Critical properties of the interacting Majorana chain

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August 2023

Delft Technology Fellowship

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Majorana fermions

- Particles that are their own antiparticles
  - $\gamma_a^\dagger = \gamma_a$ and $\gamma_a^2 = 1$
- All known fermions of the Standard model are Dirac fermions
  - ... except, perhaps, neutrinos

picture credit: imgflip.com

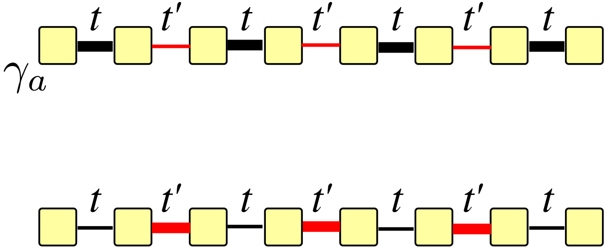
Majorana fermions

- Particles that are their own antiparticles
  - $\gamma_a^\dagger = \gamma_a$ and $\gamma_a^2 = 1$
- All known fermions of the Standard model are Dirac fermions
  - ... except, perhaps, neutrinos
- But in condensed matter
  - excitations can appeas a Majorana bound states
  - obey non-Abelian statistics Read and Green, PRB 2000; Kitaev, Uspekhi 2001; Stern, Ann.Phys. 2008.
- Quantum error corrections Kitaev '20
  - Majorana fermions are "intrinsically immune to decoherence"

A chain of non-interacting Majoranas

Majorana chain:

$$\mathcal{H} = \mathrm{i}t \sum_a \gamma_a \gamma_{a+1}$$



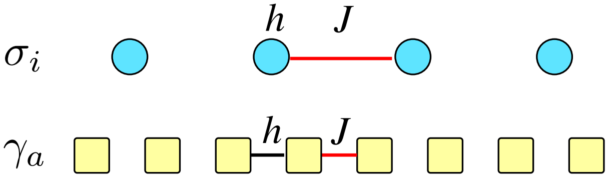
Kitaev '20

Majorana vs transverse-field Ising chains

Majorana chain:

$$\mathcal{H} = \mathrm{i}t \sum_a \gamma_a \gamma_{a+1}$$

$$\gamma_{2j} = K_j \sigma_j^x; \quad \gamma_{2j+1} = K_j \sigma_j^y; \quad \gamma_{2j}\gamma_{2j+1} = i\sigma_j^z; \quad K_j = \prod_{k=1}^{j-1} \sigma_k^z.$$



Transverse field Ising model:

$$\mathcal{H} = \sum_j J\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z$$

Scope

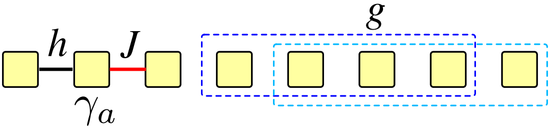
- Interacting Majorana chain
  - Ising
  - Floating
  - No generalized C-IC transition
- Interactions + Disorder
  - Resilient infinite randomness
- Competing interactions
  - $9\frac{1}{2}$ phases
  - 8-vertex criticality
  - Emergent particle-hole symmetry

Interacting Majorana chains

The model

Majorana chain:

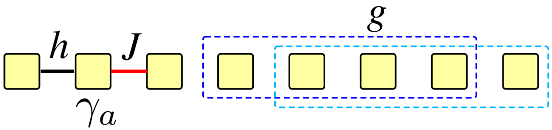
$$\mathcal{H} = \mathrm{i}t\sum_a \gamma_a\gamma_{a+1} - g\sum_a \gamma_a\gamma_{a+1}\gamma_{a+2}\gamma_{a+3}$$



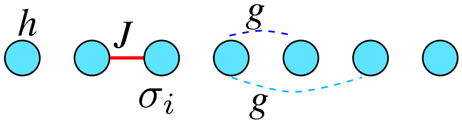
The model in terms of Pauli matrices

Majorana chain:

$$\mathcal{H} = \mathrm{i}t \sum_a \gamma_a \gamma_{a+1} - g \sum_a \gamma_a \gamma_{a+1} \gamma_{a+2} \gamma_{a+3}$$



$$\mathcal{H} = \sum_j \left[-J \sigma_j^x \sigma_{j+1}^x - h \sigma_j^z + g(\sigma_j^z \sigma_{j+1}^z + \sigma_j^x \sigma_{j+2}^x) \right]$$



Duality

Duality

$$H = \sum_i \sigma_i^x \sigma_{i+1}^x + h\sigma_i^z + g(\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$

Kramers–Wannier duality transformation:

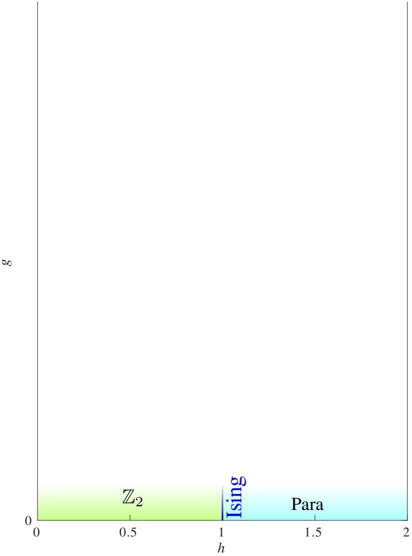
$$\sigma_i^x \sigma_{i+1}^x = \tilde{\sigma}_i^z; \qquad \sigma_i^z = \tilde{\sigma}_i^x \tilde{\sigma}_{i+1}^x$$

$$H = h\sum_i \tilde{\sigma}_i^x \tilde{\sigma}_{i+1}^x + \frac{1}{h}\tilde{\sigma}_i^z + \frac{g}{h}(\tilde{\sigma}_i^x \tilde{\sigma}_{i+2}^x + \tilde{\sigma}_i^z \tilde{\sigma}_{i+1}^z)$$

- Duality: $h \to h^{-1}$ and $g \to g/h$
- $h = 1$ is a self-dual line
- up to boundary terms

Previous studies

Starting point



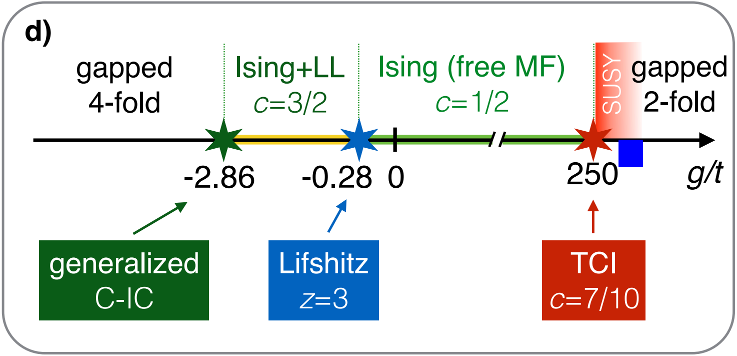
$$H = J \sum_i \sigma_i^x \sigma_{i+1}^x + h \sum_i \sigma_i^z$$

$$+g \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$

- Ising transition at $h = J$

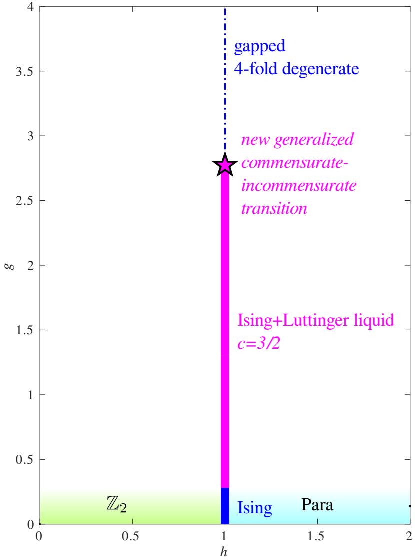
Previous studies

$$H = J\sum_i \sigma_i^x\sigma_{i+1}^x - h\sum_i \sigma_i^z - g\sum_i(\sigma_i^x\sigma_{i+2}^x + \sigma_i^z\sigma_{i+1}^z)$$



Rahmani, Zhu, Franz, Affleck, Phys. Rev. B 92, 235123 (2015);

Previous studies



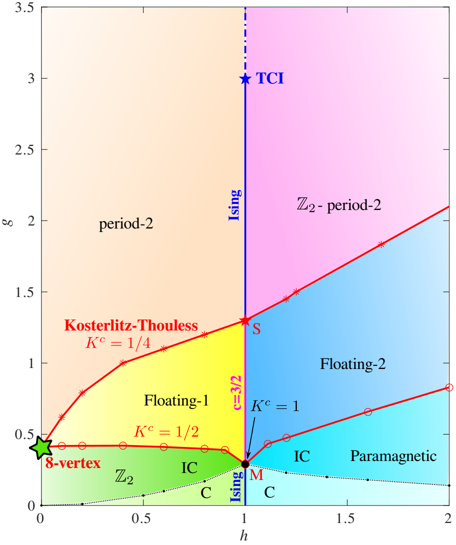
$$H = J\sum_i \sigma_i^x\sigma_{i+1}^x - h\sum_i \sigma_i^z + g\sum_i(\sigma_i^x\sigma_{i+2}^x + \sigma_i^z\sigma_{i+1}^z)$$

$h = J$

- $g < 0.28$ Ising phase
- $0.28 < g < 2.86$ Ising+LL
- $g > 2.86$ gapped, 4-fold degenerate

Rahmani, Zhu, Franz, Affleck, Phys. Rev. B 92, 235123 (2015)
see also Milsted, Seabra, Fulga, Beenakker, Cobanera Phys. Rev. B 92, 085139 (2015) with $g^c \approx 5$.

Majorana chain: an extended phase diagram

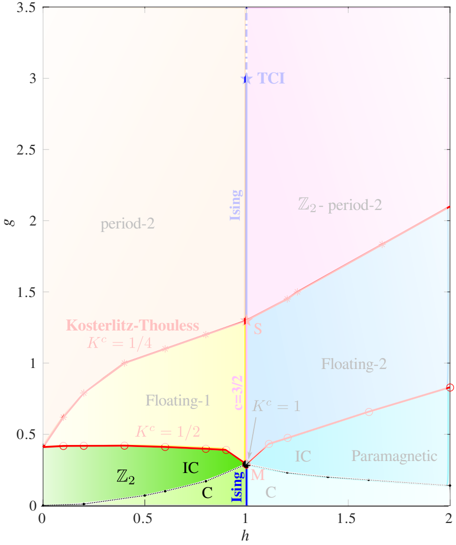


$$H = J\sum_i \sigma_i^x \sigma_{i+1}^x - h\sum_i \sigma_i^z + g\sum_i (\sigma_i^x\sigma_{i+2}^x + \sigma_i^z\sigma_{i+1}^z)$$

By looking away from the self-dual line $h = 1$ we can understand better the critical properties along it!

