Kac-Moody symmetries in one-dimensional bosonic systems

Wei Tang (UGent)

Ref: Phys. Rev. B 108, 035153 (2023)

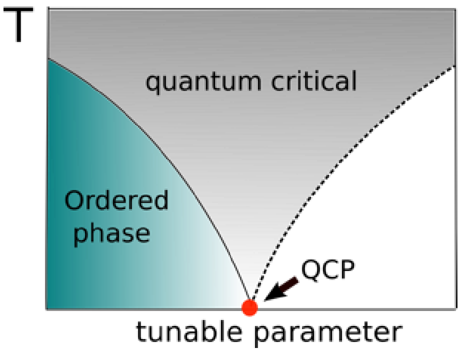
In collaboration with: Jutho Haegeman (UGent)

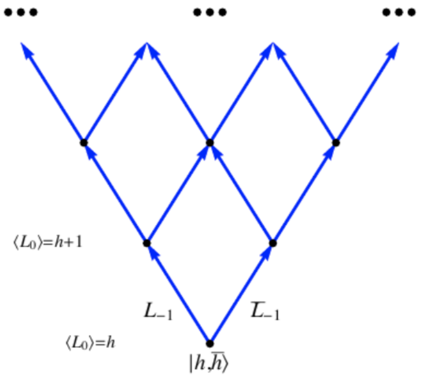
15.08.2023

Overview

CFT emerging from microscopic systems

- Critical systems: scale invariance, infinite correlation length, universal behaviour
- (1+1)D & 2D: conformal invariance, Virasoro algebra





$$[L_n^{\mathrm{CFT}}, L_m^{\mathrm{CFT}}] = (n-m)L_{n+m}^{\mathrm{CFT}} + \frac{c}{12}n(n^2-1)\delta_{n+m,0},$$
$$[L_n^{\mathrm{CFT}}, \overline{L}_m^{\mathrm{CFT}}] = 0,$$
$$[\overline{L}_n^{\mathrm{CFT}}, \overline{L}_m^{\mathrm{CFT}}] = (n-m)\overline{L}_{n+m}^{\mathrm{CFT}} + \frac{c}{12}n(n^2-1)\delta_{n+m,0},$$

microscopic degrees of freedom

$\xrightarrow{?}$ degrees of freedoms in CFT

Lattice realization of Virasoro generator

CFT $H^{\mathrm{CFT}} = \int_0^L \mathrm{d}x\, h^{\mathrm{CFT}}(x)$ | Lattice system $H = \sum_j h_j$

$h^{\mathrm{CFT}}(x) = T^{\mathrm{CFT}}(x) + \bar{T}^{\mathrm{CFT}}(x)$ $\longleftrightarrow$ local Hamiltonian term h_j

$p^{\mathrm{CFT}}(x) = T^{\mathrm{CFT}}(x) - \bar{T}^{\mathrm{CFT}}(x)$ $\longleftrightarrow$ energy current $p_j = \mathrm{i}[h_j, h_{j-1}]$

$L_m^{\mathrm{CFT}}, \bar{L}_m^{\mathrm{CFT}}$: Fourier components of $T(x), \bar{T}(x)$ $\longleftrightarrow$ lattice realization: Fourier components of $(h_j \pm p_j)/2$

Koo, Saleur, Nucl. Phys. B 426, 459 (1994)
Milsted, Vidal, Phys. Rev. B 96, 245105 (2017)
Zou, Milsted, Vidal, Phys. Rev. Lett. 121, 230402 (2018)

U(1) Kac-Moody algebra

- U(1) charge $Q^{\mathrm{CFT}} = \int_0^L \mathrm{d}x\, q^{\mathrm{CFT}}(x)$, $q^{\mathrm{CFT}}(x) = J^{\mathrm{CFT}}(x) + \bar{J}^{\mathrm{CFT}}(x)$

- Fourier modes of $J_m, \bar{J}_n$ $(m, n \in \mathbb{Z})$ of $J^{\mathrm{CFT}}(x)$, $\bar{J}^{\mathrm{CFT}}(x)$

$$\begin{cases} [J_m^{\mathrm{CFT}}, J_n^{\mathrm{CFT}}] = m\delta_{m+n,0} \\ [\bar{J}_m^{\mathrm{CFT}}, \bar{J}_n^{\mathrm{CFT}}] = m\delta_{m+n,0} \\ [J_m^{\mathrm{CFT}}, \bar{J}_n^{\mathrm{CFT}}] = 0 \end{cases}$$

