THE STANDARD MODEL OF PARTICLE PHYSICS

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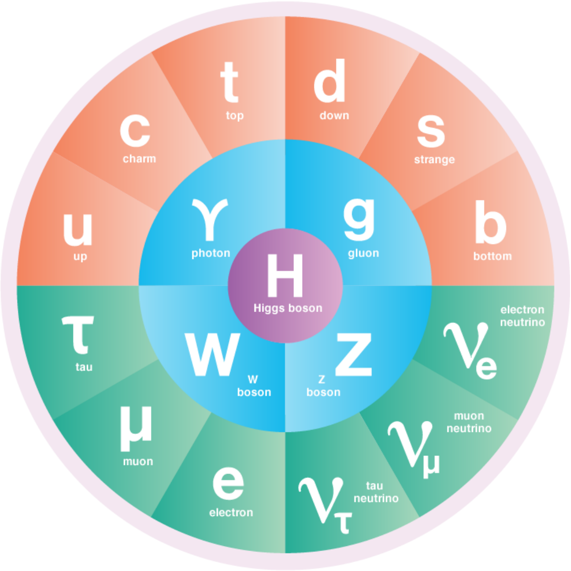
THE STANDARD MODEL OF PARTICLE PHYSICS



THE STANDARD MODEL IN A NUTSHELL

- The Standard Model describes incredibly well how Nature works to very short distances $\sim 10^{-19}$ m (or equivalently, to very high energies)

- The discovery of the Higgs boson at CERN in 2012 confirmed the existence of its last missing piece (and lead to the Nobel prize)

- However, we know that *there has to be something else* since it does not explain e.g.

  - Dark Matter
  - The matter-antimatter asymmetry of the Universe
  - Why gravity is so weak
  - Neutrino masses
  - Why fermion masses are so different

see Javi's lectures!

REFERENCES:

TEXTBOOKS:

- Paul Langacker (2017). *The Standard Model and Beyond*. Taylor & Francis. DOI: 10.1201/b22175
- Matthew D. Schwartz (2013). *Quantum Field Theory and the Standard Model*. Cambridge University Press. DOI: 10.1017/9781139540940
- John F. Donoghue, Eugene Golowich, and Barry R. Holstein (2014). *Dynamics of the Standard Model : Second edition*. Cambridge University Press. DOI: 10.1017/CBO9780511803512
- Pierre Ramond (1999). *Journeys beyond the standard model*. Perseus Books
- Ta-Pei Cheng and Ling-Fong Li (1984). *Gauge Theory of Elementary Particle Physics*. Oxford, UK: Oxford University Press
- S. Pokorski (2000). *Gauge Field Theores*. Cambridge University Press. DOI: 10.1017/CBO9780511612343
- E. Leader and E. Predazzi (1996). *An Introduction to gauge theories and modern particle physics*. Vol. 1. Cambridge University Press. DOI: 10.1017/CBO9780511622595

References:

More textbooks:

- R. Keith Ellis, W. James Stirling, and B. R. Webber (1996). *QCD and collider physics*. Cambridge University Press. DOI: 10.1017/CBO9780511628788
- Guido Altarelli (2017). *Collider Physics within the Standard Model: A Primer*. Ed. by James Wells. Vol. 937. Springer. DOI: 10.1007/978-3-319-51920-3
- Gustavo C. Branco, Luis Lavoura, and Joao P. Silva (1999). *CP Violation*. Clarendon Press

Lectures:

- José Ignacio Illana and Alejandro Jimenez Cano (2022). "Quantum field theory and the structure of the Standard Model". In: DOI: 10.22323/1.406.0314. arXiv: 2211.14636 [hep-ph]
- Antonio Pich (2012). "The Standard Model of Electroweak Interactions". In: arXiv: 1201.0537 [hep-ph]
- A. Pich (1995). "Chiral perturbation theory". In: DOI: 10.1088/0034-4885/58/6/001. arXiv: hep-ph/9502366

REFERENCES:

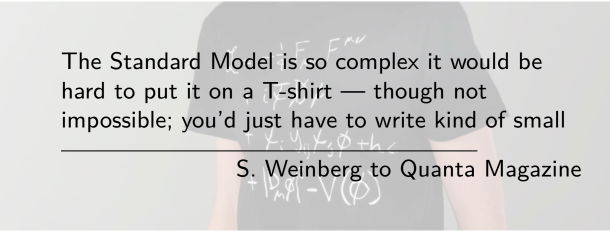
LECTURES:

- G. P. Salam (2020). "Elements of QCD for hadron colliders". In: *CERN Yellow Rep. School Proc.* 5. Ed. by M. Mulders and J. Trân Thanh Vân, pp. 1–56. DOI: 10.23730/CYRSP-2020-005.1
- Peter Skands (2013). "Introduction to QCD". In: *Theoretical Advanced Study Institute in Elementary Particle Physics: Searching for New Physics at Small and Large Scales*, pp. 341–420. DOI: 10.1142/9789814525220_0008. arXiv: 1207.2389 [hep-ph]

AND MANY MORE!

WHAT IS THE SM?

[Glashow 1961; A. Salam 1968; S. Weinberg 1967]



> The Standard Model is so complex it would be hard to put it on a T-shirt — though not impossible; you'd just have to write kind of small
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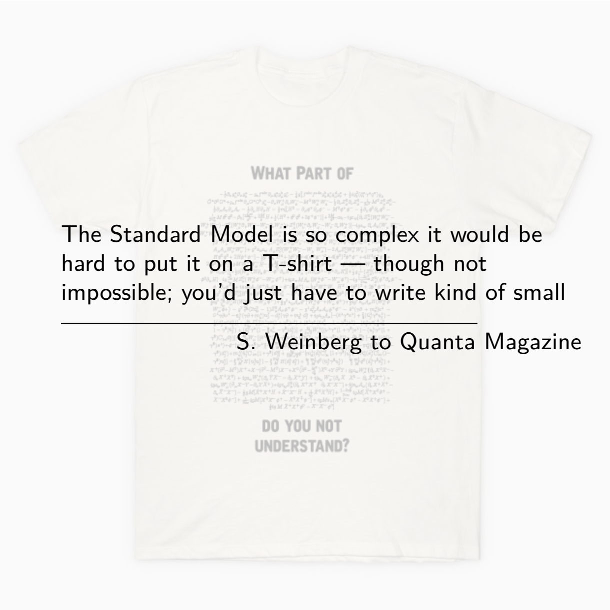
> S. Weinberg to Quanta Magazine

WHAT IS THE SM?



The Standard Model is so complex it would be hard to put it on a T-shirt — though not impossible; you'd just have to write kind of small

S. Weinberg to Quanta Magazine

What is the SM?

The SM is a local quantum field theory (QFT) defined by

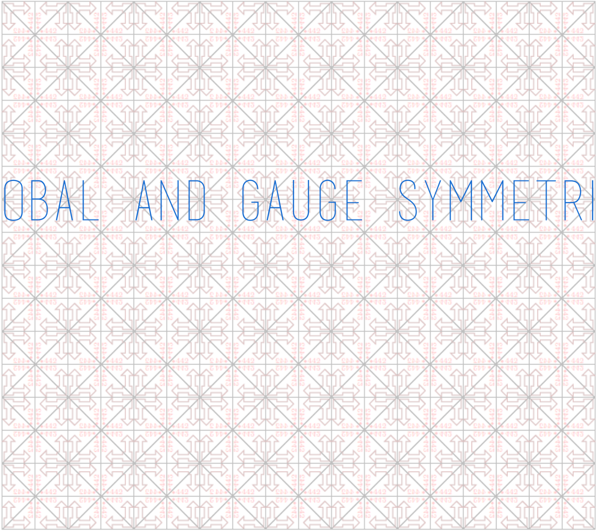
1. its (gauge) symmetry group
2. its matter content
3. the condition of renormalizability

together with a specific pattern of spontaneous symmetry breaking.

We will come back to all these ingredients one by one.

