Dualities in one-dimensional quantum lattice models[a][b]

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[a] PRX Quantum 4, 020357
[b] arXiv:2211.03777

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Kramers-Wannier duality

Consider the transverse field Ising model

$$\mathbb{H}_A = -J\sum_{\mathsf{i}} X_{\mathsf{i}-\frac{1}{2}} X_{\mathsf{i}+\frac{1}{2}} - Jg\sum_{\mathsf{i}} Z_{\mathsf{i}+\frac{1}{2}}, \quad \text{global } \bigotimes_{\mathsf{i}} Z_{\mathsf{i}+\frac{1}{2}} \text{ symmetry, broken when } g < 1$$

Kramers and Wannier introduced the following transformation:

$$X_{\mathsf{i}-\frac{1}{2}} X_{\mathsf{i}+\frac{1}{2}} \to X_{\mathsf{i}}, \quad Z_{i+\frac{1}{2}} \to Z_{\mathsf{i}} Z_{\mathsf{i}+1}$$

This defines the dual Hamiltonian

$$\mathbb{H}_B = -J\sum_{\mathsf{i}} X_{\mathsf{i}} - Jg\sum_{\mathsf{i}} Z_{\mathsf{i}} Z_{\mathsf{i}+1}, \quad \text{global } \bigotimes_{\mathsf{i}} X_{\mathsf{i}} \text{ symmetry, broken when } g > 1$$

Kramers-Wannier duality is a non-local transformation:

$$X_{\mathsf{i}-\frac{1}{2}} \to \prod_{\mathsf{j}\geq\mathsf{i}} X_{\mathsf{j}}$$

Generalities on dualities

We characterize a duality as follows:

- unitary (isometric) transformation relating two models
- symmetric local operators $\rightarrow$ dual symmetric local operators (e.g. Hamiltonians)
- non-symmetric order operators $\rightarrow$ dual non-symmetric (string) order operators

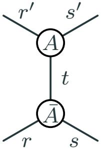
Dualities are dictated by **symmetries**

$\rightsquigarrow$ two models with the **same symmetries** admit the **same dualities**

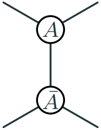
Symmetric Hamiltonians: ordinary symmetries

Define a symmetric Hamiltonian as a sum of local terms $\mathbb{H}_A = \sum_{\mathsf{i}} \mathbb{h}_{A,\mathsf{i}}$ with

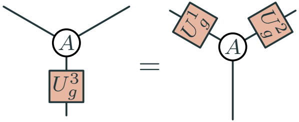
$$\mathbb{h}_{A,\mathsf{i}} = \sum_t A^t_{r's'} \bar{A}^t_{rs} \, |r', s'\rangle\langle r, s| \equiv \sum_t$$



$$|r', s'\rangle\langle r, s| \equiv$$



For instance, consider a (finite) group symmetry G, which implies



, with $U_g^i = \bigoplus_{j_i} D^{j_i}(g)$

Wigner-Eckart theorem $\rightsquigarrow$ A is built from **Clebsch-Gordan coefficients**:

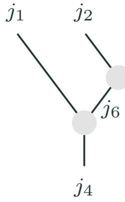
$$A^t_{rs} \equiv A^{(j_3 m_3)}_{(j_1 m_1)(j_2 m_2)} \equiv A^{j_3}_{j_1 j_2} C^{j_1 j_2 j_3}_{m_1 m_2 m_3}$$

