{\tau}^{x\,\vee}_{j^\star+1}\right)\left(\hat{\tau}^{x\,\vee}_{j^\star}\,\hat{\tau}^{x\,\vee}_{j^\star+2}\right)\right], \tag{49}$$

with the domain of definition $\mathcal{H}^{\vee}_{b'=0}$ defined in Eq. (15a).

# Reduced coupling space

On the reduced coupling space (45), the corresponding zero-temperature phase diagram is




At $(\Delta, J) = (0, 0)$, the non-degenerate ground state is

$$
\begin{aligned}
&|\mathrm{PM}\rangle := |\downarrow, \cdots, \downarrow\rangle\,,\\
&\hat{\tau}^{z\vee}_{j^\star}\ |\downarrow, \cdots, \downarrow\rangle = -\,|\downarrow, \cdots, \downarrow\rangle\,, \qquad j^\star \in \Lambda^\star\,.
\end{aligned}
\tag{50}
$$

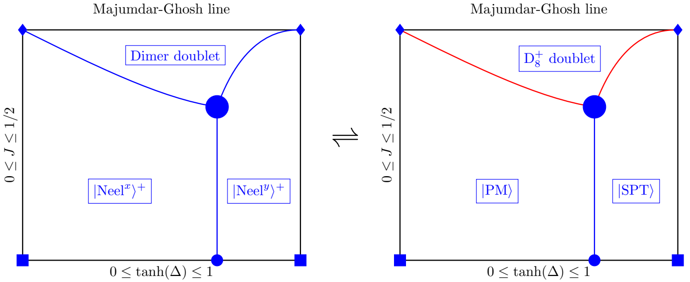
At $(\Delta, J) = (\infty, 0)$, the non-degenerate ground state is defined implicitly by

$$
\hat{\tau}^{x\vee}_{j^\star-1}\,\hat{\tau}^{z\vee}_{j^\star}\,\hat{\tau}^{x\vee}_{j^\star+1}\ |\mathrm{SPT}\rangle = +\,|\mathrm{SPT}\rangle\,, \qquad j^\star \in \Lambda^\star\,. \tag{51}
$$

Along the MG line $(\Delta, J) = (\Delta, 1/2)$, the four degenerate ground states are

$$|1\rangle = |\downarrow, \rightarrow, \downarrow, \leftarrow, \downarrow, \rightarrow, \downarrow, \leftarrow, \cdots\rangle\,, \tag{52a}$$
$$|2\rangle = |\downarrow, \leftarrow, \downarrow, \rightarrow, \downarrow, \leftarrow, \downarrow, \rightarrow, \cdots\rangle\,, \tag{52b}$$
$$|3\rangle = |\rightarrow, \downarrow, \leftarrow, \downarrow, \rightarrow, \downarrow, \leftarrow, \downarrow, \cdots\rangle\,, \tag{52c}$$
$$|4\rangle = |\leftarrow, \downarrow, \rightarrow, \downarrow, \leftarrow, \downarrow, \rightarrow, \downarrow, \cdots\rangle\,. \tag{52d}$$

# KW duality holds only after imposing the consistency conditions



## KW dual correlation functions

The correlation functions (48) KW dualize to

$$\widehat{C}^{x\,\vee}_{j,j+n} := \prod_{\ell=j^\star}^{j^\star+n-1} \hat{\tau}^{z\,\vee}_{\ell^\star}, \tag{53a}$$

$$\widehat{C}^{y\,\vee}_{j,j+n} := \hat{\tau}^{x\,\vee}_{j^\star-1}\,\hat{\tau}^{x\,\vee}_{j^\star}\left(\prod_{\ell=j^\star}^{j^\star+n-1} \hat{\tau}^{z\,\vee}_{\ell^\star}\right)\hat{\tau}^{x\,\vee}_{j^\star+n-1}\,\hat{\tau}^{x\,\vee}_{j^\star+n}, \tag{53b}$$

$$\widehat{D}^{\vee}_{j} := \frac{1}{3}\left(\hat{\tau}^{z\,\vee}_{j^\star} - \hat{\tau}^{x\,\vee}_{j^\star-1}\,\hat{\tau}^{z\,\vee}_{j^\star}\,\hat{\tau}^{x\,\vee}_{j^\star+1} + \hat{\tau}^{x\,\vee}_{j^\star-1}\,\hat{\tau}^{x\,\vee}_{j^\star+1}\right). \tag{53c}$$

We observe that operators $\widehat{C}^{x}_{j,j+n}$ and $\widehat{C}^{y}_{j,j+n}$ defined in Eqs. (48a) and (48b), respectively, dualize to non-local string operators, while the local operator $\widehat{D}_j$ defined in Eq. (58c) remains local after dualization.