GLOBAL AND GAUGE SYMMETRIES

GLOBAL AND GAUGE SYMMETRIES



SPACE - TIME SYMMETRIES

Relativistic fields in QFT span infinite-dimensional vector spaces transforming under irreps of the Poincaré group (Lorentz group + translations) $\mathrm{ISO}(1,3)$

$$[P^\mu, P^\nu] = 0, \quad [P^\mu, J^{\rho\sigma}] = i(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho), \quad J^k = \frac{1}{2}\epsilon^{klm}J^{lm}$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}), \quad K^k = J^{0k} = -J^{k0}$$

It is known that finite-dimensional representations of a simple non-compact Lie group are not unitary. This is what happens e.g. for the Poincaré group, whose *unitary* representations are *infinite-dimensional*.

The Lorentz group is locally isomorphic to $SU(2) \otimes SU(2)$ (same algebra), whose irreps can be labelled by (j_-, j_+), $j_\pm \in \{0, 1/2, 1, \ldots\}$

Quantum fields

$\phi(x), A_\mu(x), \psi(x) \ldots$

$(0,0)$ $(1/2,1/2)$ $(1/2,0),$

Fock space

$|0\rangle \quad |1\rangle, \ldots,$

SPACE - TIME SYMMETRIES

Once quantized, these fields will act on the Fock space of multiparticle states. One can consider the Hilbert subspace of one-particle states invariant under Poincaré transformations, and study the resulting irreps

$$|0\rangle \quad (\text{vacuum}), \qquad a^\dagger_{\mathbf{p},s}|0\rangle \quad (1 \text{ particle}), \qquad b^\dagger_{\mathbf{p},s}|0\rangle \quad (1 \text{ antiparticle})$$

These irreps need to be unitary, to make the scalar products invariant under a change of reference system

$$\langle\psi_1|\psi_2\rangle = \langle\psi_1|\mathcal{R}^\dagger\mathcal{R}|\psi_2\rangle.$$

This implies in particular that such irreps will be infinite-dimensional.

We can classify the one-state particles by the Casimir operators

$$m^2 \equiv P_\mu P^\mu \qquad W_\mu W^\mu, \qquad W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma.$$

since their eigenvalues will label the different 'irreps'.

SPACE - TIME SYMMETRIES

Case $m \neq 0$, we choose $P^\mu = (m, 0, 0, 0)$, which leads to

$$W^0 = 0,\ W^i = -\frac{m}{2}\epsilon^{ijk0}J^{jk} = \frac{m}{2}\epsilon^{ijk}J^{jk} = mJ^i \Rightarrow W_\mu W^\mu = -m^2 j(j+1)$$

The irreps are labeled by m and j and the vectors by $|j_3 = -j, \ldots, j\rangle$

Massive particles of spin j have $2j+1$ dof and $SU(2)$ is the little group

Case $m = 0$, we choose $P^\mu = (\omega, 0, 0, \omega)$, which leads to

$$\begin{aligned} -W^0 = W^3 = \omega J^3, \quad W^{1,2} = \omega(J^1 \pm K^2) \\ \Rightarrow W_\mu W^\mu = -\omega^2[(J^1 + K^2)^2 + (J^2 - K^1)^2] \end{aligned}$$

Now the little group is $SO(2)$ and the irreps are unidimensional and are labeled by the helicity $h \in \{0, \pm 1/2, \pm 1, \ldots\}$.

SPACE - TIME SYMMETRIES

If we want to build a QFT with a masless vector field $V_\mu = \mathbf{4}$ $(= \mathbf{1} \oplus \mathbf{3}$ under the rotation group $SO(3))$, we need to be sure that just propagate transverse polarizations.

Indeed polarizations related by a multiple of p_μ are related by a Poincaré transformation and should be considered equivalent

$$\epsilon_\mu \to \epsilon_\mu + \alpha p_\mu$$

This looks very similar to

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

Gauge redundancy! It is a redundancy of our QFT but not a symmetry. There is no conserved charge.