Symmetric Hamiltonians: ordinary symmetries

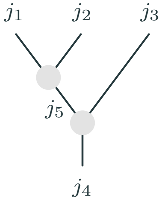
Clebsch-Gordan coefficients are recoupled using F-symbols:

$$\sum_{m_6} C^{j_2 j_3 j_6}_{m_2 m_3 m_6} C^{j_1 j_6 j_4}_{m_1 m_6 m_4} = \sum_{j_5, m_5} \left(F^{j_1 j_2 j_3}_{j_4}\right)^{j_6}_{j_5} C^{j_1 j_2 j_5}_{m_1 m_2 m_5} C^{j_1 j_6 j_4}_{m_1 m_6 m_4}$$

which up to a phase are the $6j$ symbols; graphically,



$$= \sum_{j_5} \left(F^{j_1 j_2 j_3}_{j_4}\right)^{j_6}_{j_5}$$

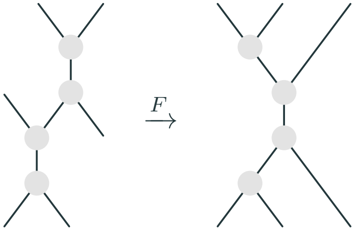


This data defines an “input” **fusion category** $\mathcal{D}$

Algebra of symmetric operators

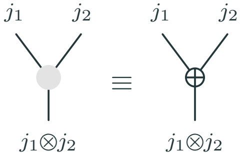
Recoupling theory allows computation of symmetric operator products:

$$\mathcal{O}_x \mathcal{O}_y = \sum_z f_{xy}^z(F) \mathcal{O}_z,$$



Symmetric operators generate an algebra; this algebra is known as the bond algebra, and it is characterized by F-symbols of the input fusion category $\mathcal{D}$.

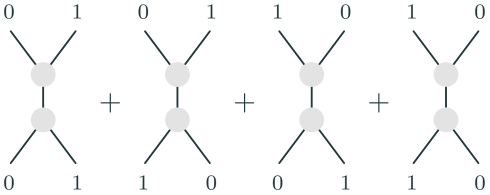
Ising model: $\mathcal{D} = \mathsf{Vec}_{\mathbb{Z}_2}$, two 1d irreps $\{0, 1\}$, $j_1 \otimes j_2 = j_1 + j_2 \mod 2$, with Clebsch-Gordan coefficients given by



$$= |j_1\rangle|j_2\rangle\langle j_1 \otimes j_2|$$

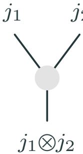
In terms of the CG coefficients, we can write

$$X_{\mathsf{i}-\frac{1}{2}} X_{\mathsf{i}+\frac{1}{2}} =$$

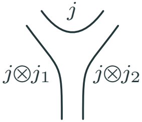


Symmetric Hamiltonians: Kramers-Wannier dual of Ising model

Given recoupling equation for $\mathcal{D} = \mathsf{Vec}_{\mathbb{Z}_2}$ there is another solution

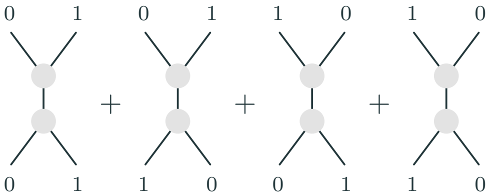


$\equiv \sum_j$



$= \sum_j |j \otimes j_1\rangle |j\rangle |j \otimes j_2\rangle \langle j \otimes j_1| \langle j \otimes j_2|$

Same linear combination that yielded $X_{\mathsf{i}-\frac{1}{2}} X_{\mathsf{i}+\frac{1}{2}}$ now gives the dual operator

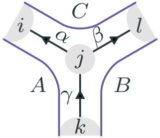


$= X_{\mathsf{i}}$

Dual realizations of operators obtained in this way generate the same algebra!