The JW dual to Eq. (44) is

$$\begin{aligned}\widehat{H}^{\vee}_{f=1} := J_1 \sum_{j\in\Lambda} \Big(&\Delta_x\, \mathrm{i}\hat{\beta}^{\vee}_{j+1}\, \hat{\alpha}^{\vee}_{j} + \Delta_y\, \mathrm{i}\hat{\beta}^{\vee}_{j}\, \hat{\alpha}^{\vee}_{j+1} + \Delta_z\, \hat{\beta}^{\vee}_{j}\, \hat{\beta}^{\vee}_{j+1}\, \hat{\alpha}^{\vee}_{j}\, \hat{\alpha}^{\vee}_{j+1}\Big) \\ + J_2 \sum_{j=1}^{2N} \Big(&\Delta_x\, \hat{\beta}^{\vee}_{j+1}\, \hat{\beta}^{\vee}_{j+2}\, \hat{\alpha}^{\vee}_{j}\, \hat{\alpha}^{\vee}_{j+1} + \Delta_y\, \hat{\alpha}^{\vee}_{j+1}\, \hat{\alpha}^{\vee}_{j+2}\, \hat{\beta}^{\vee}_{j}\, \hat{\beta}^{\vee}_{j+1} \\ &+ \Delta_z\, \hat{\beta}^{\vee}_{j}\, \hat{\beta}^{\vee}_{j+2}\, \hat{\alpha}^{\vee}_{j}\, \hat{\alpha}^{\vee}_{j+2}\Big),\end{aligned} \tag{54}$$

with the domain of definition $\mathcal{H}^{\vee}_{f=1}$ defined in Eq. (25a).

On the reduced coupling space (45), the corresponding zero-temperature phase diagram has three gapped phases, each of which is controlled by the fixed points:

- At $(\Delta, J) = (0, 0)$, the non-degenerate ground state is defined by

$$\begin{aligned} \mathrm{i}\hat{\beta}^{\vee}_{1}\,\hat{\alpha}^{\vee}_{2N}\,|\mathrm{Kitaev}\rangle &= +\,|\mathrm{Kitaev}\rangle\,, \\ \mathrm{i}\hat{\beta}^{\vee}_{j+1}\,\hat{\alpha}^{\vee}_{j}\,|\mathrm{Kitaev}\rangle &= -\,|\mathrm{Kitaev}\rangle\,, \quad j = 1, \cdots, 2N-1. \end{aligned} \tag{55}$$

- At $(\Delta, J) = (\infty, 0)$, the non-degenerate ground state is defined by

$$\begin{aligned} \mathrm{i}\hat{\beta}^{\vee}_{2N}\,\hat{\alpha}^{\vee}_{1}\,|\overline{\mathrm{Kitaev}}\rangle &= +\,|\overline{\mathrm{Kitaev}}\rangle\,, \\ \mathrm{i}\hat{\beta}^{\vee}_{j}\,\hat{\alpha}^{\vee}_{j+1}\,|\overline{\mathrm{Kitaev}}\rangle &= -\,|\overline{\mathrm{Kitaev}}\rangle\,, \quad j = 1, \cdots, 2N-1. \end{aligned} \tag{56}$$

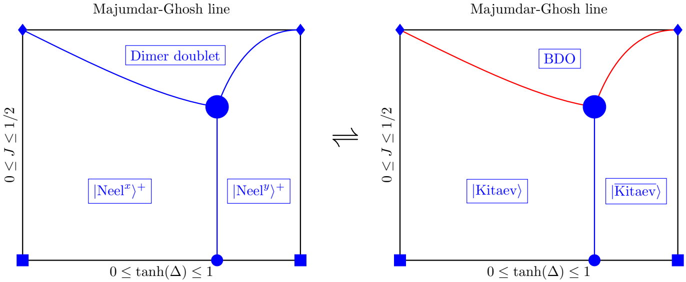
- At $(\Delta, J) = (1, 1/2)$, the two degenerate ground states are

$$|\mathrm{Bonding}^{\vee}_{\mathrm{o}}\rangle := \left[\prod_{j=1}^{N} \frac{1}{\sqrt{2}} \left(\hat{c}^{\vee\,\dagger}_{2j-1} + \hat{c}^{\vee\,\dagger}_{2j}\right)\right]|0\rangle, \qquad |\mathrm{Bonding}^{\vee}_{\mathrm{e}}\rangle := \left[\prod_{j=1}^{N} \frac{1}{\sqrt{2}} \left(\hat{c}^{\vee\,\dagger}_{2j} + \hat{c}^{\vee\,\dagger}_{2j+1}\right)\right]|0\rangle, \tag{57a}$$

  where the complex fermion operators are defined as

$$\hat{c}^{\vee\,\dagger}_{j} := \frac{1}{2}(\hat{\alpha}^{\vee}_{j} - \mathrm{i}\hat{\beta}^{\vee}_{j}), \qquad \hat{c}^{\vee}_{j} := \frac{1}{2}(\hat{\alpha}^{\vee}_{j} + \mathrm{i}\hat{\beta}^{\vee}_{j}). \tag{57b}$$