GLOBAL SYMMETRIES

For the sake of concreteness, let us consider a free-fermion theory

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\Psi, \qquad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu, \quad \overline{\psi} = \psi^\dagger \gamma^0$$

This Lagrangian density and the resulting action are invariants under a transformation

$$\psi(x) \to e^{-iq\theta}\psi(x), \qquad q, \theta \in \mathbb{R}$$

Noether theorem tell us that there is a conserved current

$$j^\mu = q\bar{\psi}\gamma^\mu q, \qquad \partial_\mu j^\mu = 0$$

as well as a Noether charge $[Q, H] = 0$

$$Q = q \int \mathrm{d}^3x \, : \bar{\psi}\gamma^0\psi \, := q \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s=1,2} \left(a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} - b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s} \right)$$

$$Q a^\dagger_{\mathbf{k},s}|0\rangle = +q a^\dagger_{\mathbf{k},s}|0\rangle \quad (\text{particle}), \quad Q b^\dagger_{\mathbf{k},s}|0\rangle = -q b^\dagger_{\mathbf{k},s}|0\rangle \quad (\text{antiparticle}),$$

GLOBAL SYMMETRIES

In general, global symmetries will be described by N-dimensional compact Lie groups

$$g \in G, \qquad g(\theta) = e^{-iT^a\theta^a} \qquad \theta^a \in \mathbb{R}, \qquad a = 1, \dots, N,$$

where T^a are the generators, satisfying the Lie algebra

$$[T^a, T^b] = if^{abc}T^c, \qquad \mathbf{Tr}(T^a \cdot T^b) = \frac{1}{2}\delta^{ab}.$$

They will have associated a conserved current and fields will transform

$$\psi(x) \to U(\theta)\psi(x) = \exp(-iT^a\theta^a)\psi(x), \qquad \psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{pmatrix}$$

where T^a is represented by an hermitian d-dimensional square matrix.

THE GLOBAL SYMMETRIES OF THE SM

The SM has the following global symmetry

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$$

- $U(1)$: 1 generator (q), one-dimensional irreps only

- $SU(2)$: 3 generators, $f^{abc} = \epsilon_{abc}$, Levi-Civita symbol
  - Fundamental representation ($d = 2$), $\mathbf{2}$, $T^a = \sigma^a/2,\ a = 1, 2, 3$
  - Adjoint representation ($d = N = 3$), $\mathbf{3}$, $(T^a)_{bc} = -if^{abc}$

Note that the fundamental is pseudo-real since if $\phi \sim \mathbf{2}$, $i\sigma^2\phi^* \sim \mathbf{2}$

- $SU(3)$: 8 generators

$$f^{123} = 1, f^{458} = f^{678} = \frac{\sqrt{3}}{2}, f^{147} = f^{156} = f^{246} = f^{247} = f^{345} = -f^{367}$$

  - Fundamental representation ($d = 3$), $\mathbf{3}$, $T^a = \lambda^a/2$, $a = 1, \ldots, 8$
  - Adjoint representation ($d = N = 8$), $\mathbf{8}$, $(T^a)_{bc} = -if^{abc}$

WHAT ABOUT GAUGE 'SYMMETRIES'?

One can turn any global symmetry of the theory into something local by correcting the difference from point x to point y with a connection.

When we talk about gauge theories we just mean that the connection is indeed physical.

In practice,

$$\theta^a = \theta^a(x) \Rightarrow U(\theta) = \exp\left(-iT^a\theta^a(x)\right), \quad \psi(x) \to \exp\left(-iT^a\theta^a(x)\right)\psi(x)$$

The covariant derivative of the field

$$D_\mu\psi(x) = (\partial_\mu - igT^aA^a_\mu(x))\psi(x) = (\partial_\mu - ig\hat{A}_\mu(x))\psi(x)$$

needs to transform as the field, i.e., $D_\mu\phi(x) \to UD_\mu\phi(x)$. This is true if

$$\hat{A}_\mu(x) \to U\hat{A}_\mu(x)U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger, \qquad A^a_\mu \to A^a_\mu - f^{abc}A^b_\mu\theta^c - \frac{1}{g}\partial_\mu\theta^a.$$

Then, $\bar{\psi}i\not{D}\psi$ is invariant under a local transformation U.