- connection with bosonization

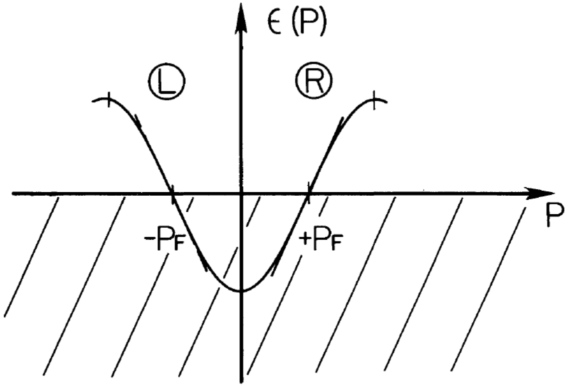
$$J_n^{\mathrm{CFT}} \sim \sqrt{n}\, b_{2\pi n/L} \quad \text{annihilation}$$

$$J_{-n}^{\mathrm{CFT}} \sim \sqrt{n}\, b_{2\pi n/L}^{\dagger} \quad \text{creation}$$

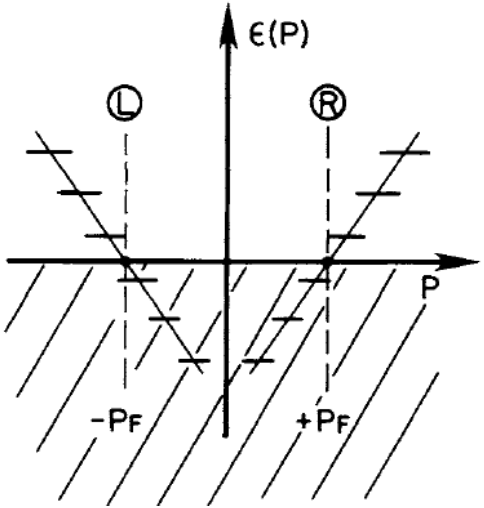
$$\Rightarrow \quad J_{-1}^{k_1} J_{-2}^{k_2} \ldots \bar{J}_{-1}^{\bar{k}_1} \bar{J}_{-2}^{\bar{k}_2} \ldots |\alpha\rangle,$$

Example. Free-Fermion Chain

$$H_0 = -\frac{t}{2}\sum_{j=0}^{N-1} (f_j^\dagger f_{j+1} + f_{j+1}^\dagger f_j)$$

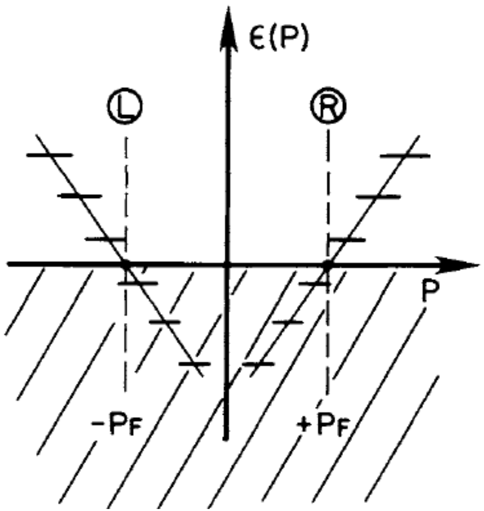


$$\varepsilon(p) = -t\cos p$$



Ludwig, in Low-dimensional quantum field theories for condensed matter physicists (1995)
von Delft, Schoeller, Ann. Phys. 7, 225 (1998)

Example. Free-Fermion Chain



$$J^{\mathrm{CFT}}(x) \sim \psi_R^\dagger(x)\psi_R(x)$$

$$\bar{J}^{\mathrm{CFT}}(x) \sim \psi_L^\dagger(x)\psi_L(x)$$

particle-hole excitations

$$J_m = \frac{2\pi}{L} \sum_{n \in \mathbb{Z}+1/2} :\psi_{R,n}^\dagger \psi_{R,n+m}:$$

$$\bar{J}_m = \frac{2\pi}{L} \sum_{n \in \mathbb{Z}+1/2} :\psi_{L,n}^\dagger \psi_{L,n+m}:$$

satisfies U(1) Kac-Moody algebra

Ludwig, in Low-dimensional quantum field theories for condensed matter physicists (1995)
von Delft, Schoeller, Ann. Phys. 7, 225 (1998)

Lattice realization of Kac-Moody generators

CFT $Q^{\mathrm{CFT}} = \int_0^L \mathrm{d}x\, q^{\mathrm{CFT}}(x)$ | Lattice system $H = \sum_j h_j$

$q^{\mathrm{CFT}}(x) = J^{\mathrm{CFT}}(x) + \bar{J}^{\mathrm{CFT}}(x)$ $\longleftrightarrow$ local charge density term n_j