NC, Laflorencie, SciPost Phys. 14, 152 (2023)

$\mathbb{Z}_2$ phase

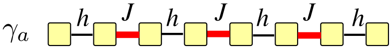


NC, Laflorencie, SciPost Phys. 14, 152 (2023)

$$H = \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z + g \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$

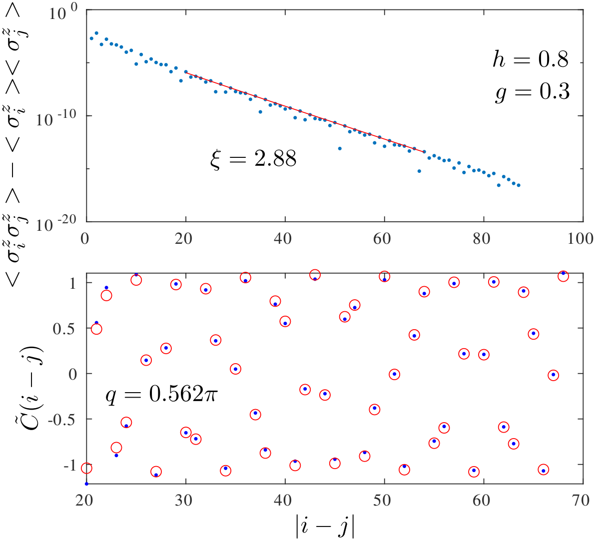
$\mathbb{Z}_2$ phase:

- Topologically non-trivial

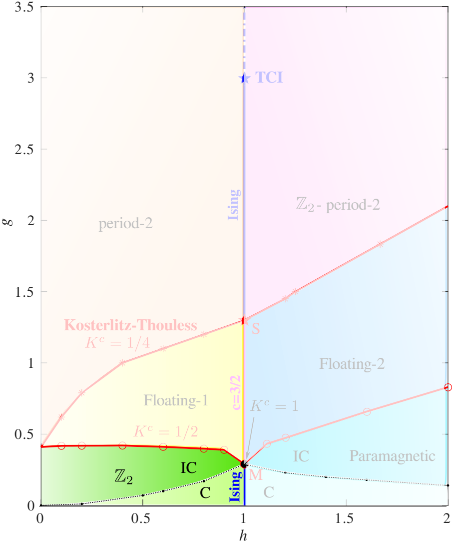


- Beyond **the disorder line** correlations are incommensurate

Incommensurate correlations



Exact zero modes

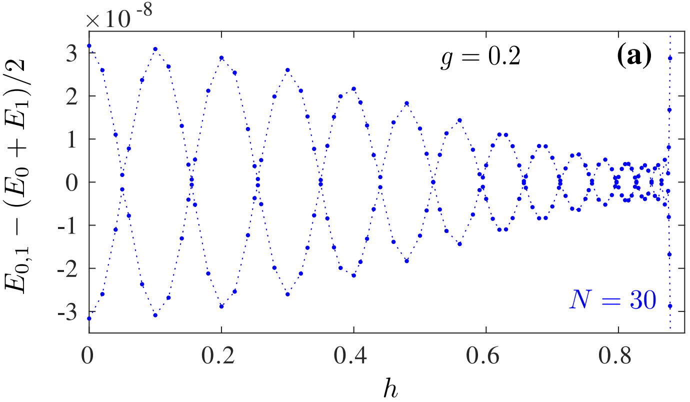


$$H = \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z + g \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$

- Beyond **the disorder line** correlations are incommensurate
- IC + egde states = Exact zero modes

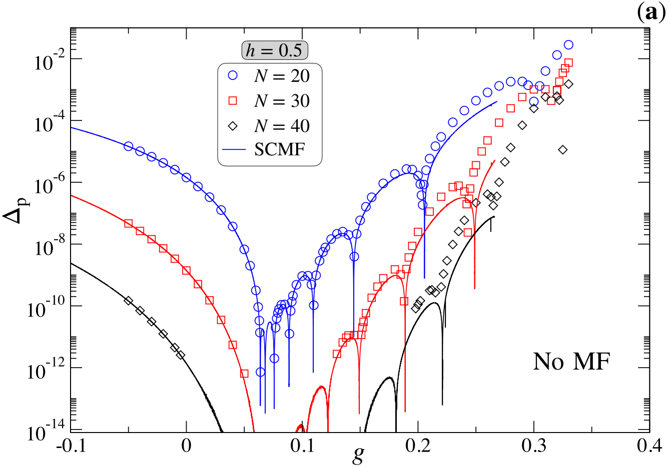
Toskovic et al. Nat. Phys. 12, 656 (2016);
Vionnet, Kumar, Mila, PRB 95, 174404 (2017);
NC, Mila, PRB 96, 060409 (2017);
NC, Mila, PRB 97, 174434 (2018)

Exact zero modes: $g = 0.2$

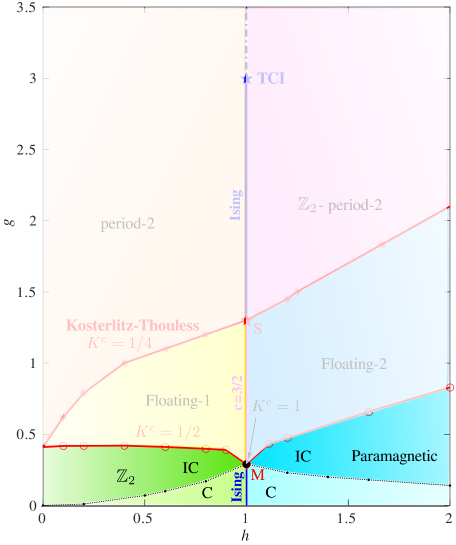


NC, Laflorencie, SciPost Phys. 14, 152 (2023)

Exact zero modes: $h = 0.5$



Paramagnetic phase



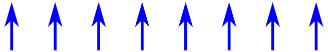
NC, Laflorencie, SciPost Phys. 14, 152 (2023)

$$H = \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z$$
$$+ g \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$

$\mathbb{Z}_2$ phase:

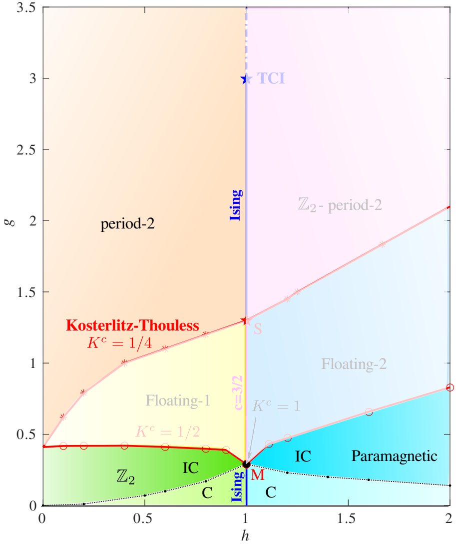
- Topologically non-trivial
- Beyond **the disorder line** correlations are incommensurate

Paramagnetic:



- Dual to $\mathbb{Z}_2$
- Incommensurate beyond the disorder line

Period-2 phase



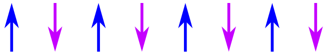
NC, Laflorencie, SciPost Phys. 14, 152 (2023)

$$H = \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z + g \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$

$\mathbb{Z}_2$ phase

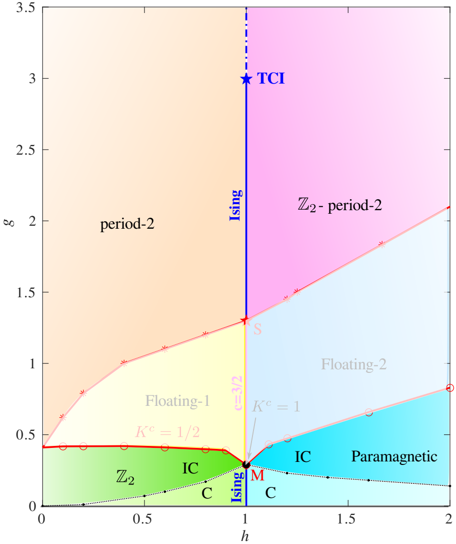
Paramagnetic

Period-2:



- Spontaneously broken translation symmetry

Period-2-$\mathbb{Z}_2$ phase



NC, Laflorencie, SciPost Phys. 14, 152 (2023)

$$H = \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z + g \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$

$\mathbb{Z}_2$ phase

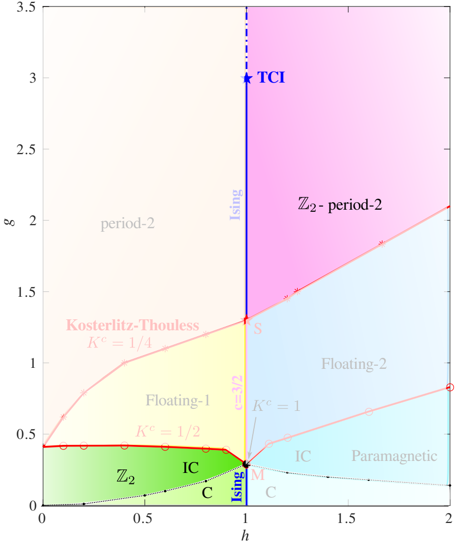
Paramagnetic

Period-2

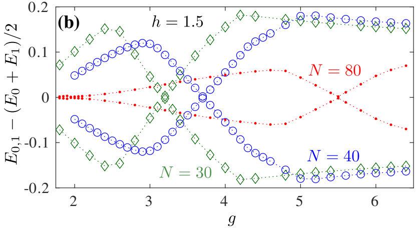
Period-2-$\mathbb{Z}_2$

- Spontaneously broken translation and parity symmetries
- Dual to the period-2 phase
- Topologically non-trivial

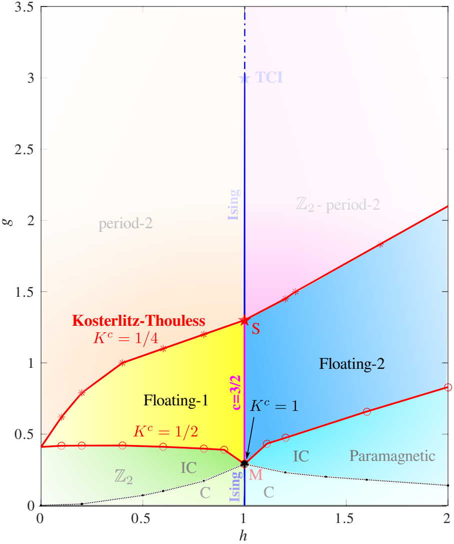
Exact zero modes in $\mathbb{Z}_2$-period-2 phase



$$H = \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z + g \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$



Floating phases

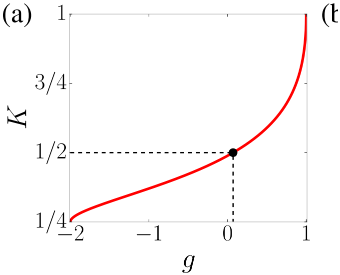


$$H = \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z + g \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$

- Floating phases
- Incommensurate Luttinger liquids
- Kosterlitz-Thouless transitions

Floating phases

Stability of the Luttinger liquid

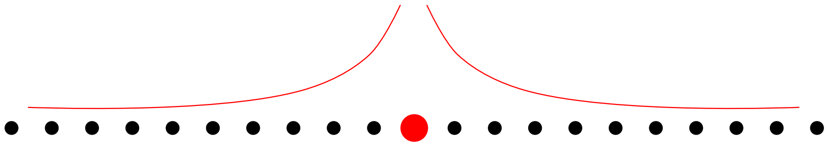


- Superconducting instability: $p^2/(4K)$ then $K^c = 1/2$
- Density wave: $K^c = (1 - \rho_0)^2 = 1/4$
- Stable Luttinger liquid for $1/4 < K < 1/2$
- Emergent U(1) symmetry

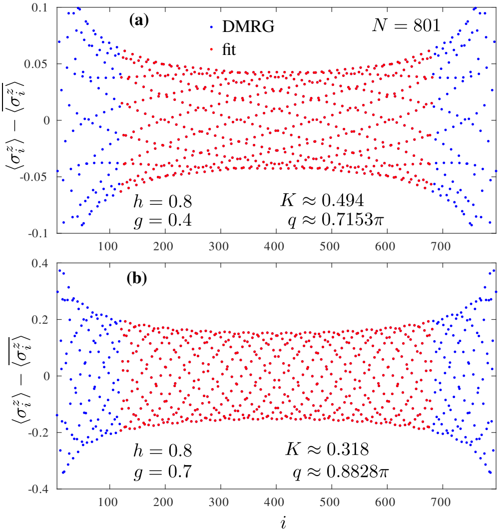
Verresen, Vishwanath, Pollmann, arXiv:1903.09179

Friedel oscillations

- Response of the system to an impurity
- In the gapped phase it decays exponentially
- At the critical point - with the corresponding critical exponent
- Open boundary conditions = impurity
- Prediction by boundary-CFT