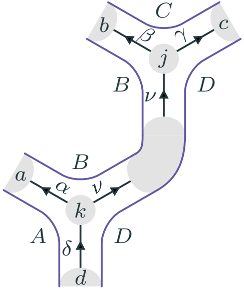
General result

Generalized Clebsch-Gordan coefficients are determined by the data ${}^{\triangleleft}F$ of a chosen **module category** $\mathcal{M}$ over the input fusion category $\mathcal{D}$:

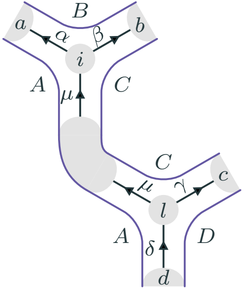


$$:= \left({}^{\triangleleft}F^{A\alpha\beta}_{B}\right)^{\gamma,jk}_{C,il}$$

satisfying



$$= \sum_{\mu} \sum_{i,l} \left(F^{\alpha\beta\gamma}_{\delta}\right)^{\nu,jk}_{\mu,il}$$

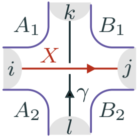


Dual models

- **Dual models are characterized by the same fusion category $\mathcal{D}$, with the same recoupling theory, but different choices of module category $\mathcal{M}$.** Duality is an isomorphism of the algebra of local symmetric operators.
- Dualities are implemented by **matrix product operator (MPO) intertwiners** that can be constructed from the categorical data.
- Dual models have equivalent but distinct realizations of (MPO) symmetries, determined by the choice of $\mathcal{M}$.

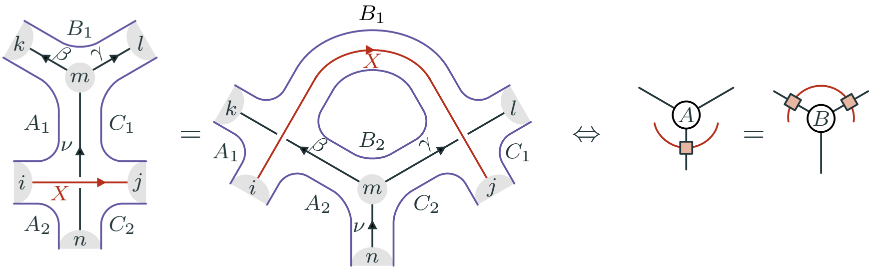
MPO intertwiners

For different choices of module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ one can compute an MPO intertwiner as a $\mathcal{D}$-module functor $X \in \mathsf{Fun}_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$:



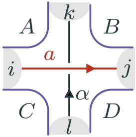
$$:= \left({}^{X}\omega^{A_2\gamma}_{B_1}\right)^{B_2,lj}_{A_1,ik}$$

where ${}^{X}\omega$ is determined by:



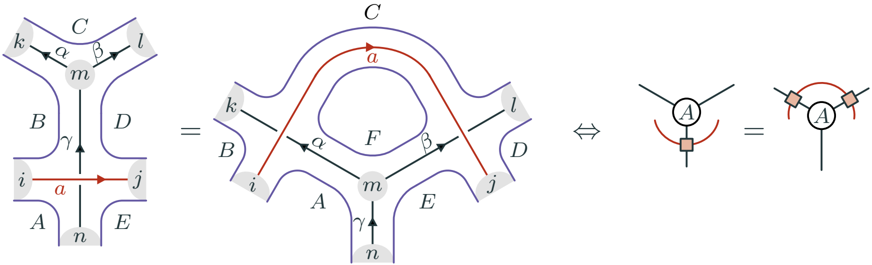
MPO symmetries

For the case where $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, we get an MPO symmetry $a \in \text{End}_{\mathcal{D}}(\mathcal{M})$:



$$= \left({}^{\bowtie}F^{aC\alpha}_{B}\right)^{A,jk}_{D,il}$$

such that



Examples

1. $\mathcal{D} = \mathsf{Vec}_{\mathbb{Z}_2}$: $\mathbb{Z}_2$ symmetry
    - $\mathcal{M} = \mathsf{Vec}$: transverse field Ising model
    - $\mathcal{M} = \mathsf{Vec}_{\mathbb{Z}_2}$: Kramers-Wannier dual
    - $\mathcal{M} = \mathsf{sVec}/\langle\psi \simeq \mathbb{1}\rangle$: free fermion
2. $\mathcal{D} = \mathsf{Vec}_{\mathbb{Z}_2\times\mathbb{Z}_2}$: $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry
    - $\mathcal{M} = \mathsf{Vec}$: spin 1 Heisenberg model, non-trivial SPT (Haldane phase)
    - $\mathcal{M} = \mathsf{Vec}^{\psi}$: Kennedy-Tasaki dual (trivial SPT), related to SPT entangler
3. $\mathcal{D} = \mathsf{Ising}$: $\mathbb{Z}_2$ symmetry $+$ Kramers-Wannier self-duality
    - $\mathcal{M} = \mathsf{Ising}$: critical transverse field Ising model
    - $\mathcal{M} = \mathsf{Ising}/\langle\psi \simeq \mathbb{1}\rangle$: massless free fermion
4. $\mathcal{D} = \mathsf{Ising}^{\boxtimes 2}$: $(\mathbb{Z}_2 + \text{Kramers-Wannier self-duality})^{\otimes 2}$
    - $\mathcal{M} = \mathsf{Ising}^2$: two decoupled critical transverse field Ising models
    - $\mathcal{M} = \mathsf{Ising}$: critical XY model
    - $\mathcal{M} = \mathsf{Ising}/\langle\psi \simeq \mathbb{1}\rangle$: massless Dirac fermion

Examples

5. $\mathcal{D} = \text{Vec}_{\mathbb{Z}_2}$: $\mathbb{Z}_2$ symmetry
   - $\mathcal{M} = \text{Vec}$: XXZ model
   - $\mathcal{M} = \text{sVec}$: t-J_z model
6. $\mathcal{D} = \text{Rep}(\mathrm{U}_q(\mathfrak{sl}_2))$: quantum deformed $SU(2)$ symmetry
   - $\mathcal{M} = \text{Rep}(\mathrm{U}_q(\mathfrak{sl}_2))$: solid-on-solid (SOS) models
   - $\mathcal{M} = \text{Vec}$: 6-vertex model (XXZ)
7. $\mathcal{D} = \mathcal{H}_3$: exotic fusion category, "Haagerup subfactor"
   - $\mathcal{M} = \mathcal{H}_3$: ?[1][2]
   - $\mathcal{M} = \mathcal{M}_{3,2}$: ?
   - $\mathcal{M} = \mathcal{M}_{3,1}$: ?

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[1] Huang, Lin, Ohmori, Tachikawa, Tezuka, *Numerical evidence for a Haagerup conformal field theory*, Phys. Rev. Lett. 128, 231603

[2] Vanhove, LL, Van Damme, Wolf, Osborne, Haegeman, Verstraete, *A critical lattice model for a Haagerup conformal field theory*, Phys. Rev. Lett. 128, 231602

Examples

8. $\mathcal{D} = \text{Rep}(S_3)$
   - $\mathcal{M} = \text{Vec}$: XXZ model
   - $\mathcal{M} = \text{Rep}(\mathbb{Z}_2)$:

$$\mathbb{H} = \sum_{\mathsf{i}} Z_{\mathsf{i}-1} Z_{\mathsf{i}+1} + Z_{\mathsf{i}-1} X_{\mathsf{i}} Z_{\mathsf{i}+1} + \Delta X_{\mathsf{i}}$$

     Interestingly, this model has a non-invertible $\text{Rep}(S_3)$ symmetry!
   - $\mathcal{M} = \text{Rep}(\mathbb{Z}_3)$: modified 3-state Potts model
   - $\mathcal{M} = \text{Rep}(S_3)$: $\text{Rep}(S_3)$ anyonic spin chain

Examples for any duality can be systematically generated

Alternatively, given a Hamiltonian with some (categorical) symmetry, all its duals can be obtained (generalized Wigner-Eckart theorem[3])