$m^{\mathrm{CFT}}(x) = J^{\mathrm{CFT}}(x) - \bar{J}^{\mathrm{CFT}}(x)$ $\longleftrightarrow$ U(1) current m_j: $\mathrm{i}[H, q_j] = m_j - m_{j-1}$

$J_m^{\mathrm{CFT}}, \bar{J}_m^{\mathrm{CFT}}$: Fourier components of $J^{\mathrm{CFT}}(x), \bar{J}^{\mathrm{CFT}}(x)$ $\longleftrightarrow$ lattice realization: Fourier components of $(q_j \pm m_j)/2$

Wang, Zou, Vidal, Phys. Rev. B 106, 115111 (2022)

Overview

Kac-Moody generator in 1D boson systems

- One-dimensional continuous boson systems with particle-number conservation

$$H = \int \mathrm{d}x \, \partial_x \psi^\dagger(x) \partial_x \psi(x) + \int \mathrm{d}x \int \mathrm{d}x' \, u(x - x') \rho(x) \rho(x)$$

- Microscopic realization of Kac-Moody generator
  - $q^{\mathrm{CFT}}(x) \to \rho(x)$
  - $m^{\mathrm{CFT}}(x) \to j(x) = -\mathrm{i}[\psi^\dagger(x)\partial_x\psi(x) - \partial_x\psi^\dagger(x)\psi(x)]$

$$\mathrm{i}[H, \rho(x)] = -\partial_x j(x)$$

$$J_n = \int_0^L dx \, \mathrm{e}^{-2\pi n \mathrm{i} x/L} \frac{\Delta\rho(x) + j(x)/v}{2},$$

$$\bar{J}_n = \int_0^L dx \, \mathrm{e}^{2\pi n \mathrm{i} x/L} \frac{\Delta\rho(x) - j(x)/v}{2}.$$

Phenomenological bosonization

$$H = \int \mathrm{d}x\, \partial_x \psi^\dagger(x) \partial_x \psi(x) + \int \mathrm{d}x \int \mathrm{d}x'\, u(x-x')\, \rho(x)\, \rho(x)$$

$$\begin{cases} \psi^\dagger(x) = \sqrt{\rho(x)}\mathrm{e}^{-\mathrm{i}\phi(x)} \\ \psi(x) = \mathrm{e}^{\mathrm{i}\phi(x)}\sqrt{\rho(x)} \end{cases}$$

$$\begin{aligned} \rho(x) &= [\partial_x \Theta(x)] \sum_{n\in\mathbb{Z}} \delta(\Theta(x) - n\pi) \\ &= (1/\pi)[\partial_x \Theta(x)] \sum_{m\in\mathbb{Z}} \exp[2m\mathrm{i}\Theta(x)] \end{aligned}$$

Long-wavelength limit:

$$\rightarrow \quad H_{\text{eff}} = (v/2\pi) \int_0^L \mathrm{d}x \left[K(\partial_x \phi(x))^2 + \frac{1}{K}(\partial_x \Theta(x) - \pi\rho_0)^2 \right]$$

$$\rightarrow \quad \rho(x) \approx (1/\pi)\partial_x \Theta(x),\ j(x) \approx 2\rho_0 \partial_x \phi(x)$$

Haldane, Phys. Rev. Lett. 47, 1840 (1981); J. Phys. C 14, 2585 (1981)
Cazalilla, J. Phys. B 37, S1 (2004)

Phenomenological bosonization

$$\begin{cases} \Theta(x) = \theta_0 + \dfrac{\pi N x}{L} - \mathrm{i}\sum_{q\neq 0} \sqrt{\left|\dfrac{\pi K}{2qL}\right|}\mathrm{sgn}(q)\mathrm{e}^{\mathrm{i}qx}(b_q^\dagger + b_{-q}) \\ \\ \phi(x) = \phi_0 + \dfrac{\pi J x}{L} - \mathrm{i}\sum_{q\neq 0} \sqrt{\left|\dfrac{\pi}{2qLK}\right|}\mathrm{e}^{\mathrm{i}qx}(b_q^\dagger - b_{-q}) \end{cases}$$

$$\rightarrow H_{\mathrm{eff}} = \sum_{q\neq 0} v|q| b_q^\dagger b_q + \frac{\pi v}{2LK}(N - N_0)^2 + \frac{\pi v K}{2L} J^2 + \mathrm{const.}$$