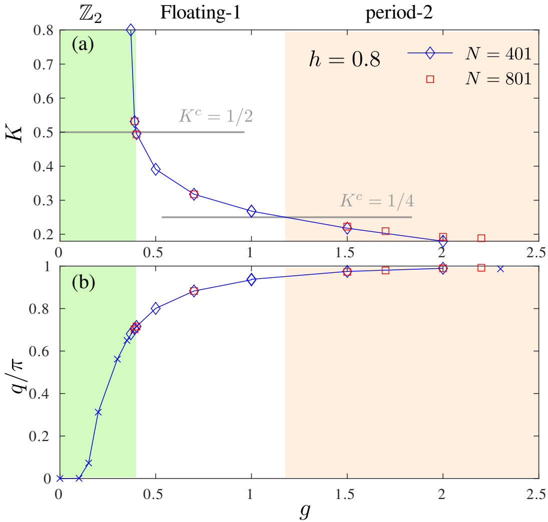


Friedel oscillations inside the floating phase

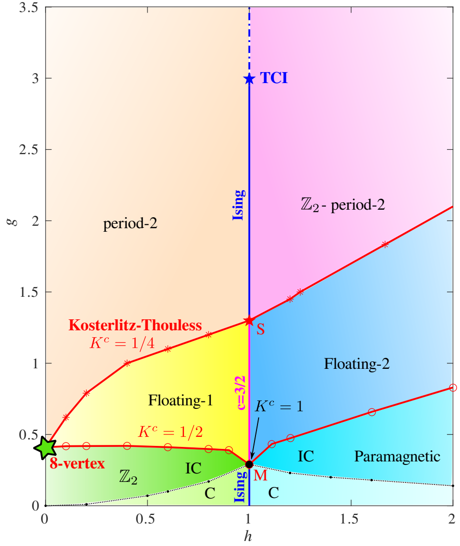


- Edges polarized along z!
- bCFT: $\sigma_i^z \propto \frac{\cos(qj)}{[(N/\pi)\sin(\pi j/N)]^K}$
- Scaling dimension = Luttinger liquid parameter K

Luttinger liquid exponent K & wave-vector q

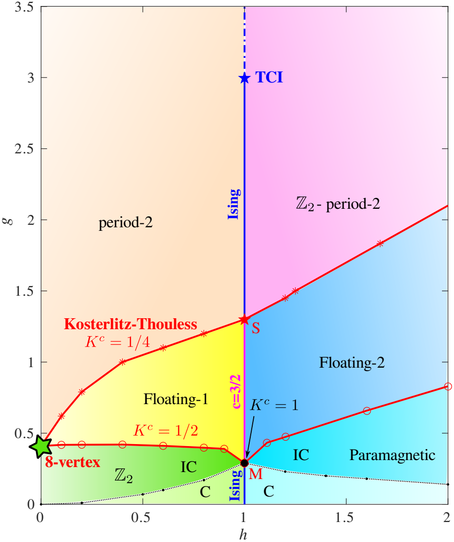


- Stable Luttinger liquid for $1/4 < K < 1/2$
- Incommensurate in both $\mathbb{Z}_2$ and period-2 phases
- IC-IC Kosterlitz-Thouless transition



Floating-1 vs Floating-2

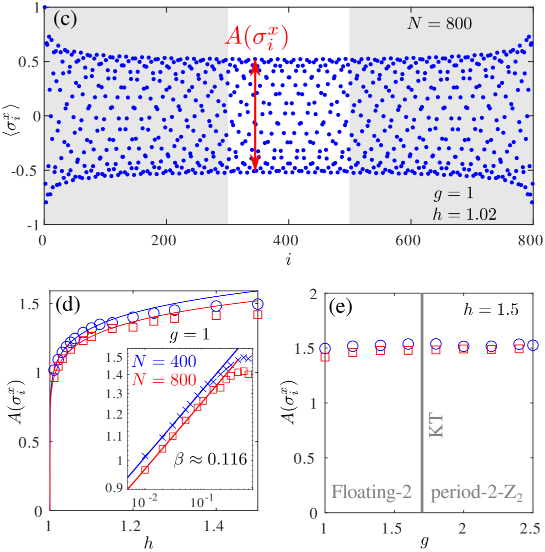
What is the difference?



Floating-1 vs Floating-2

What is the difference?
Broken $\mathbb{Z}_2$ symmetry

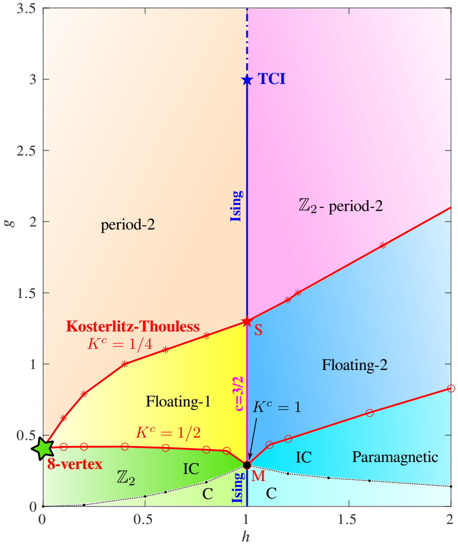
Floating-2 with broken $\mathbb{Z}_2$ symmetry



Order parameter

- Commensurate Ising: $|\sigma_i^x - \sigma_{i+1}^x|$
- Generalization to IC: $A(\sigma_i^x)$
- Decay towards $h = 1$ with $\beta \approx 0.116$
- Ising $\beta = 1/8$
- Insensitive to Kosterlitz-Thouless transition

Self-dual line



Lifshitz transition:

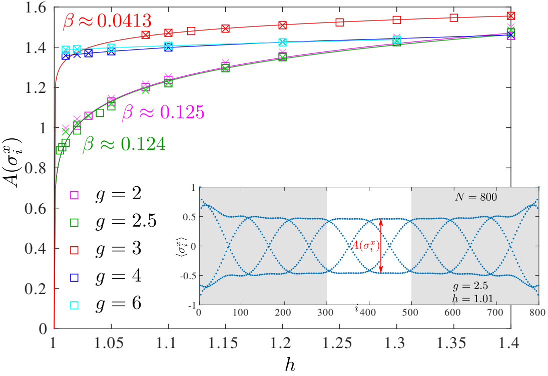
- Pairing has $K^c = 1/2$
- In paramagnet domain walls appear in pairs
- ... thus $K^c = 1/2$
- **BUT** none of the two are relevant along $h = 1$
- $K^c = 1$ as in the free-fermion theory

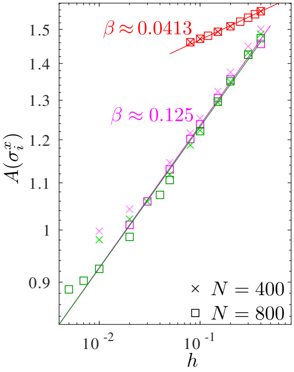
Point S - Kosterlitz-Thouless

- translation symmetry is broken everywhere: $K^c = 1/4$

Tri-critical Ising end point

Evidences of the tri-critical Ising end point

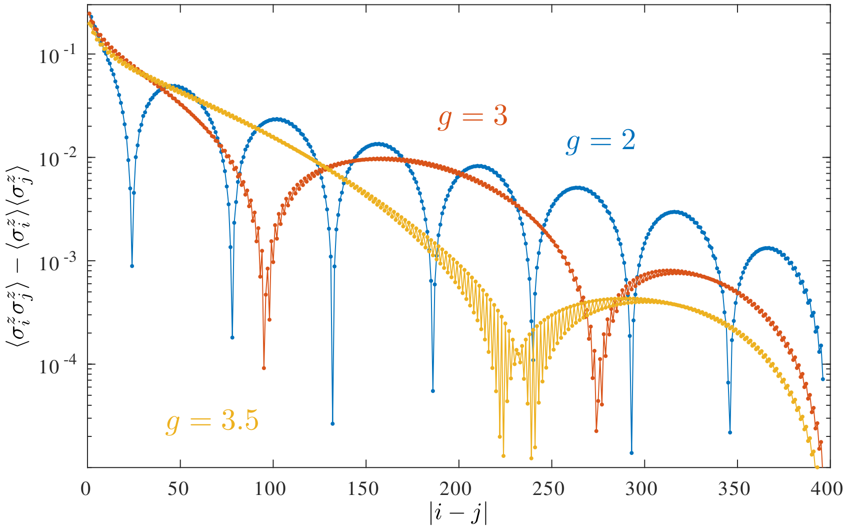




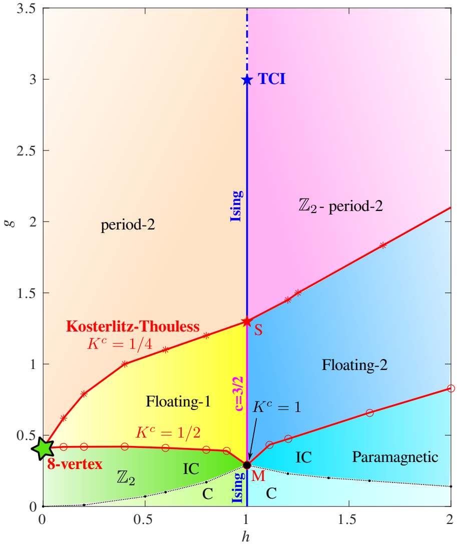
Ising: $\beta = 1/8 = 0.125$

Tri-critical Ising: $\beta = 1/24 \approx 0.0417$

Incommensurability persists beyond TCI



Self-dual line



- 1st order with 6x GS
- Tri-critical Ising (TCI) $g \approx 3$
- Ising for $1.3 \lesssim g \lesssim 3$
- Kosterlitz-Thouless (S) $g \approx 1.3$
  probably emergent SUSY
- Ising + LL $0.3 \lesssim < g \lesssim 1.3$
- Lifshitz transition (M) $g \approx 0.3$
- Ising for $g \lesssim 0.3$

Disorder

Resilient infinite randomness

Majorana chain in the presence of disorder

$$\mathcal{H} = \sum_j J_i \sigma_j^x \sigma_{j+1}^x - h_i \sigma_j^z + g(\sigma_j^z \sigma_{j+1}^z + \sigma_j^x \sigma_{j+2}^x); \qquad J_i, h_i \in [0.1, 1.9] \;\; \& \;\; \overline{\ln J} = \overline{\ln h}$$

Majorana chain in the presence of disorder

$$\mathcal{H} = \sum_j J_i \sigma_j^x \sigma_{j+1}^x - h_i \sigma_j^z + g(\sigma_j^z \sigma_{j+1}^z + \sigma_j^x \sigma_{j+2}^x); \qquad J_i, h_i \in [0.1, 1.9] \;\;\&\;\; \overline{\ln J} = \overline{\ln h}$$

Non-interacting case:

- Infinite randomness critical point

Fisher, PRL (1992); Fisher, PRB (1995); Igloi, Monthus, Phys. Rep. (2005)

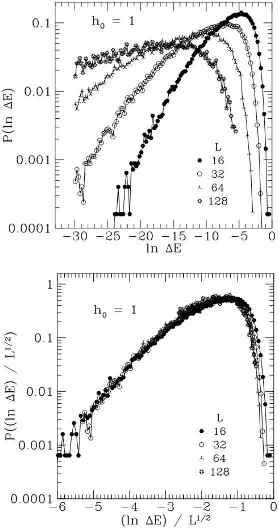
Majorana chain in the presence of disorder

$$\mathcal{H} = \sum_j J_i \sigma_j^x \sigma_{j+1}^x - h_i \sigma_j^z + g(\sigma_j^z \sigma_{j+1}^z + \sigma_j^x \sigma_{j+2}^x); \qquad J_i, h_i \in [0.1, 1.9] \ \& \ \overline{\ln J} = \overline{\ln h}$$