DYNAMICS OF THE GAUGE FIELDS

Since we said that the connection is physical, we should add kinetic terms for them. This is the so-called Yang-Mills Lagrangian [Yang and Mills 1954]

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{Tr}(\hat{A}_{\mu\nu}\hat{A}^{\mu\nu}) = -\frac{1}{4}A^a_{\mu\nu}A^{a\mu\nu}$$

where

$$\begin{aligned}\hat{A}_{\mu\nu} &= A^a_{\mu\nu}T^a = D_\mu\hat{A}_\nu - D_\nu\hat{A}_\mu = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - ig[\hat{A}_\mu, \hat{A}_\nu] \\ &\Rightarrow A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu\end{aligned}$$

Then, we can write the first terms of our SM t-shirt

$$\mathcal{L}_{\mathrm{gauge}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W^{i\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

where G^a_μ, W^i_μ, B_μ with $a = 1, \dots, 8$, $i = 1, 2, 3$, are the gauge bosons of $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$, respectively.

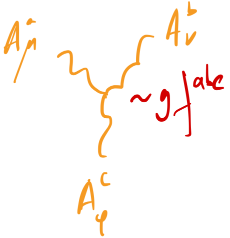
DYNAMICS OF THE GAUGE FIELDS

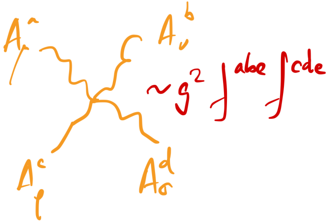
The Yang-Mills Lagrangian includes cubic and quartic self interactions

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})$$

$$\mathcal{L}_{\text{cubic}} = -\frac{1}{2} g f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu}_\mu A^{c\nu}$$

$$\mathcal{L}_{\text{quartic}} = -\frac{1}{4} g^2 f^{abe} f^{cde} A^a_\mu A^b_\nu A^{c\mu} A^{d\nu}$$





Dynamics of the gauge fields

Is this everything? What happens with terms like the one below?

$$\mathcal{L}_\theta = \theta \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \widetilde{G}^{a\mu\nu}, \qquad \widetilde{G}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma}$$

The short answer is that it depends. One can write down such a term like the derivative of a current

$$\mathcal{L}_\theta = \theta \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \widetilde{G}^{a\mu\nu} = \partial_\mu \mathcal{K}^\mu$$

When this is the case, due to the generalized Stokes theorem

$$S_\theta = \int_{\mathcal{M}_4} \mathrm{d}^4x \, \mathcal{L}_\theta = \oint_{\partial \mathcal{M}_4} dS \, \mathcal{K}_\mu n^\mu$$

If $\mathcal{K}^\mu$ goes to zero fast enough when go to the infinite, there is no contribution and we can forget about such term. We will come back to this later (strong CP problem).

QUANTIZING A GAUGE THEORY

Within the path-integral formalism, Green functions are obtained by derivating the generating functional

$$\mathcal{Z}[J] = \mathcal{N} \int \mathcal{D}\phi \exp\left\{ i \int \mathrm{d}^4x \left[\mathcal{L}(\phi) + J\phi \right] \right\}, \qquad \mathcal{Z}[0] = 1$$

(one should define $\mathcal{Z}[J]$ in the Euclidean space and then continue it analytically to the Minkowsky space). In particular

$$G^{(N)}(x_1, x_2, \ldots, x_N) = \frac{1}{i^N} \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_N)} \mathcal{Z}[J] \bigg|_{J=0}$$

Then, cross-sections are obtained through the LSZ theorem [Lehmann, Symanzik, and Zimmermann 1955].