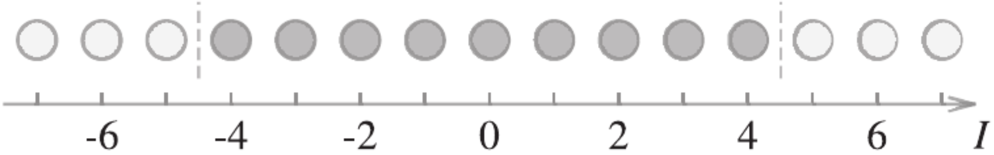
$$\rightarrow J_n^{\mathrm{CFT}} = J_n^{\mathrm{micro}} / \sqrt{K},\ \bar{J}_n^{\mathrm{CFT}} = \bar{J}_n^{\mathrm{mirco}} / \sqrt{K}$$

Haldane, Phys. Rev. Lett. 47, 1840 (1981); J. Phys. C 14, 2585 (1981)
Cazalilla, J. Phys. B 37, S1 (2004)

Lieb-Liniger model: Bethe-Ansatz analysis

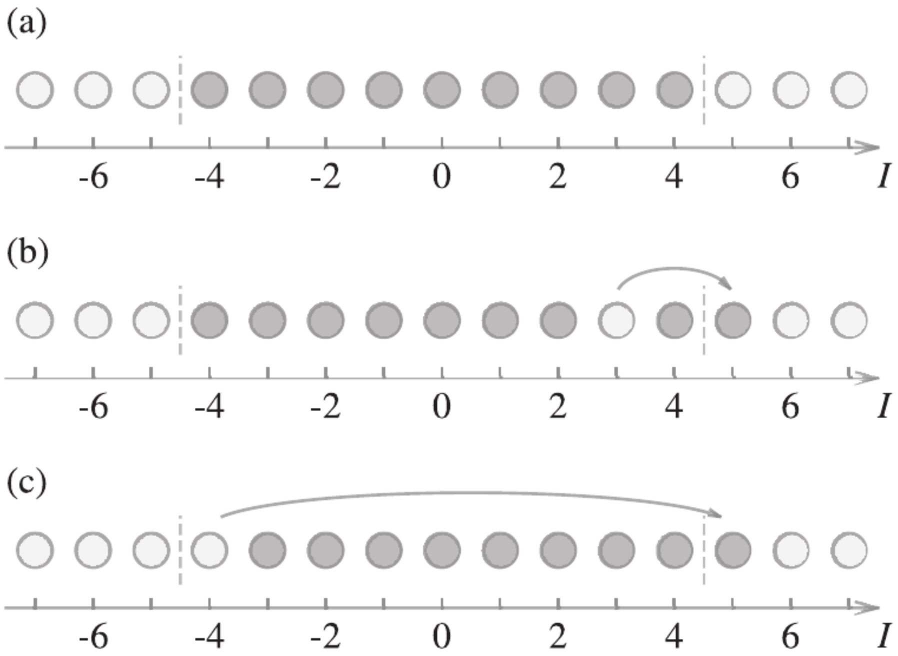
$$H = \int \mathrm{d}x\, \partial_x \psi^\dagger(x) \partial_x \psi(x) + \int \mathrm{d}x\, \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x)$$

- $\psi_{\{\lambda_j\}}(\boldsymbol{x}) = \sum_{\mathcal{P}} a(\mathcal{P}) \exp(\mathrm{i} \sum_j \lambda_{\mathcal{P}(j)} x_j)$
- quasi-momenta $\{\lambda_j\} \leftrightarrow$ quantum numbers $\{I_j\}$: integers/half-integers
- $E = \sum_j \lambda_j^2$, $P = \sum_j 2\pi I_j / L$



Yang, Yang, J. Math. Phys. 10, 1115 (1969)
Jiang, Chen, Guan, Chin. Phys. B 24, 050311 (2015)