Non-interacting case:

- Infinite randomness critical point
- Dynamical critical exponent $z = \infty$
- Energy gap is broadly distributed: $\overline{\ln \Delta} \sim -\sqrt{N}$

Fisher, PRL (1992); Fisher, PRB (1995); Igloi, Monthus, Phys. Rep. (2005); Young, Rieger, PRB (1996)



Majorana chain in the presence of disorder

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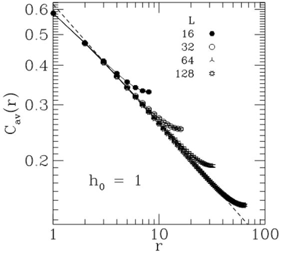
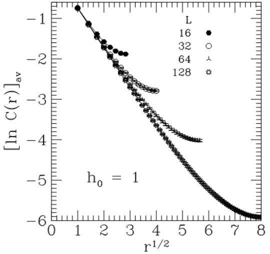


Non-interacting case:

- Infinite randomness critical point
- Dynamical critical exponent $z = \infty$
- Energy gap is broadly distributed: $\overline{\ln \Delta} \sim -\sqrt{N}$
- Lack of self-averaging:
  $\overline{\langle \sigma_\ell^x \sigma_{\ell+r}^x \rangle} \sim r^{\frac{-3+\sqrt{5}}{2}}$ and $\overline{\ln \langle \sigma_\ell^x \sigma_{\ell+r}^x \rangle} \sim -\sqrt{r}$



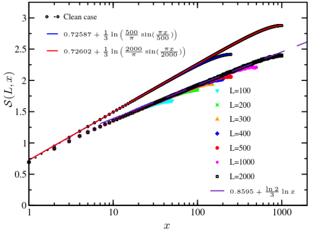
Fisher, PRL (1992); Fisher, PRB (1995); Igloi, Monthus, Phys. Rep. (2005); Young, Rieger, PRB (1996)

Majorana chain in the presence of disorder

$$\mathcal{H} = \sum_j J_i \sigma^x_j \sigma^x_{j+1} - h_i \sigma^z_j + g(\sigma^z_j \sigma^z_{j+1} + \sigma^x_j \sigma^x_{j+2}); \qquad J_i, h_i \in [0.1, 1.9] \;\; \& \;\; \overline{\ln J} = \overline{\ln h}$$

Non-interacting case:

- Infinite randomness critical point
- Dynamical critical exponent $z = \infty$
- Energy gap is broadly distributed: $\overline{\ln \Delta} \sim -\sqrt{N}$
- Lack of self-averaging:
  $\overline{\langle \sigma^x_\ell \sigma^x_{\ell+r} \rangle} \sim r^{\frac{-3+\sqrt{5}}{2}}$ and $\overline{\ln \langle \sigma^x_\ell \sigma^x_{\ell+r} \rangle} \sim -\sqrt{r}$
- Entanglement entropy grows logarithmically
  $\overline{S_q(n)} = \frac{c \ln 2}{6} \ln(n) + s_q$



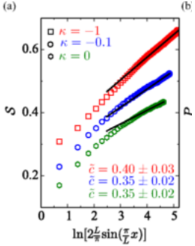
Fisher, PRB (1995); Young, Rieger, PRB (1996); Laflorencie, PRB (2005)

Majorana chain in the presence of disorder

$$\mathcal{H} = \sum_j J_i \sigma^x_j \sigma^x_{j+1} - h_i \sigma^z_j + g(\sigma^z_j \sigma^z_{j+1} + \sigma^x_j \sigma^x_{j+2}) \qquad J_i, h_i \in [0.1, 1.9] \ \& \ \overline{\ln J} = \overline{\ln h}$$

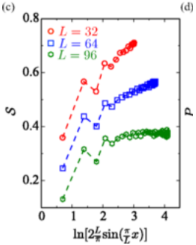
Non-interacting case - Infinite randomness fixed point:

- Energy gap is broadly distributed: $\overline{\ln \Delta} \sim -\sqrt{N}$
- Lack of self-averaging:
  $\overline{\langle \sigma^x_\ell \sigma^x_{\ell+r} \rangle} \sim r^{\frac{-3+\sqrt{5}}{2}}$ and $\overline{\ln \langle \sigma^x_\ell \sigma^x_{\ell+r} \rangle} \sim -\sqrt{r}$
- Entanglement entropy grows logarithmically
  $\overline{S_q(n)} = \frac{c \ln 2}{6} \ln(n) + s_q$



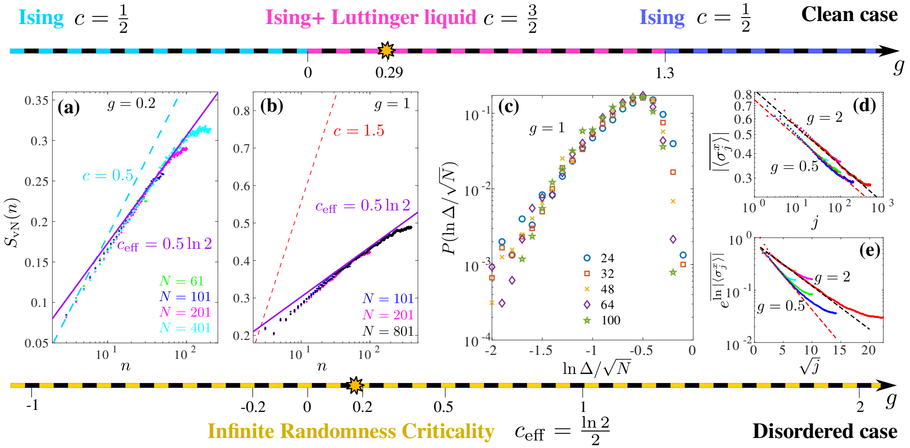
Interacting case - very controversial

- Fisher (1995): interactions are irrelevant
- Milsted et al (2005): Infinite randomness for $g < 0$
- Karcher et al (2019): Clean physics for $g < 0$
- both: Saturation of Ent. entropy for $g > 0$



Majorana chain in the presence of disorder

$$\mathcal{H} = \sum_j J_i \sigma_j^x \sigma_{j+1}^x - h_i \sigma_j^z + g(\sigma_j^z \sigma_{j+1}^z + \sigma_j^x \sigma_{j+2}^x); \qquad J_i, h_i \in [0.1, 1.9] \;\; \& \;\; \overline{\ln J} = \overline{\ln h}$$

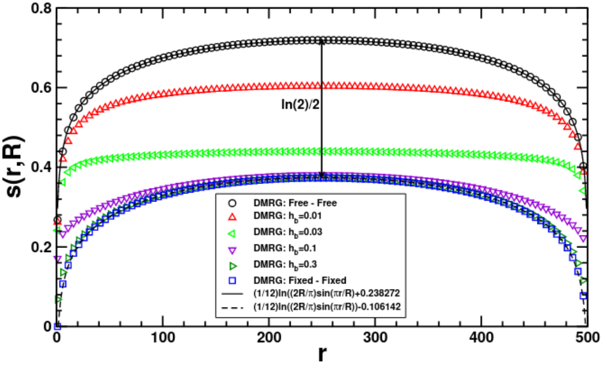


Resilient infinite randomness criticality!

Trick #1:

Fixed boundary conditions

Fixed BC = Lower entanglement

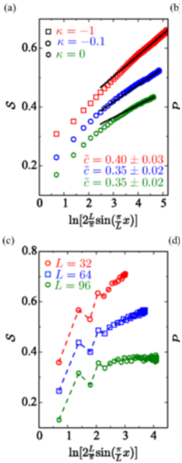


Affleck, Laflorencie, Sorensen, J. Phys. A: Math. Theor. 42 504009 (2009)

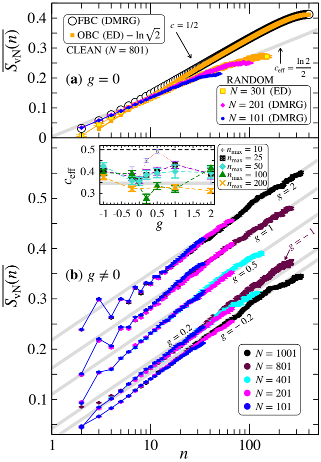
Lower entanglement = lower computational costs

Majorana chain in the presence of disorder

$$\mathcal{H} = \sum_j J_i \sigma^x_j \sigma^x_{j+1} - h_i \sigma^z_j + g(\sigma^z_j \sigma^z_{j+1} + \sigma^x_j \sigma^x_{j+2}) \qquad J_i, h_i \in [0.1, 1.9] \;\; \& \;\; \overline{\ln J} = \overline{\ln h}$$



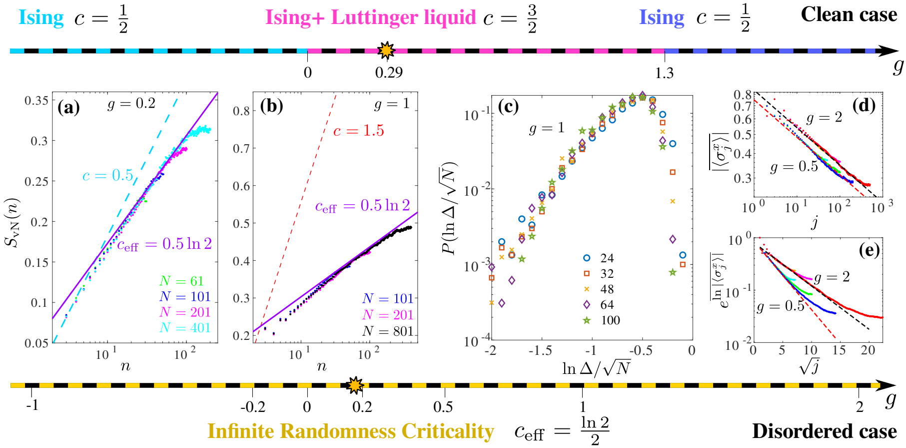
Milsted et al (2015): OBC



NC, Laflorencie (2023): FBC

Majorana chain in the presence of disorder

$$\mathcal{H} = \sum_j {\color{red}J_i}\sigma^x_j\sigma^x_{j+1} - {\color{red}h_i}\sigma^z_j + g(\sigma^z_j\sigma^z_{j+1} + \sigma^x_j\sigma^x_{j+2}); \qquad J_i, h_i \in [0.1, 1.9] \;\; \& \;\; \overline{\ln J} = \overline{\ln h}$$

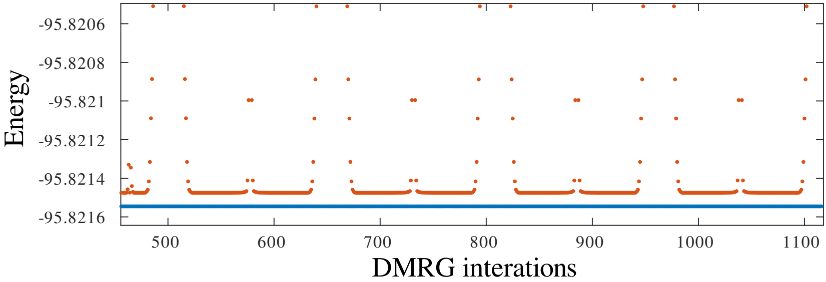


Resilient infinite randomness criticality!