QUANTIZING A GAUGE THEORY

In gauge theories, redundancy makes everything more complicated. Indeed, since $S_{\mathrm{YM}}[\hat{A}_\mu] = S_{\mathrm{YM}}[\hat{A}_\mu^U]$, where

$$\hat{A}_\mu^U = U\hat{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$$

Therefore, in evaluating

$$\mathcal{Z}[J] = \mathcal{N}\int \mathcal{D}A_\mu \exp\Big(iS_{\mathrm{YM}}[\hat{A}_\mu] + i\int d^4x\ J_\mu^a A^{a\mu}\Big)$$

we would like to integrate over just one representative A_μ^U of each equivalence class defined by gauge transformations, where

$$\int \mathcal{D}A_\mu = \int \mathcal{D}U \int \mathcal{D}A_\mu^U.$$

QUANTIZING A GAUGE THEORY

Therefore, we fix a particular gauge

$$F[A_\mu] = 0$$

with $F[A_\mu]$ any functional of A_μ, like e.g. $F[A_\mu] = \partial_\mu A^\mu$. Then

$$\mathcal{D}A_\mu^U = \mathcal{D}A_\mu \delta(A_\mu \sim A_\mu^U) = \mathcal{D}A_\mu \delta[F[A_\mu]] \mathrm{det} M$$

with

$$M(x,y) = \left. \frac{\delta F[A_\mu(x)]}{\delta U(y)} \right|_{F=0},$$

analogously to

$$\delta(x - x_0) = \delta(f(x)) \left| \frac{\partial f}{\partial x} \right|_{f(x_0)=0}.$$

QUANTIZING A GAUGE THEORY

If we consider gauge-fixing conditions of the form

$$F[A_\mu] - C(x) = 0$$

with C an arbitrary, independent function of A_μ, $\det M$ becomes independent of $C(x)$ and we can write

$$\begin{aligned}\int \mathcal{D}A_\mu^U &= \int \mathcal{D}A_\mu \delta[F[A_\mu] - C(x)]\,\det M \\ &= \mathcal{N} \int \mathcal{D}A_\mu \,\det M \int \mathcal{D}C\, \delta[F[A_\mu] - C(x)] G[C] \\ &= \mathcal{N} \int \mathcal{D}A_\mu \,\det M\, G[F[A_\mu]]\end{aligned}$$

with $G[C]$ some arbitrary functional like

$$G[C] = \exp\Big\{ -\frac{i}{\xi} \int d^4x\ C^2(x) \Big\}.$$

QUANTIZING A GAUGE THEORY

Therefore, we can avoid summing over equivalent gauge configurations by replacing

$$\int \mathcal{D}A_\mu \to \int \mathcal{D}A_\mu \, \det M \exp\Big\{ -\frac{i}{\xi} \int d^4x \;\, F^2[A_\mu] \Big\}.$$

This expression can be further simplified by taking into account that

$$\det M = \int \mathcal{D}\bar{\eta}\, \mathcal{D}\eta \exp\Big\{ -i \int \mathrm{d}^4x\, \mathrm{d}^4y \bar{\eta}(x) M(x,y) \eta(y) \Big\},$$

$$M(x,y) = \left| \frac{\delta F[A_\mu(x)]}{\delta U(y)} \right|_{F=0}$$

where η and $\bar{\eta}$ are Grassmann variables.

QUANTIZING A GAUGE THEORY

This leads to [Faddeev and Popov 1967]

$$\mathcal{Z}[J] = \mathcal{N}' \int \mathcal{D}A_\mu \, \mathcal{D}\bar{\eta} \, \mathcal{D}\eta \exp\Big\{ i \int d^4x [\mathcal{L}_{\text{total}} + J^\mu A_\mu] \Big\},$$

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} = -\frac{1}{4} A^a_{\mu\nu} A^{a\mu\nu} - \frac{1}{2\xi} F^2[A_\mu] - \bar{\eta} M \eta,$$

where η and $\bar{\eta}$ are ghosts, anti-conmuting scalar fields. They are *unphysical* but need to be included in loops. For the Lorentz gauge,

$$F[A^a_\mu] = \partial_\mu A^{a\mu} \Rightarrow \mathcal{L}_{\text{GF}} = -\sum_a \frac{1}{2\xi_a} (\partial^\mu A^a_\mu)^2$$

and

$$F[A^a_\mu] \mapsto F[A^a_\mu] - f^{abc} \partial^\mu (A^b_\mu \theta^c) - \frac{1}{g} \Box \theta^a$$

where we have used that under a gauge transformation $U = e^{-iT^a \theta^a(x)}$,

$$A^a_\mu \mapsto A^a_\mu - f^{abc} A^b_\mu \theta^c - \frac{1}{g} \partial_\mu \theta^a.$$