Lieb-Liniger model: Bethe-Ansatz analysis



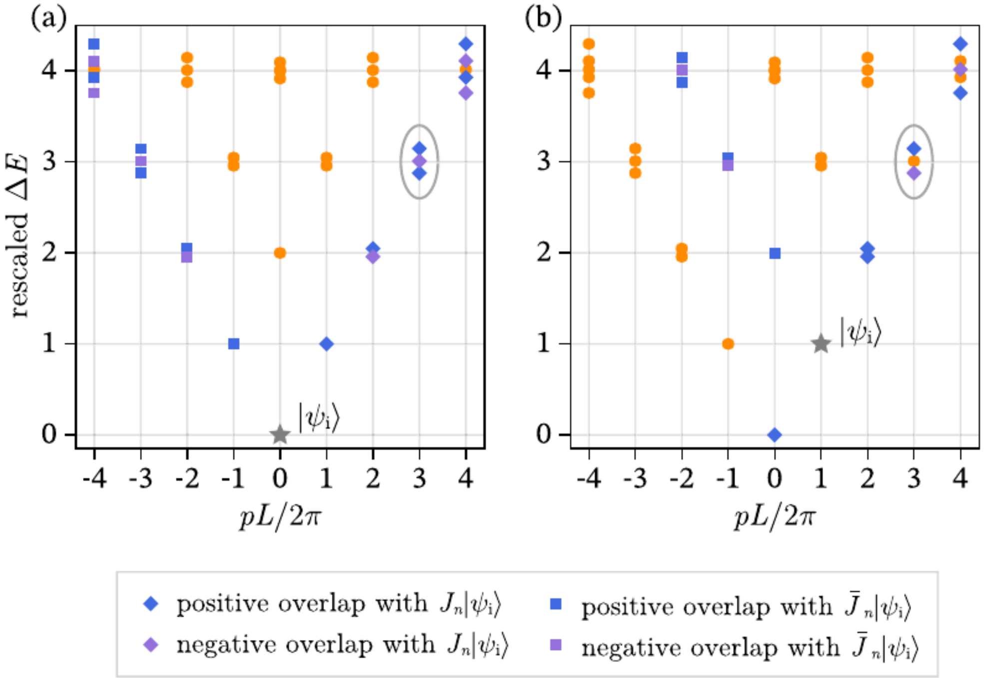
ground state, “Fermi sea”

low-energy, long-wavelength

⋆ correspond to Kac-Moody generators J_{-n}, $\bar{J}_{-n}$

low-energy, short-wavelength

Lieb-Liniger model: Bethe-Ansatz analysis

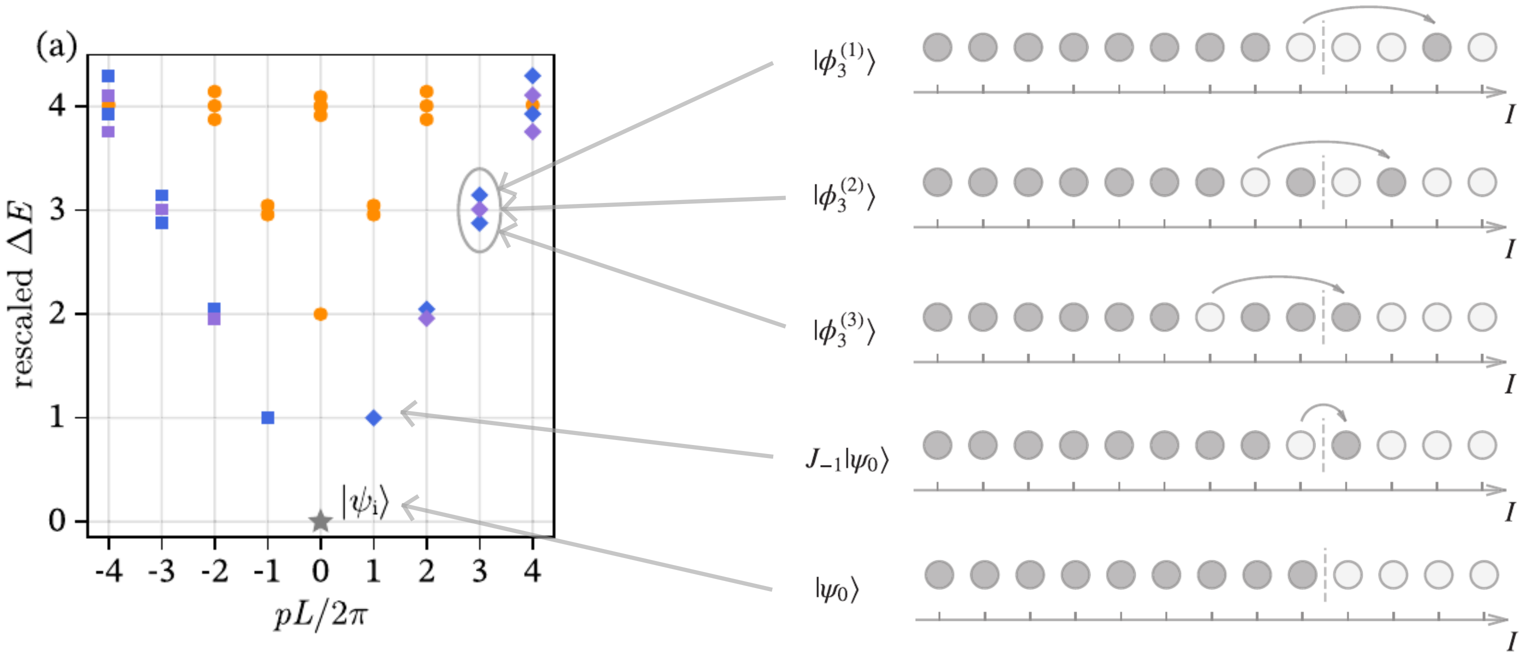


$$C_{\mu,\nu}[X] \equiv \frac{\langle\mu|X|\lambda\rangle}{\langle\mu|\mu\rangle\langle\lambda|\lambda\rangle}$$