Trick #2:

Target multiple states in DMRG

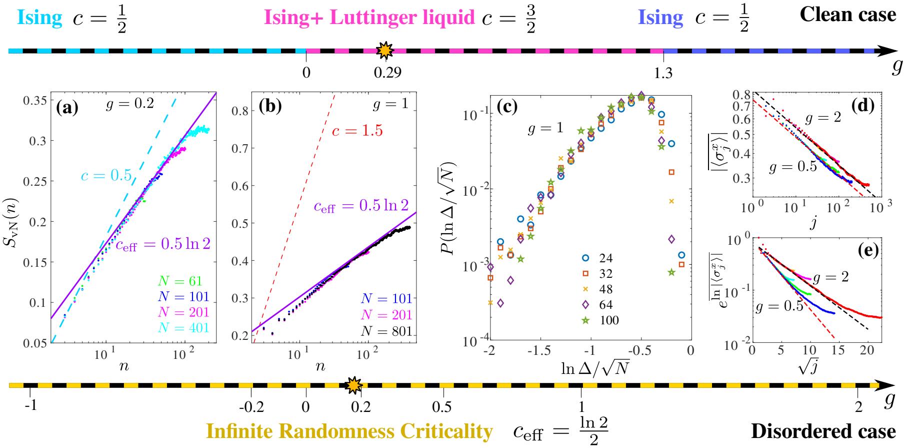
Flat intervals = trustworthy excitations



NC, Mila (2017); NC, Laflorencie (2023)

Majorana chain in the presence of disorder

$$\mathcal{H} = \sum_j J_i \sigma_j^x \sigma_{j+1}^x - h_i \sigma_j^z + g(\sigma_j^z \sigma_{j+1}^z + \sigma_j^x \sigma_{j+2}^x); \qquad J_i, h_i \in [0.1, 1.9] \;\; \& \;\; \overline{\ln J} = \overline{\ln h}$$



Resilient infinite randomness criticality!

Trick #3:

Distant correlations

$$\overline{\langle \sigma_\ell^x \sigma_{\ell+r}^x \rangle} \sim r^{-\eta} \quad \text{and} \quad \overline{\ln \langle \sigma_\ell^x \sigma_{\ell+r}^x \rangle} \sim -\sqrt{r}$$

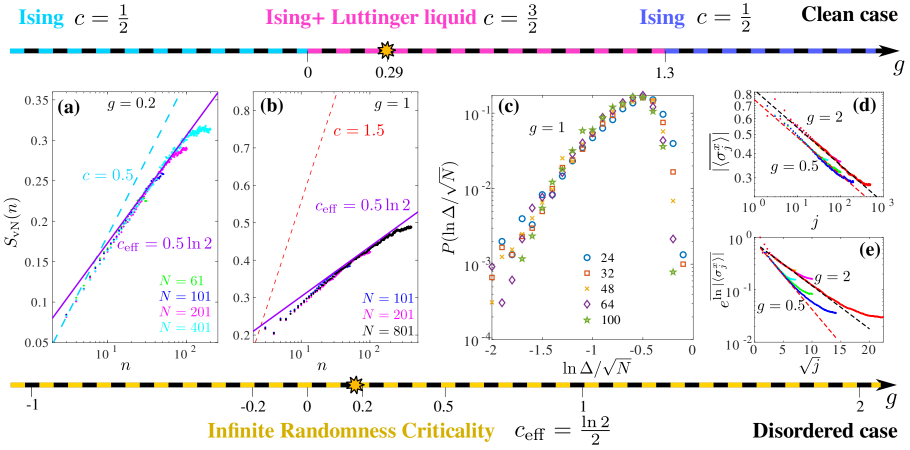
$$\Downarrow$$

Friedel oscillations

$$\overline{|\langle \sigma_j^x \rangle|} \sim j^{-\eta/2} \quad \text{and} \quad \overline{\ln |\langle \sigma_j^x \rangle|} \sim -\sqrt{j}$$

Majorana chain in the presence of disorder

$$\mathcal{H} = \sum_j J_i \sigma_j^x \sigma_{j+1}^x - h_i \sigma_j^z + g(\sigma_j^z \sigma_{j+1}^z + \sigma_j^x \sigma_{j+2}^x); \qquad J_i, h_i \in [0.1, 1.9] \;\; \& \;\; \overline{\ln J} = \overline{\ln h}$$

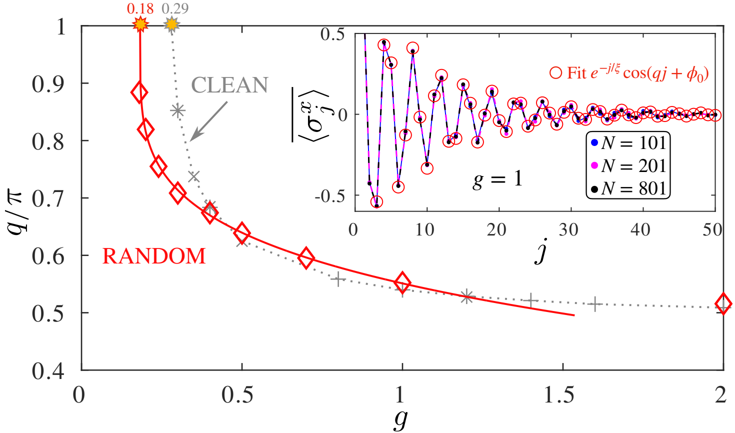


Resilient infinite randomness criticality!

Surprise

Incommensurability in the disordered chain

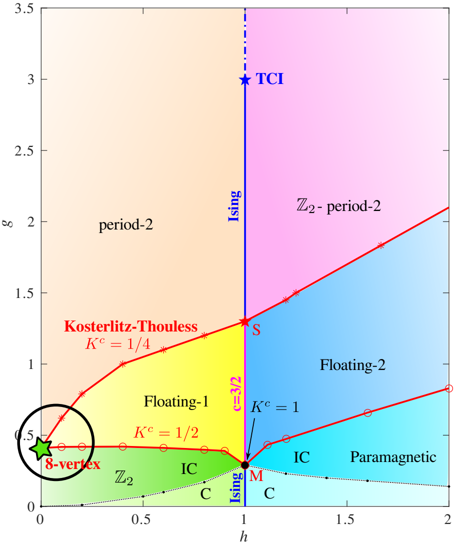
$$\mathcal{H} = \sum_j J_i \sigma^x_j \sigma^x_{j+1} - h_i \sigma^z_j + g(\sigma^z_j \sigma^z_{j+1} + \sigma^x_j \sigma^x_{j+2}); \qquad J_i, h_i \in [0.1, 1.9] \;\; \& \;\; \overline{\ln J} = \overline{\ln h}$$



Incommensurate correlations beyond $g \approx 0.18$!

Back to the clean case

Multi-critical point



$$H = \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z + g \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$

- Floating phase collapses
- ... along a particle-hole symmetry line $h = 0$
- Direct transition between $\mathbb{Z}_2$ and period-2 phases
- **8-vertex critical point?**

Multicritical point in the

8-vertex universality class

Universality classes =
Family of quantum phase transitions characterized by the same *universal* rules of the asymptotic scaling

In simplest cases this simply means
the universal critical exponents

Ising, 3-state Potts, Wess-Zumino-Witten, Kosterlitz-Thouless, Pokrovsky-Talapov...

... But sometimes it only means
the universal ratio/function of the critical exponents

Ashkin-Teller, 8-vertex, chiral transitions, ...

Eight-vertex criticality. Integrable model

$$H = \sum_i J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z - B\sigma_i^z,$$

- Critical, in 8-vertex universality class at $J_x = -J_z$ and $B = 0$
- Control parameter:

$$\rho = \mathrm{acos}[J_y/J_x].$$

- Critical exponents

$$\nu = \pi/(2\rho), \quad \beta = (\pi - \rho)/(4\rho),$$

- Scaling dimenision

$$d = \beta/\nu = \pi - \rho/(2\pi)$$

Baxter, Annals of Physics 70, 193 (1972)

Eight-vertex criticality. Integrable model

$$H = \sum_i J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z - B\sigma_i^z$$

$$H_{\mathrm{NN}} = \sum_i -t(d_i^\dagger d_{i+1} + \mathrm{h.c.}) + \lambda(d_i^\dagger d_{i+1}^\dagger + \mathrm{h.c.}) - \mu n_i + V n_i n_{i+1}$$

- Critical, in 8-vertex universality class at $\lambda = (V - 2t)/2$ and $V = \mu$
- Control parameter:

$$\rho = \mathrm{acos}[(1-\lambda)/(1+\lambda)].$$

- Critical exponents

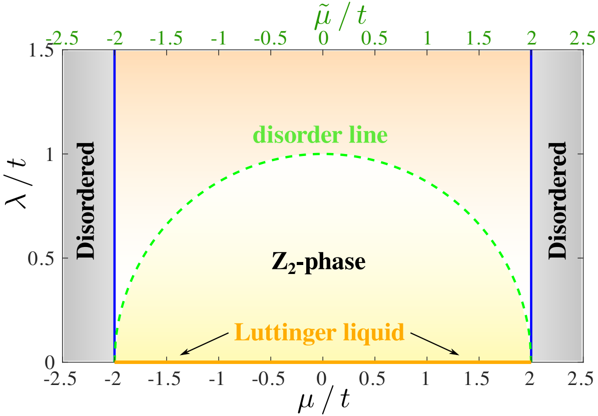
$$\nu = \pi/(2\rho), \quad \beta = (\pi - \rho)/(4\rho),$$

- Scaling dimenision

$$d = \beta/\nu = \pi - \rho/(2\pi)$$

Non-interacting Kitaev chain

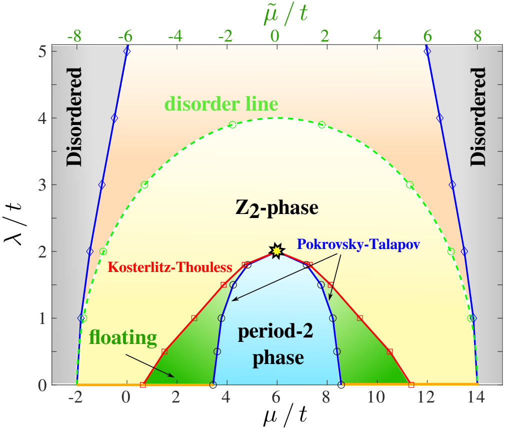
$$H_{\rm NN} = \sum_i -t(d_i^\dagger d_{i+1} + \text{h.c.}) + \lambda(d_i^\dagger d_{i+1}^\dagger + \text{h.c.}) - \mu n_i + V n_i n_{i+1}$$



NC, Mila, PRB (2023)

Phase diagram. $V > 2$

$$H_{\text{NN}} = \sum_i -t(d_i^\dagger d_{i+1} + \text{h.c.}) + \lambda(d_i^\dagger d_{i+1}^\dagger + \text{h.c.}) - \mu n_i + V n_i n_{i+1}$$



NC, Mila, PRB (2023)

Eight-vertex criticality. Integrable model

$$H = \sum_i J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z - B\sigma_i^z$$

$$H_{\mathrm{NN}} = \sum_i -t(d_i^\dagger d_{i+1} + \mathrm{h.c.}) + \lambda(d_i^\dagger d_{i+1}^\dagger + \mathrm{h.c.}) - \mu n_i + V n_i n_{i+1}$$