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[3] Bridgeman, LL, Verstraete, *Invertible bimodule categories and generalized Schur orthogonality*, Communications in Mathematical Physics, 1-24 (2023)

Duality as an isometry

Hilbert space and Hamiltonian split into superselection sectors, which have to match between models:

$$\mathcal{H}_A = \bigoplus_i^n \mathcal{H}_{A,i} \quad \text{and} \quad \mathcal{H}_B = \bigoplus_i^n \mathcal{H}_{B,i}\,,$$

$$\mathbb{H}_A = \bigoplus_i^n \mathbb{H}_{A,i} \quad \text{and} \quad \mathbb{H}_B = \bigoplus_i^n \mathbb{H}_{B,i}\,.$$

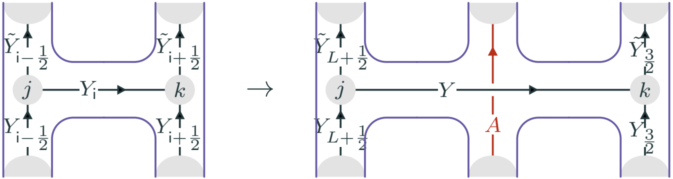
although they need not be the same size (different degeneracies). Dualities are isometries that interchange these sectors:

$$\mathbb{U}_i : \mathcal{H}_{A,i} \to \mathcal{H}_{B,i}, \quad \text{s.t.} \quad \mathbb{U}_i(\mathbb{H}_{A,i})\mathbb{U}_i^\dagger = \mathbb{H}_{B,i}$$

Here, superselection sectors refer to symmetry charges *and* boundary conditions

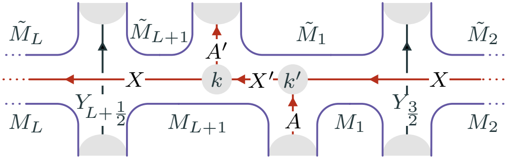
Symmetry twisted boundary conditions

Symmetry twisted boundary conditions locally change the bonds in the Hamiltonian



in such a way that translation invariance is preserved up to local unitaries. The symmetry operators now act as symmetry "tubes":

$$\mathfrak{T}^{A,A',X,X',k,k'}_{\mathcal{M}|\mathcal{M}} =$$

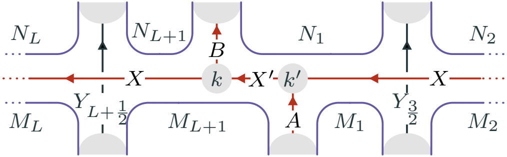


Superselection sectors

These symmetry tubes span an algebra, known as the tube algebra. Superselection sectors are irreducible representations of this algebra, which are described by the Drinfel'd center $\mathcal{Z}(\mathcal{C})$ of the fusion category $\mathcal{C}$ that describes the symmetry operators.

We can similarly define intertwiner tubes, that will implement the duality operator in the presence of a symmetry twist:

$$\mathfrak{T}^{A,B,X,X',k,k'}_{\mathcal{M}|\mathcal{N}} =$$



Superselection sectors

These intertwiner tubes allow us to relate dual topological sectors labeled by $\mathcal{Z}(\mathcal{C}_i) \simeq \mathcal{Z}(\mathcal{C}_j)$, which in general involves a permutation of the topological sectors.

The simplest example is the interchange of charges and fluxes for the $\mathbb{Z}_2$ Kramers-Wannier duality:

$$(\text{periodic}, \mathbb{Z}_2 - \text{even}) \to (\text{periodic}, \mathbb{Z}_2 - \text{even})$$
$$(\text{periodic}, \mathbb{Z}_2 - \text{odd}) \to (\text{anti-periodic}, \mathbb{Z}_2 - \text{even})$$
$$(\text{anti-periodic}, \mathbb{Z}_2 - \text{even}) \to (\text{periodic}, \mathbb{Z}_2 - \text{odd})$$
$$(\text{anti-periodic}, \mathbb{Z}_2 - \text{odd}) \to (\text{anti-periodic}, \mathbb{Z}_2 - \text{odd})$$