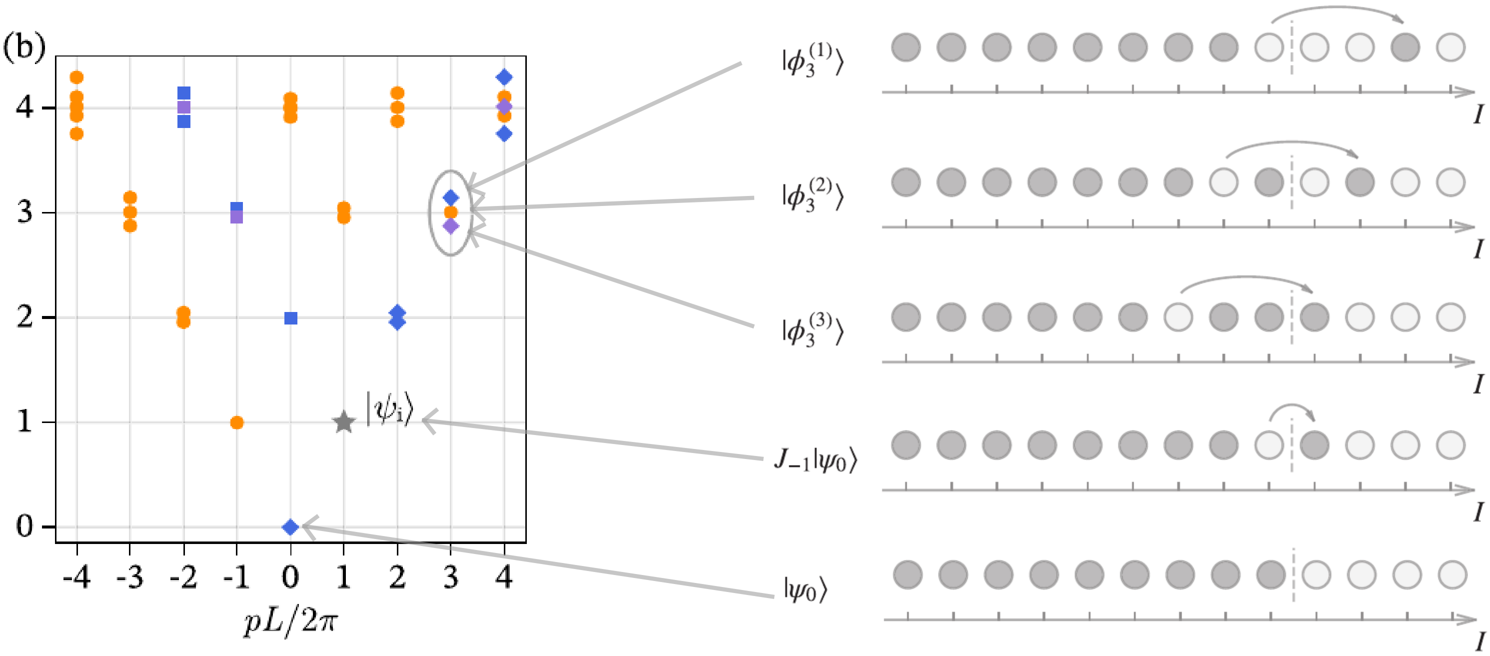
$$X = J_n \text{ or } J_m$$

De Nardis, Panfil, J. Stat. Mech. 2015, P02019 (2015)