- Critical, in 8-vertex universality class at $\lambda = (V - 2t)/2$ and $V = \mu$
- Control parameter:

$$\rho = \mathrm{acos}[(1 - \lambda)/(1 + \lambda)].$$

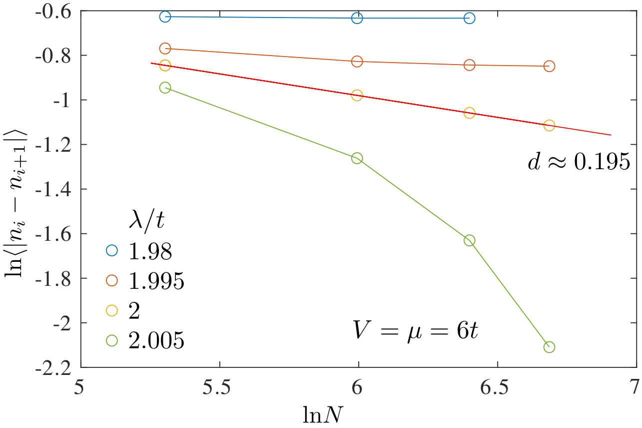
- Critical exponents

$$\nu = \pi/(2\rho), \quad \beta = (\pi - \rho)/(4\rho),$$

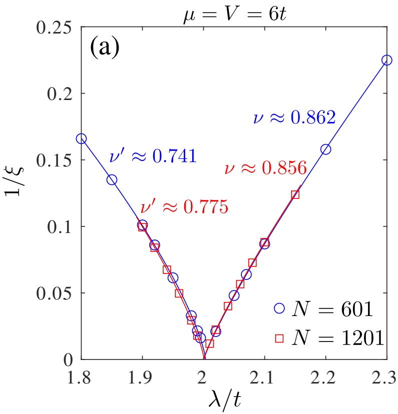
- Scaling dimenision

$$d = \beta/\nu = \pi - \rho/(2\pi)$$

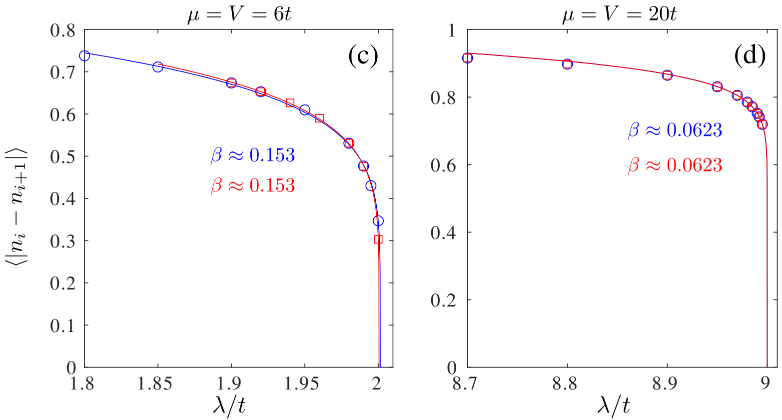
Scaling dimension



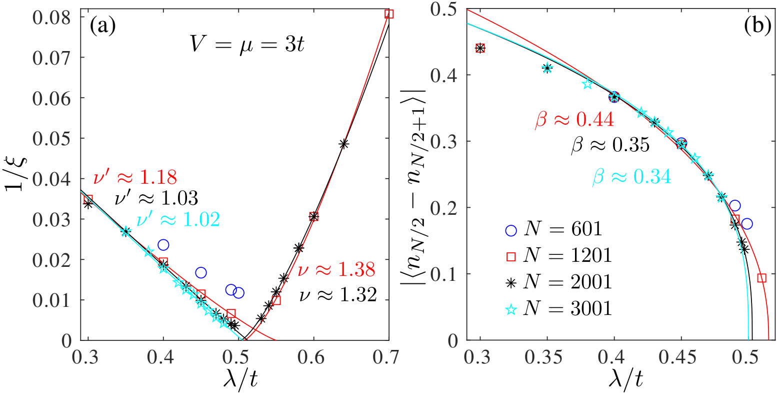
Critical exponent ν



Critical exponent β

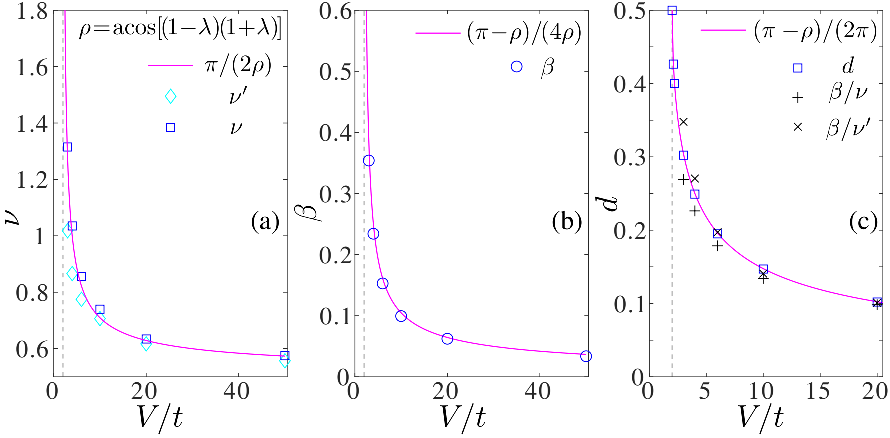


… sometimes tricky

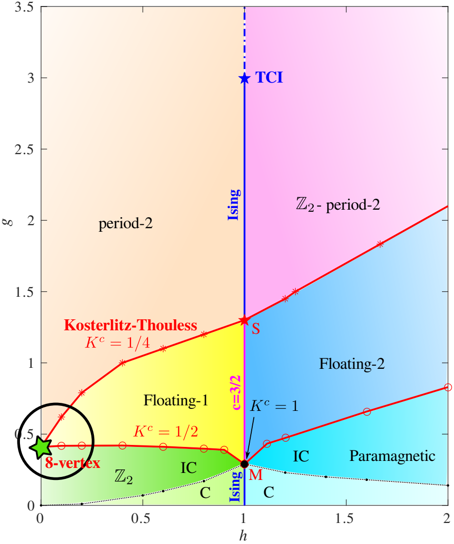


8-vertex critical point:

Numerical results vs theory predictions



Back to Majorana



$$H = \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z + g \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$

- Floating phase collapses
- ... along a particle-hole symmetry line $h = 0$
- Direct transition between $\mathbb{Z}_2$ and period-2 phases
- **8-vertex critical point?**

Direct transition between $\mathbb{Z}_2$ and period-2 phases; Collapse of the floating phase

Eight-vertex criticality. Majorana chain

$$H = \sum_i \sigma_i^x \sigma_{i+1}^x \cancel{+ J_y \sigma_i^y \sigma_{i+1}^y} + g\sigma_i^x \sigma_{i+2}^x + g\sigma_i^z \sigma_{i+1}^z - h\sigma_i^z$$

- Particle-hole symmetry at $h = 0$
- 8-vertex universality class?
- Control parameter: no prediction!

$$\cancel{\rho = \mathrm{acos}[J_y/J_x]}$$

- Critical exponents: still valid

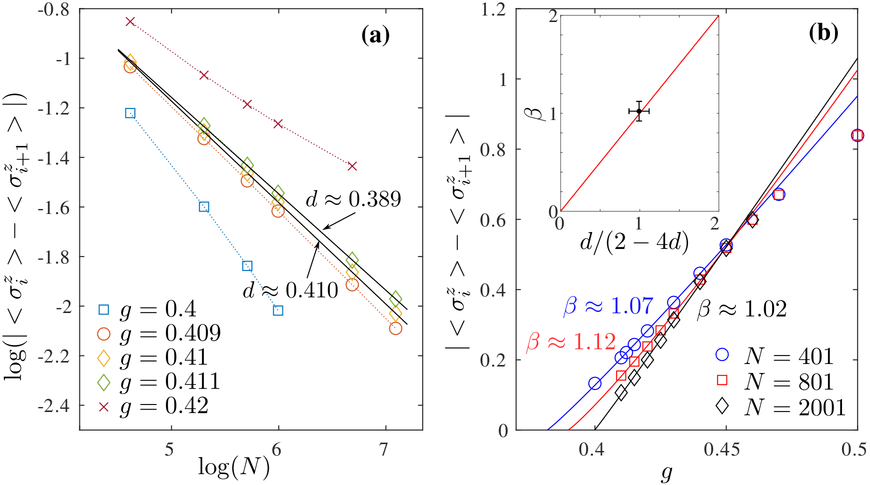
$$\nu = \pi/(2\rho), \quad \beta = (\pi - \rho)/(4\rho),$$

$$d = \pi - \rho/(2\pi)$$

- Express one exponent in terms of another:

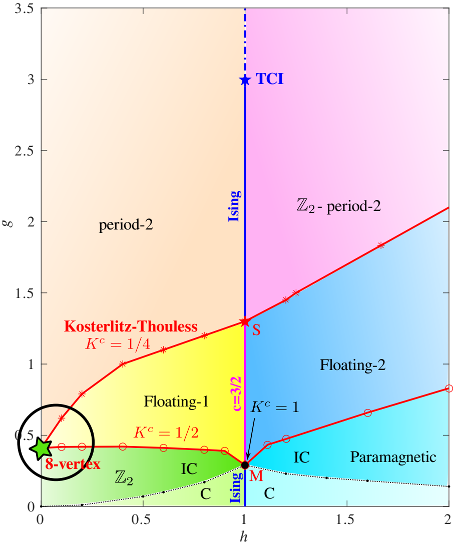
$$\beta = d/(2 - 4d)$$

Eight-vertex criticality. Majorana chain



NC, Laflorencie, SciPost Phys. 14, 152 (2023)

Dual point



$$H = \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z + g \sum_i (\sigma_i^x \sigma_{i+2}^x + \sigma_i^z \sigma_{i+1}^z)$$

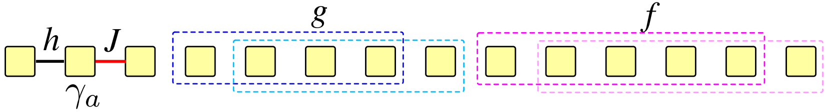
- Floating phase collapses
- ... along a particle-hole symmetry line $h = 0$
- Direct transition between $\mathbb{Z}_2$ and period-2 phases
- 8-vertex critical point
- There is a dual point at $h = \infty$

Majorana chain
with competing interactions

The model

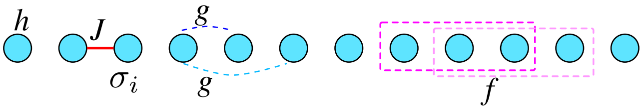
Majorana chain:

$$\mathcal{H} = \mathrm{i}t \sum_a \gamma_a \gamma_{a+1} - g \sum_a \gamma_a \gamma_{a+1} \gamma_{a+2} \gamma_{a+3} - f \sum_a \gamma_a \gamma_{a+1} \gamma_{a+3} \gamma_{a+4},$$



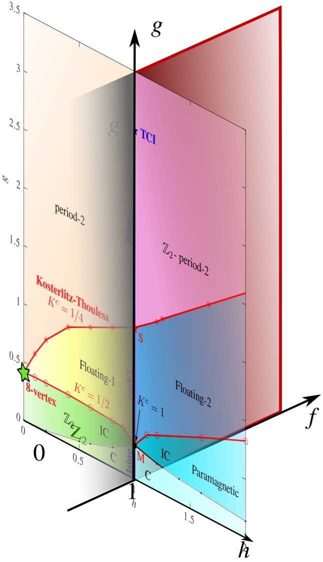
$$\mathcal{H} = \sum_j \left[-J \sigma_j^x \sigma_{j+1}^x - h \sigma_j^z + g(\sigma_j^z \sigma_{j+1}^z + \sigma_j^x \sigma_{j+2}^x) \right.$$

$$\left. + f(\sigma_j^z \sigma_{j+1}^x \sigma_{j+2}^x + \sigma_j^x \sigma_{j+1}^x \sigma_{j+2}^z) \right].$$



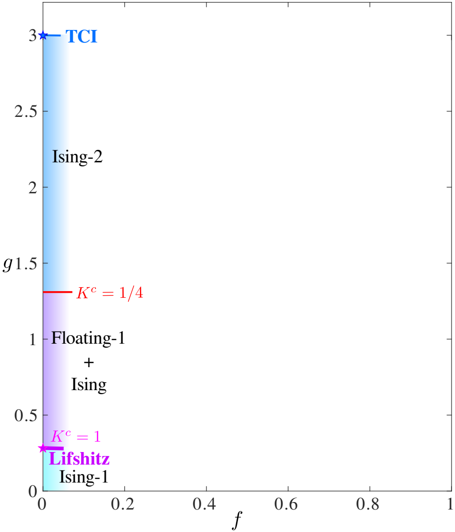
Self-dual at $h/J = 1$!

An extended phase diagram.