QUANTIZING A GAUGE THEORY

In this case, the Faddeev-Popov determinant reads

$$M_{ab} = \frac{\delta F[A^a_\mu]}{\delta\theta^b} = f^{abc}(\partial^\mu A^c_\mu + A^c_\mu \partial^\mu) - \frac{1}{g}\delta^{ab}\Box = -\frac{1}{g}(\delta^{ab}\Box - g f^{abc} A^c_\mu \partial^\mu).$$

We can get rid of the $1/g$ factor by redefining η and $\bar{\eta}$, leading to

$$\begin{aligned}\det M &= \int \mathcal{D}\bar{\eta}\,\mathcal{D}\eta \exp\Big\{-i\int \mathrm{d}^4x\,\bar{\eta}^a(\delta^{ab}\Box - g f^{abc} A^c_\mu \partial^\mu)\eta^b\Big\} \\ &\overset{\text{IBP}}{=} \int \mathcal{D}\bar{\eta}\,\mathcal{D}\eta \exp\Big\{i\int \mathrm{d}^4x\,[\partial^\mu\bar{\eta}^a \partial_\mu \eta^a - g f^{abc}(\partial^\mu \bar{\eta}^a)\eta^b A^c_\mu]\Big\}\end{aligned}$$

Then

$$\mathcal{L}_{\text{FP}} = (\partial^\mu \bar{\eta}^a)(\partial_\mu \eta^a - g f^{abc}\eta^b A^c_\mu) = (\partial^\mu\bar{\eta}^a)(D^{\text{Adj}}_\mu)^{ab}\eta^b.$$

QUANTIZING A GAUGE THEORY

The full Lagrangian for a Yang-Mills theory

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}$$

is invariant under the following BRS transformations [Becchi, Rouet, and Stora 1976] (with θ a Grassmann variable, $\theta^2 = 0$)

$$\delta A_\mu^a = -\frac{1}{g}\theta (D_\mu^{\text{Adj}})^{ab}\eta^b, \quad \delta\bar{\eta}^a = \frac{1}{g}\theta\frac{1}{\xi}\partial^\nu A_\nu^a, \quad \delta\eta^a = \frac{1}{2}\theta f^{abc}\eta^b\eta^c,$$

where $\varphi \to \varphi + \delta\varphi = \varphi + \theta\Delta(\varphi)$, $\varphi = A_\mu^a, \eta^a, \bar{\eta}^a$. BRS invariance of the effective action Γ (generation functional of all 1PI diagrams – see later)

$$\Delta(\Gamma) = 0 = \int d^4x \left[\frac{\delta\Gamma}{\delta\varphi}\delta\varphi + \ldots\right] \Rightarrow \left.\frac{\delta\Delta(S)}{\delta\varphi_j \cdots}\right|_{\varphi=0} = 0 \quad \text{ST identities.}$$

Slavnov-Taylor (ST) [Slavnov 1972; Taylor 1971] identities are relations between vertex functions. They summarize all Ward identities and are essential for renormalization (BRS do not change in $D = 4 - 2\varepsilon$).

THE FULL SM GAUGE LAGRANGIAN

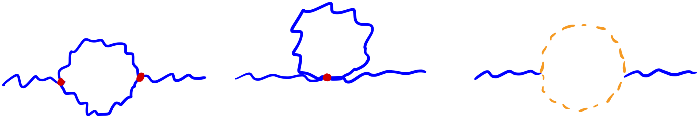
Then, the complete SM gauge Lagrangian reads (calling c_g^a, c_w^i, c_b the FP ghost fields of $SU(3)_C, SU(2)_L$ and $U(1)_Y$, respectively)

$$\mathcal{L}_G^{\text{tot}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} = -\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{4}W_{\mu\nu}^i W^{i\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

$$- \frac{1}{2\xi_a}(\partial^\mu G_\mu^a)^2 - \frac{1}{2\xi_i}(\partial^\mu W_\mu^i)^2 - \frac{1}{2\xi}(\partial^\mu B_\mu)^2$$

$$+ (\partial^\mu \bar{c}_g^a)(\partial_\mu c_g^a - g_s f^{abc} c_g^b G_\mu^c) \ + (\partial^\mu \bar{c}_w^i)(\partial_\mu c_w^i - g\epsilon^{ijk} c_w^i W_\mu^k)$$