Lieb-Liniger model: Bethe-Ansatz analysis



Lieb-Liniger model: Bethe-Ansatz analysis

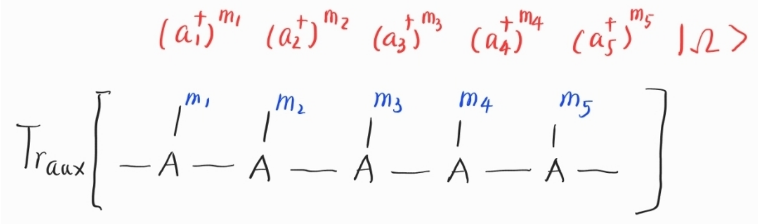


Overview

Continuous matrix product state

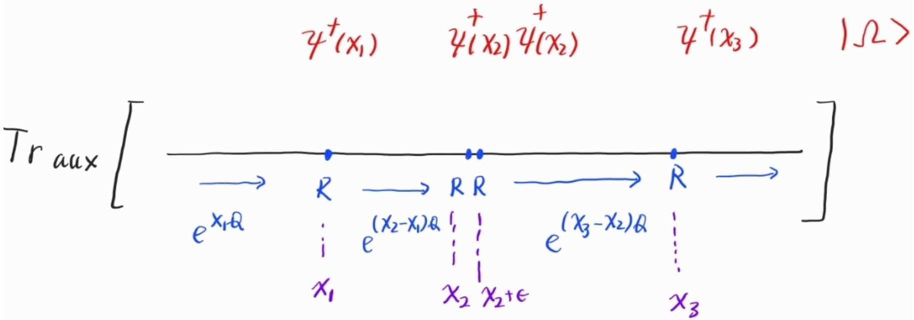
- Matrix product state

$$|\psi\rangle = \mathrm{Tr}(A^{m_1}A^{m_2}\ldots A^{m_N})(a_1^\dagger)^{m_1}\ldots(a_N^\dagger)^{m_N}|\Omega\rangle$$



- Taking continuous limit $\epsilon \to 0$, number of sites $L/\epsilon \to \infty$

$$A^0 = I + \epsilon Q$$
$$A^1 = \sqrt{\epsilon} R$$

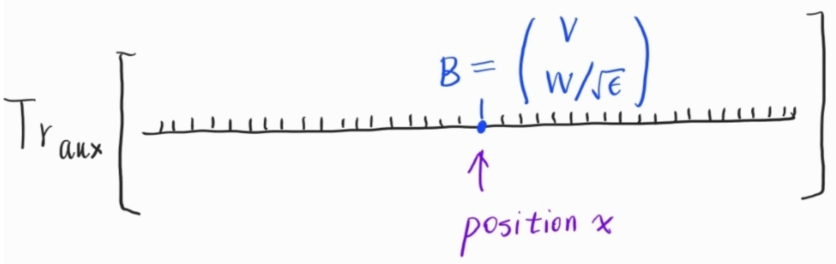


$$|\psi\rangle = \mathrm{Tr}_{\mathrm{aux}}(\mathcal{P}\exp[\int \mathrm{d}x\,(Q\otimes \mathbb{1} + R\otimes \psi^\dagger(x))])|\Omega\rangle$$

Verstraete, Cirac, Phys. Rev. Lett. 104, 190405 (2010)
Haegeman, Cirac, Osborne, Verstraete, Phys. Rev. B 88, 085118 (2013)

Continuous matrix product state

$$|\Phi_p(V, W)\rangle = \int \mathrm{d}x \exp(\mathrm{i}px)$$

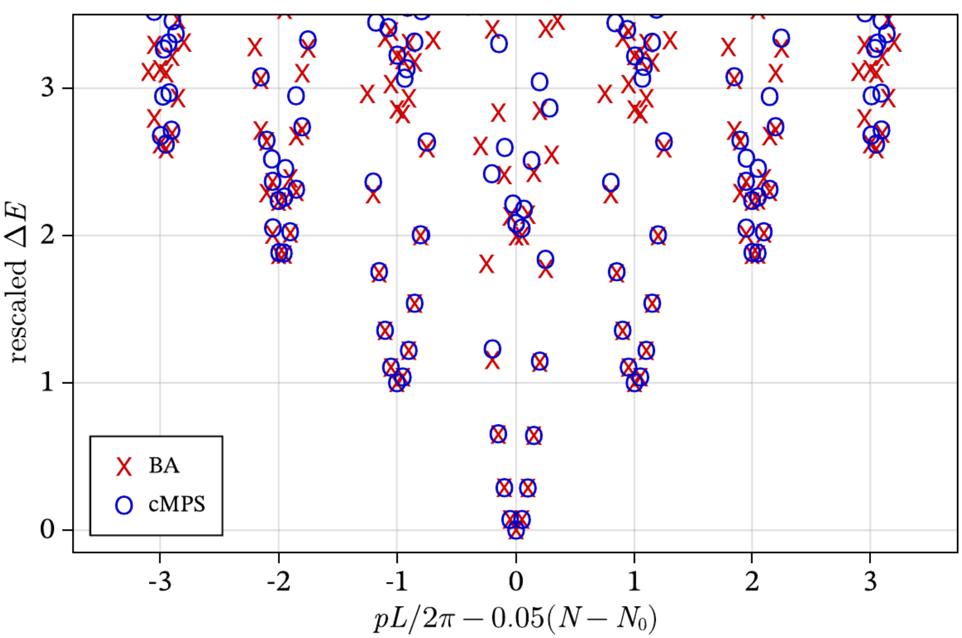


$$\langle \Phi_p(\bar{V}, \bar{W}) | \hat{H} - E_0 | \Phi_{p'}(V, W) \rangle = 2\pi\delta(p - p') [\, V^\dagger W^\dagger \,] H_p \begin{bmatrix} V \\ W \end{bmatrix},$$

$$\langle \Phi_p(\bar{V}, \bar{W}) | \Phi_{p'}(V, W) \rangle = 2\pi\delta(p - p') [\, V^\dagger W^\dagger \,] N_p \begin{bmatrix} V \\ W \end{bmatrix},$$