# JW duality holds only after imposing the consistency conditions



The correlation functions (48) JW dualize to

$$\widehat{C}^{x\,\vee}_{j,j+n} := \prod_{\ell=j}^{j+n-1} \mathrm{i}\hat{\beta}^{\vee}_{\ell+1}\,\hat{\alpha}^{\vee}_{\ell} \tag{58a}$$

$$\widehat{C}^{y\,\vee}_{j,j+n} := \prod_{\ell=j}^{j+n-1} \mathrm{i}\hat{\beta}^{\vee}_{\ell}\,\hat{\alpha}^{\vee}_{\ell+1}, \tag{58b}$$

$$\widehat{D}^{\vee}_{j} := \frac{1}{3}\left(\mathrm{i}\hat{\beta}^{\vee}_{j+1}\,\hat{\alpha}^{\vee}_{j} + \mathrm{i}\hat{\beta}^{\vee}_{j}\,\hat{\alpha}^{\vee}_{j+1} + \hat{\beta}^{\vee}_{j}\,\hat{\beta}^{\vee}_{j+1}\,\hat{\alpha}^{\vee}_{j}\,\hat{\alpha}^{\vee}_{j+1}\right). \tag{58c}$$

As was the case with the KW dualization, we observe that operators $\widehat{C}^{x}_{j,j+n}$ and $\widehat{C}^{y}_{j,j+n}$ defined in Eqs. (48a) and (48b), respectively, dualize to non-local string operators, while the local operator $\widehat{D}_{j}$ defined in Eq. (58c) remains local after dualization.

## KW dual to the quantum spin-1/2 $XYZ$ chain with OPBCs

The Kramers-Wannier dual of the Hamiltonian (1) when open boundary conditions are imposed in the reduced coupling space is

$$
\begin{aligned}
\widehat{H}^{\vee}_{\tau} = \sum_{j=1}^{2N-1} \hat{\tau}^{z\,\vee}_{j+\frac{1}{2}} - \Delta \Bigg( & \underline{\hat{\tau}^{z\,\vee}_{1+\frac{1}{2}}\,\hat{\tau}^{x\,\vee}_{2+\frac{1}{2}}} + \sum_{j=2}^{2N-2} \hat{\tau}^{x\,\vee}_{j-1+\frac{1}{2}}\,\hat{\tau}^{z\,\vee}_{j+\frac{1}{2}}\,\hat{\tau}^{x\,\vee}_{j+1+\frac{1}{2}} + \underline{\underline{\hat{\tau}^{x\,\vee}_{2N-2+\frac{1}{2}}\,\hat{\tau}^{z\,\vee}_{2N-1+\frac{1}{2}}}} \Bigg) \\
+ J \Bigg\{ & \sum_{j=1}^{2N-2} \hat{\tau}^{z\,\vee}_{j+\frac{1}{2}}\,\hat{\tau}^{z\,\vee}_{j+1+\frac{1}{2}} + \Delta \Bigg[ \underline{\left(\hat{\tau}^{z\,\vee}_{1+\frac{1}{2}}\,\hat{\tau}^{x\,\vee}_{2+\frac{1}{2}}\right)\left(\hat{\tau}^{x\,\vee}_{1+\frac{1}{2}}\,\hat{\tau}^{z\,\vee}_{2+\frac{1}{2}}\,\hat{\tau}^{x\,\vee}_{3+\frac{1}{2}}\right)} \\
& + \sum_{j=2}^{2N-3} \left(\hat{\tau}^{x\,\vee}_{j-\frac{1}{2}}\,\hat{\tau}^{z\,\vee}_{j+\frac{1}{2}}\,\hat{\tau}^{x\,\vee}_{j+\frac{3}{2}}\right)\left(\hat{\tau}^{x\,\vee}_{j+\frac{1}{2}}\,\hat{\tau}^{z\,\vee}_{j+\frac{3}{2}}\,\hat{\tau}^{x\,\vee}_{j+\frac{5}{2}}\right) \\
& + \underline{\underline{\left(\hat{\tau}^{x\,\vee}_{2N-3+\frac{1}{2}}\,\hat{\tau}^{z\,\vee}_{2N-2+\frac{1}{2}}\,\hat{\tau}^{x\,\vee}_{2N-1+\frac{1}{2}}\right)\left(\hat{\tau}^{x\,\vee}_{2N-2+\frac{1}{2}}\,\hat{\tau}^{z\,\vee}_{2N-1+\frac{1}{2}}\right)}} \Bigg] \Bigg\}.
\end{aligned}
\tag{59}
$$

Evidently, the dual of the Hamiltonian (1) when open boundary conditions are imposed is not the Hamiltonian (49) when open boundary conditions are imposed.