Focus on $g - f$ plane at $h = 1$

An extended phase diagram. The starting point



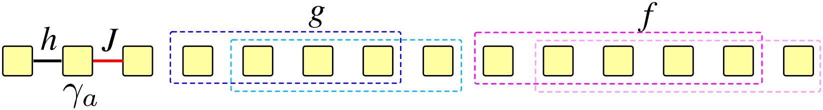
$f = 0$:

- $g \approx 0.29$ Lifshitz point
- $g \approx 1.3$ Kosterlitz-Thouless
- $g \approx 3$ Tri-critical Ising

Previous studies: f-interaction

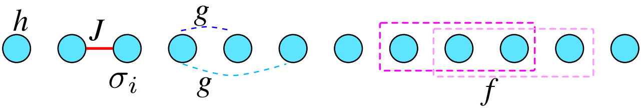
Majorana chain:

$$\mathcal{H} = \mathrm{i}t\sum_a \gamma_a\gamma_{a+1} - g\sum_a \gamma_a\gamma_{a+1}\gamma_{a+2}\gamma_{a+3} - f\sum_a \gamma_a\gamma_{a+1}\gamma_{a+3}\gamma_{a+4}$$



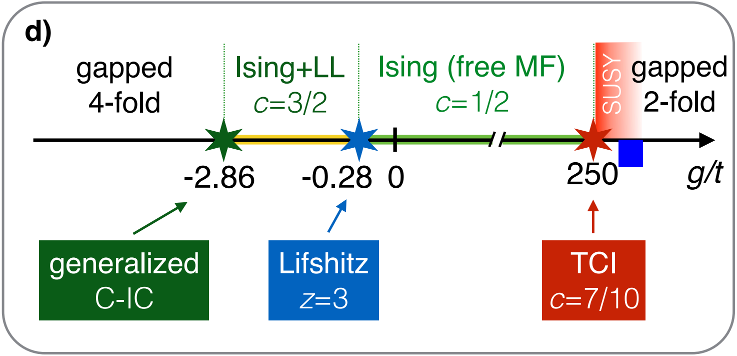
$$\mathcal{H} = \sum_j \left[-J\sigma_j^x\sigma_{j+1}^x - h\sigma_j^z + g(\sigma_j^z\sigma_{j+1}^z + \sigma_j^x\sigma_{j+2}^x)\right.$$

$$\left. + f(\sigma_j^z\sigma_{j+1}^x\sigma_{j+2}^x + \sigma_j^x\sigma_{j+1}^x\sigma_{j+2}^z)\right]$$



Previous studies: g-interaction

$$H = J\sum_i \sigma_i^x\sigma_{i+1}^x - h\sum_i \sigma_i^z - g\sum_i(\sigma_i^x\sigma_{i+2}^x + \sigma_i^z\sigma_{i+1}^z)$$



Rahmani, Zhu, Franz, Affleck, Phys. Rev. B 92, 235123 (2015);

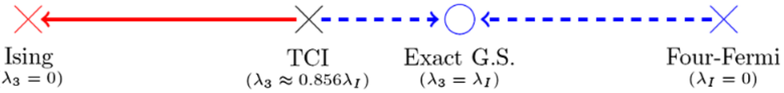
The model: f-interaction

PHYSICAL REVIEW LETTERS **120,** 206403 (2018)

Editors' Suggestion

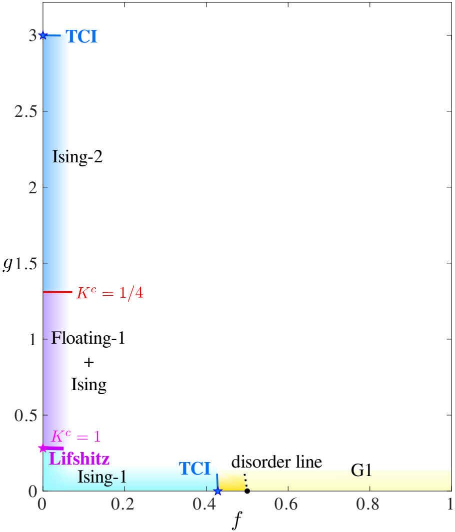
Lattice Supersymmetry and Order-Disorder Coexistence in the Tricritical Ising Model

Edward O'Brien[1] and Paul Fendley[1,2]
[1] *Rudolf Peierls Centre for Theoretical Physics, 1 Keble Road, Oxford OX1 3NP, United Kingdom*
[2] *All Souls College, Oxford OX1 4AL, United Kingdom*



$$\mathcal{H} = \sum_j -2\lambda_I \sigma_j^x \sigma_{j+1}^x - 2\lambda_I \sigma_j^z + \lambda_3(\sigma_j^z \sigma_{j+1}^x \sigma_{j+2}^x + \sigma_j^x \sigma_{j+1}^x \sigma_{j+2}^z)$$

Previous studies:



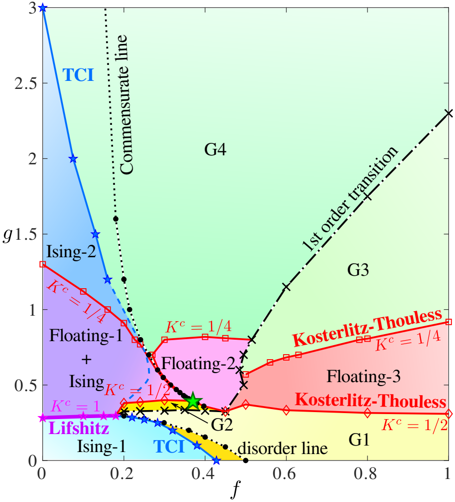
$f = 0$:

- $g \approx 0.29$ Lifshitz point
- $g \approx 1.3$ Kosterlitz-Thouless
- $g \approx 3$ Tri-critical Ising

$g = 0$:

- $f \approx 0.428$ Tri-critical Ising
- $f = 0.5$ Frustration-free and disorder point with 3-fold degenerate GS

Phase diagram



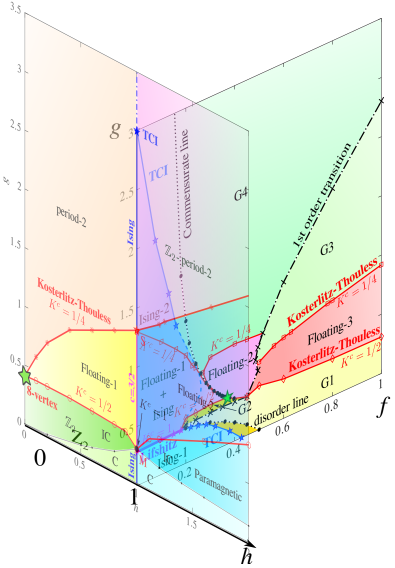
Gapped phases:

- 3-fold degenerate
  - G1
  - G2
- 6-fold degenerate
  - G3
  - G4

Critical phases

- Floating-2
- Floating-3
- Ising+Floating-1
- Ising-1
- Ising-2

Phase diagram



Gapped phases:

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  - G4

Critical phases

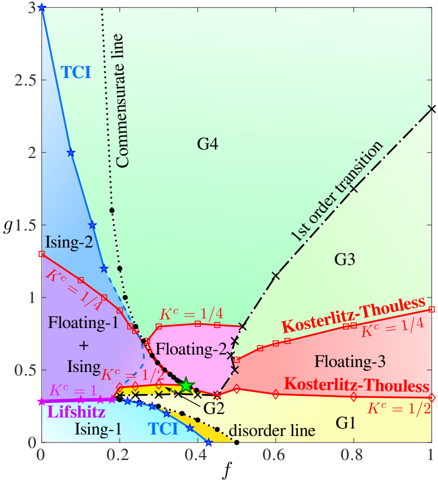
- Floating-2
- Floating-3

First order transition

- Ising+Floating-1
- Ising-1
- Ising-2

Ising

Phase diagram



Gapped phases:

- 3-fold degenerate
  - G1
  - G2
- 6-fold degenerate
  - G3
  - G4

Critical phases

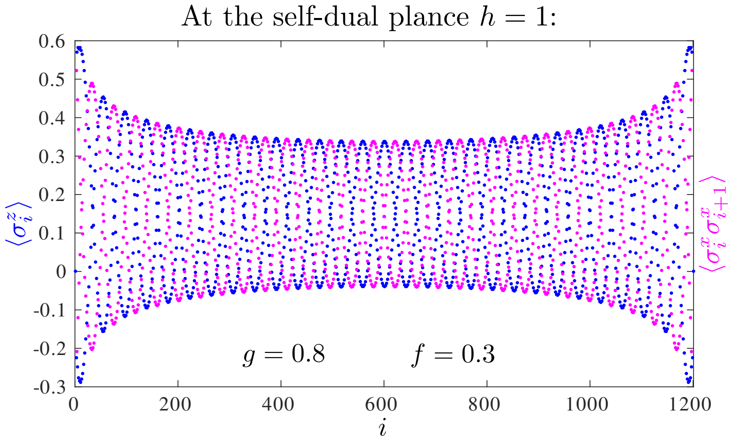
- Floating-2
- Floating-3

First order transition

- Ising+Floating-1
- Ising-1
- Ising-2

Ising

Duality in action



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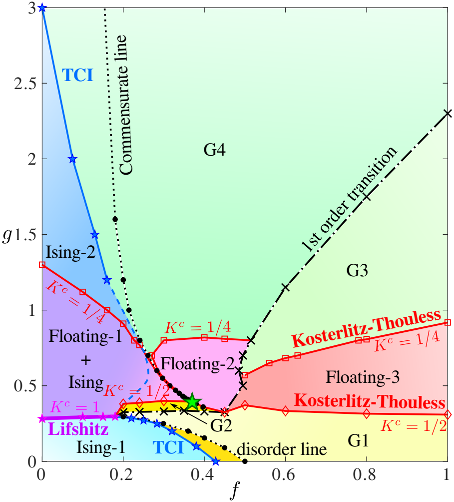
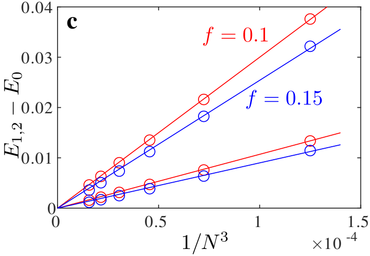
Lifshitz transition



- **Lifshitz line**
  - $K^c = 1$ + C-IC transition
  - Dynamical exponent $z = 3$



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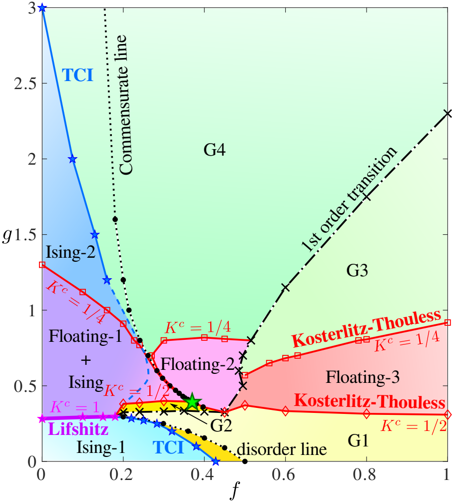
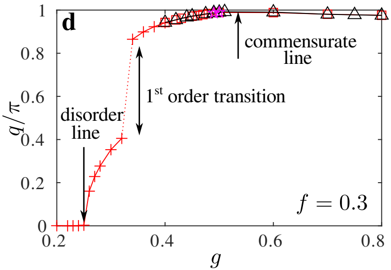
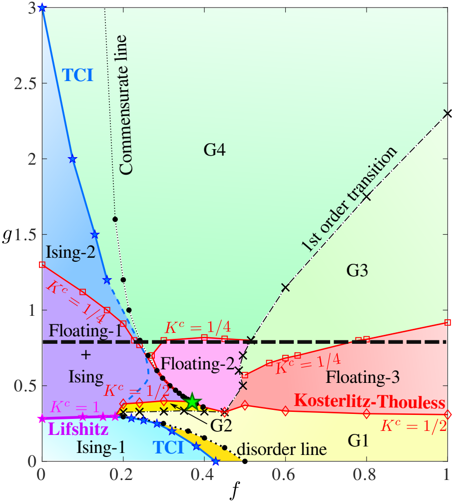
Phase diagram