These maps can be computed explicitly for any duality[4]

[4]LL, Delcamp, Verstraete, *Dualities in one-dimensional quantum lattice models: topological sectors* 2211.03777

Conclusion

- Given Hamiltonian $\mathbb{H}$, its duals can be systematically constructed via identification of underlying categorical structures describing its symmetries:

$$\text{Fun}_{\mathcal{D}}(\mathcal{M},\mathcal{M}) \circlearrowright \mathbb{H}(\mathcal{D},\mathcal{M}) \xrightarrow{\text{Fun}_{\mathcal{D}}(\mathcal{M},\mathcal{M}')} \mathbb{H}(\mathcal{D},\mathcal{M}') \circlearrowleft \text{Fun}_{\mathcal{D}}(\mathcal{M}',\mathcal{M}')$$

- Dualities and symmetries are realized as MPOs
- Generalization to **higher dimensions** is **systematic**[5][6]

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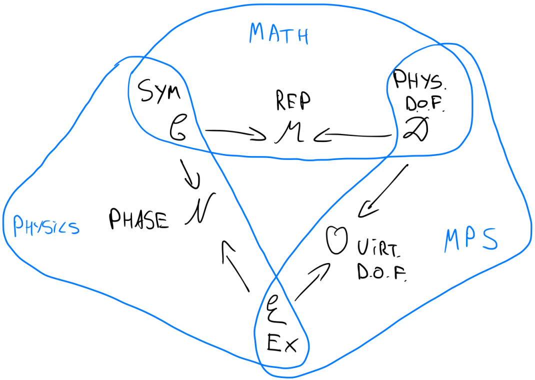
[5] Delcamp, *Tensor network approach to electromagnetic duality in (3+1)d topological gauge models*, JHEP 149 (2022)

[6] Delcamp, Tiwari, *Higher categorical symmetries and gauging in two-dimensional spin systems*, 2301.01259

Application: symmetric MPS

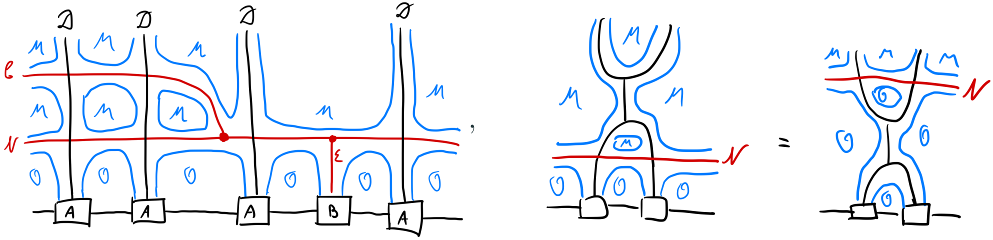
Dualities provide framework for understanding MPS with symmetries[7]:



where $\mathcal{N} = \mathrm{Fun}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$

[7]LL et. al, *Variational MPS with (generalized) symmetries*, in preparation

Given a symmetry described by $(\mathcal{C}, \mathcal{M}, \mathcal{D})$, we can write a generic MPS in the phase given by $\mathcal{N}$[8] with excitations $\mathcal{E}$ as



The action of the Hamiltonian $\mathbb{H}(\mathcal{D}, \mathcal{M})$ on this MPS is given by the dual Hamiltonian $\mathbb{H}(\mathcal{D}, \mathcal{O})$; this dual model typically has a constrained Hilbert space

This ansatz provides a characterization of an MPS in a given phase with the smallest number of variational parameters $\rightsquigarrow$ numerically more efficient

[8]Garre-Rubio, LL, Molnar, *Classifying phases protected by matrix product operator symmetries using matrix product states*, Quantum 7, 927 (2023)