## JW dual to the quantum spin-1/2 *XYZ* chain with OPBCs

The Jordan-Wigner dual of the Hamiltonian (1) when open boundary conditions are imposed in the reduced coupling space is

$$\begin{aligned}\widehat{H}^{\vee}_{\beta\alpha} = & \sum_{j=1}^{2N-1} \left(\mathrm{i}\hat{\beta}^{\vee}_{j}\,\hat{\alpha}^{\vee}_{j+1} + \Delta\,\mathrm{i}\hat{\alpha}^{\vee}_{j}\,\hat{\beta}^{\vee}_{j+1}\right) \\ & + J\sum_{j=1}^{2N-2}\left(\mathrm{i}\hat{\beta}^{\vee}_{j}\,\hat{\beta}^{\vee}_{j+1}\,\hat{\alpha}^{\vee}_{j+1}\,\hat{\alpha}^{\vee}_{j+2} + \Delta\,\mathrm{i}\hat{\alpha}^{\vee}_{j}\,\hat{\alpha}^{\vee}_{j+1}\,\hat{\beta}^{\vee}_{j+1}\,\hat{\beta}^{\vee}_{j+2}\right).\end{aligned} \tag{60}$$

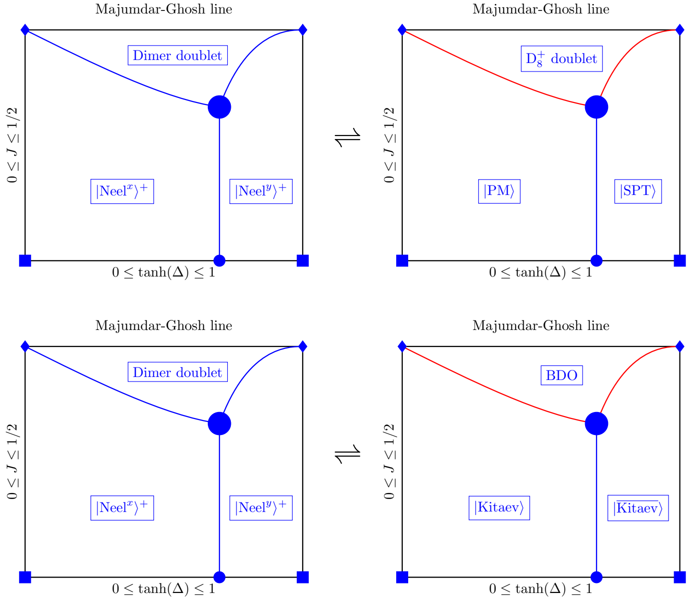
If we do the unitary transformation

$$\hat{\beta}^{\vee}_{j} \mapsto +\hat{\alpha}^{\vee}_{j}, \qquad \hat{\alpha}^{\vee}_{j} \mapsto -\hat{\beta}^{\vee}_{j}, \tag{61}$$

we recover Hamiltonian (54) in the reduced coupling space with open boundary conditions.

The two-fold degeneracy of the $\mathrm{Neel}_x$ or $\mathrm{Neel}_y$ phases is now interpreted by the existence of a single Majorana zero mode localized at the left and right ends of the open chain.

# Summary


Majumdar-Ghosh line
Dimer doublet
$|\mathrm{Neel}^x\rangle^+$
$|\mathrm{Neel}^y\rangle^+$
$0 \leq J \leq 1/2$
$0 \leq \tanh(\Delta) \leq 1$
Majumdar-Ghosh line
$\mathrm{D}_8^+$ doublet
$|\mathrm{PM}\rangle$
$|\mathrm{SPT}\rangle$
$0 \leq J \leq 1/2$
$0 \leq \tanh(\Delta) \leq 1$
Majumdar-Ghosh line
Dimer doublet
$|\mathrm{Neel}^x\rangle^+$
$|\mathrm{Neel}^y\rangle^+$
$0 \leq J \leq 1/2$
$0 \leq \tanh(\Delta) \leq 1$
Majumdar-Ghosh line
BDO
$|\mathrm{Kitaev}\rangle$
$|\overline{\mathrm{Kitaev}}\rangle$
$0 \leq J \leq 1/2$
$0 \leq \tanh(\Delta) \leq 1$