- TCI and disorder lines
- **Lifshitz line**
  - $K^c = 1$ + C-IC transition
  - $z = 3$
  - Continues as a 1st order transition

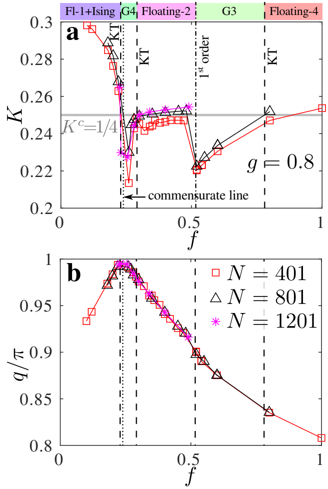


Commensurate line: suppression of the floating phase

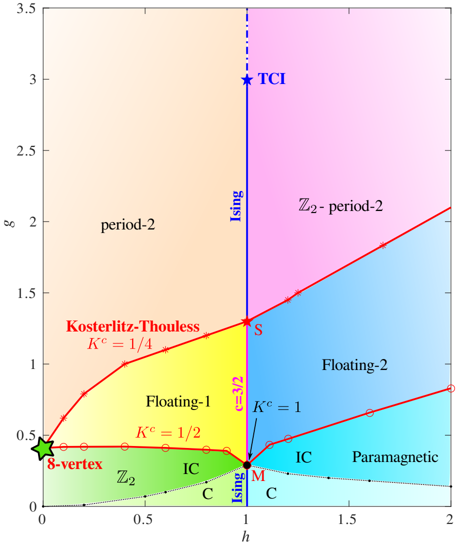


Floating phases

- Stable when $1/4 < K < 1/2$



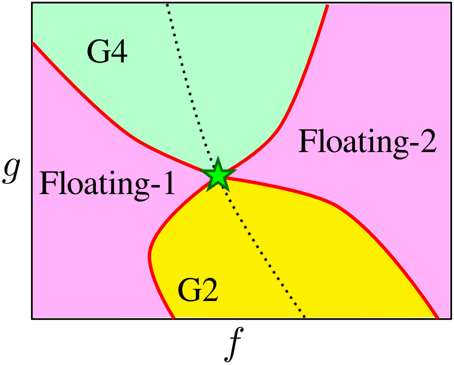
Commensurate line = collapse of the floating phase



- **Commensurate line**
  - No floating phase
  - Direct transition in the 8-vertex universality class?

Let's check!

Commensurate line



- Lifshitz line
- Floating phases
- **Commensurate line**
  - No floating phase
  - Direct transition in the 8-vertex universality class?

Let's check!

... no exact particle-hole symmetry line
... but there is a commensurate line

Eight-vertex criticality along the commensurate line

$$\mathcal{H} = \sum_j \left[-J\sigma_j^x\sigma_{j+1}^x - h\sigma_j^z + g(\sigma_j^z\sigma_{j+1}^z + \sigma_j^x\sigma_{j+2}^x) + f(\sigma_j^z\sigma_{j+1}^x\sigma_{j+2}^x + \sigma_j^x\sigma_{j+1}^x\sigma_{j+2}^z)\right]$$

- No explicit particle-hole symmetry
- 8-vertex universality class?
- Control parameter: no prediction!

$$\rho = \text{acos}[J_y/J_x]$$

- Critical exponents: still valid

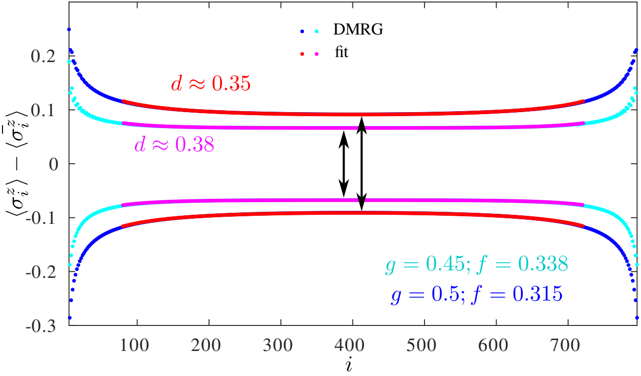
$$\nu = \pi/(2\rho), \quad \beta = (\pi - \rho)/(4\rho),$$

$$d = \pi - \rho/(2\pi)$$

- Express one exponent in terms of another:

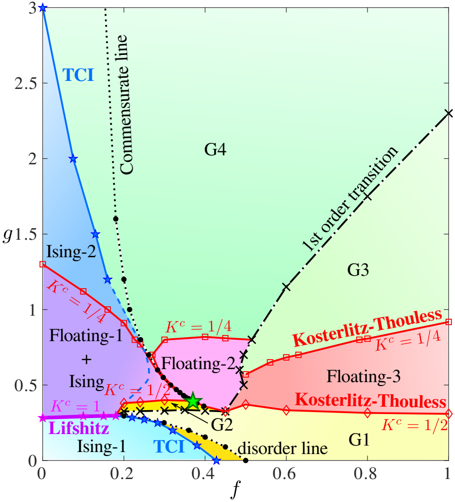
$$d = 2\beta/(1 + 4\beta)$$

Friedel oscillations along the commensurate line



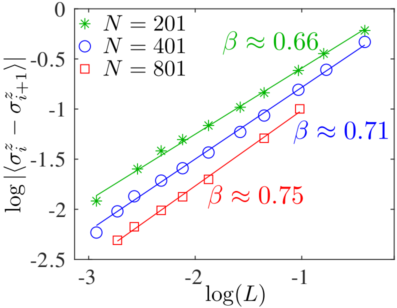
Note that it is not mandatory to sit exactly at the commensurate line, but since the G4-phase is too narrow for small g the commensurate line is the simplest choice.

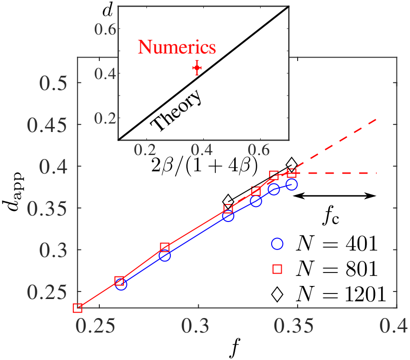
Scaling towards the transition



- Commensurate line is not straight!
- ... and there is no straight-line cut within G4 phase
- Compute the distance to the transition **along** the commensurate line

Multicritical point





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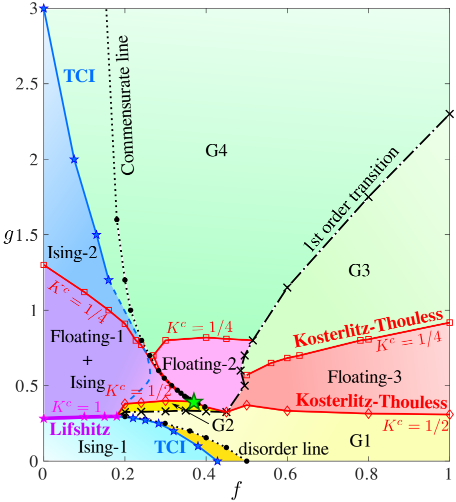
Summary

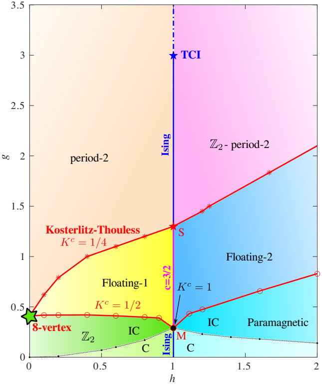
Interacting Majorana chains have extremely rich critical behavior:

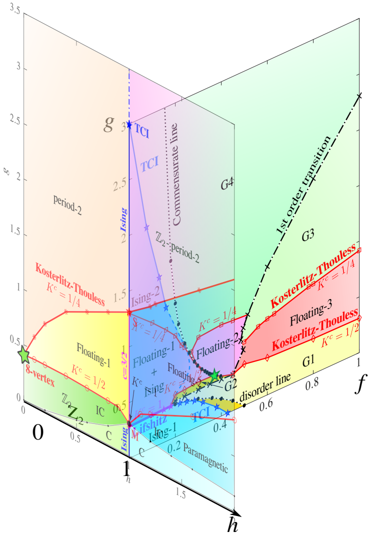
- Ising
- Disorder: Infinite randomness
- Tri-critical Ising
- Floating
- Lifshitz transition: $z = 3$
- 8-vertex multicritical point
  - Commensurate line
  - Emergent particle-hole symetry
  - Collapse of the floating phase



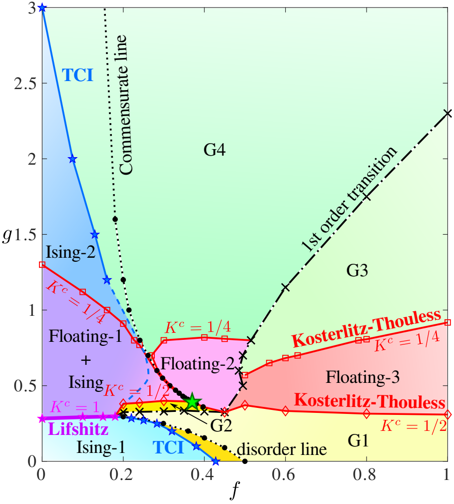
Outlook

- Two copies of 8-vertex point
- Finite f seems to bring them together





Outlook



- Two copies of 8-vertex point
- Finite f seems to bring them together
- Particle-hole symmetric surface?
- Supersymmetry
  - Two superconformal TCI lines

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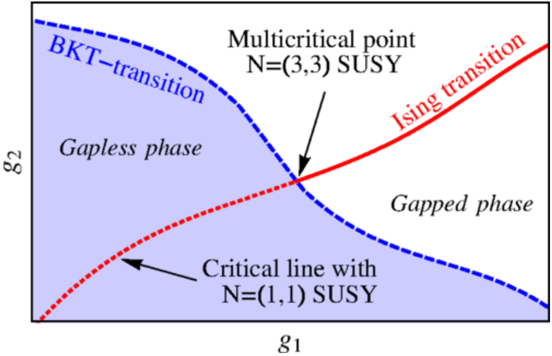
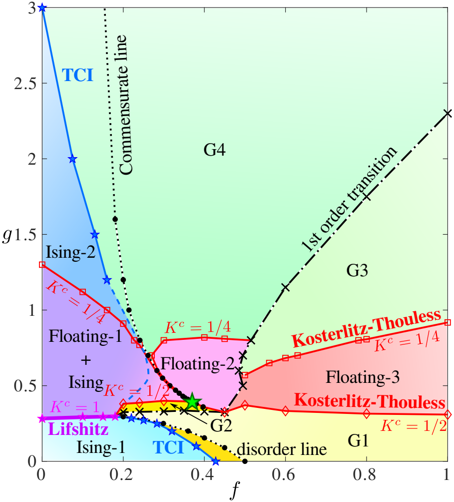
SUSY when Ising dives into Luttinger liquid



Huijse, Bauer, Berg, PRL 114, 090404; Sitte, Rosch, Meyer, Matveev, Garst, PRL 102, 176404 (2009)

Outlook



- Two copies of 8-vertex point
- Emergent particle-hole symmetry along the commensurate line
- End point of the Lifshitz line?
  - Two TCI lines merge into Lifshitz?
- Supersymmetry
  - Two superconformal TCI lines
  - SUSY in Floating-1+Ising?
  - SUSY at the KT transition?

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