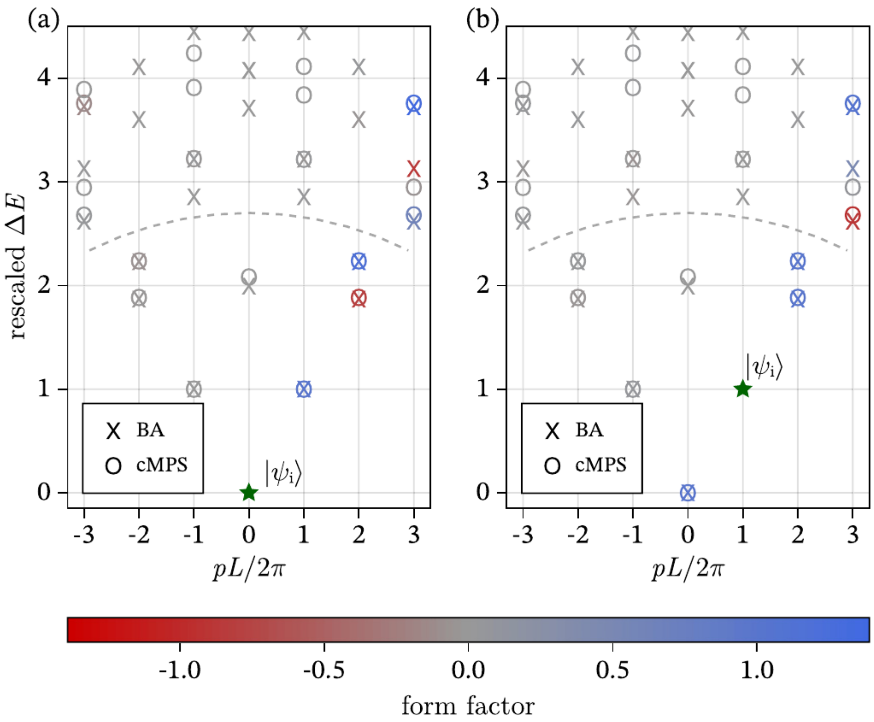
Rommer, Östlund, Phys. Rev. B 55, 2164 (1997)
Haegeman, Pirvu, Weir, Cirac, Osborne, Verschelde, Verstraete, Phys. Rev. B 85, 100408 (2012)
Pirvu, Haegeman, Verstraete, Phys. Rev. B 85, 035130 (2012)
Draxler, Haegeman, Osborne, Stojevic, Vanderstraeten, Verstraete, Phys. Rev. Lett. 111, 020402 (2013)

Continuous matrix product state: results



Lieb-Liniger with $L = 16, c = 1, \mu = 1.426, N_0 = 16$.

Continuous matrix product state: results



Overview

- Introduction
  - Lattice realization for Virasoro generators
  - Lattice realization for Kac-Moody generators
- $\mathrm{U}(1)$ Kac-Moody generators in one-dimensional boson systems
  - Phenomenological bosonization
  - Lieb-Liniger model: Bethe ansatz analysis
- Continuous matrix product state approach
- Conclusion and outlook

Conclusion and Outlook

- We performed a detailed study of the microscopic realization of $\mathrm{U}(1)$ Kac-Moody generators in 1D boson systems
  - Phenomenological bosonization
  - Bethe ansatz study of the integrable Lieb-Liniger model

- We proposed a cMPS approach to compute the Kac-Moody generators in generic 1D boson systems

Conclusion and Outlook

- We performed a detailed study of the microscopic realization of $\mathrm{U}(1)$ Kac-Moody generators in 1D boson systems
  - Phenomenological bosonization
  - Bethe ansatz study of the integrable Lieb-Liniger model

    $\longrightarrow$ Other integrable models? Calogero-Sutherland, Haldane-Shastry…

    $\longrightarrow$ Different boundary conditions?

- We proposed a cMPS approach to compute the Kac-Moody generators in generic 1D boson systems

  $\longrightarrow$ Interacting bosons with long-range interactions

  $\longrightarrow$ Multi boson systems or spin-1/2 fermions, other Kac-Moody algebras

Thank You!