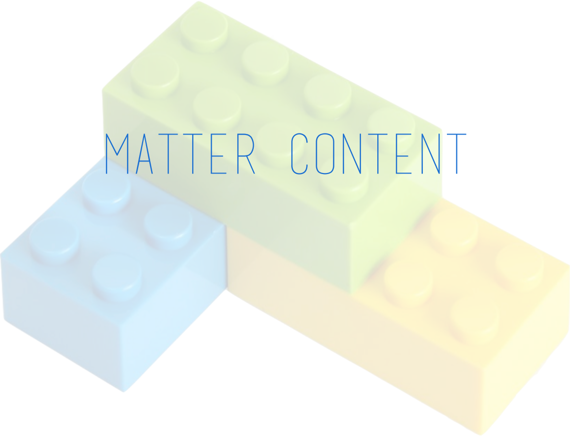
$$+ \partial^\mu \bar{c}_b \partial_\mu c_b$$



$$\Pi_{\mu\nu} = i\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\Pi(k^2)$$

$$k^\mu \Pi_{\mu\nu} = 0$$

MATTER CONTENT



MATTER CONTENT

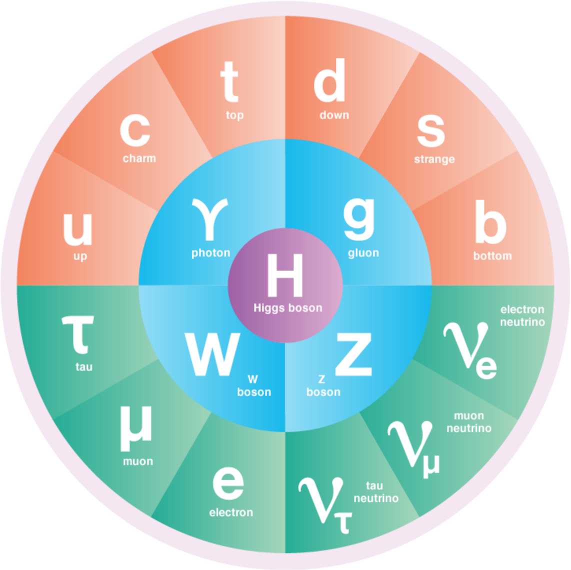
THE SM BUILDING BLOCKS

The matter of the SM is made of chiral fermions, transforming under $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$

Multiplet	Quantum Numbers	I	II	III	Q
q_L^i	$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})$	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$	$\begin{matrix} +2/3 \\ -1/3 \end{matrix}$
u_R^i	$(\mathbf{3}, \mathbf{1}, +\frac{2}{3})$	u_R	c_R	t_R	$+2/3$
d_R^i	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})$	d_R	s_R	b_R	$-1/3$
ℓ_L^i	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$	$\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}$	$\begin{pmatrix} \nu_L^\mu \\ \mu_L \end{pmatrix}$	$\begin{pmatrix} \nu_L^\tau \\ \tau_L \end{pmatrix}$	$\begin{matrix} 0 \\ -1 \end{matrix}$
e_R^i	$(\mathbf{1}, \mathbf{1}, -1)$	e_R	μ_R	τ_R	-1

As we will see later, $Q = T_L^3 + Y$

THE SM BUILDING BLOCKS



THE SM MATTER CONTENT

With these ingredients we can now write

$$\mathcal{L}_{\rm ferm} = \bar{q}_L^i i\not{D} q_L^i + \bar{u}_R^i i\not{D} u_R^i + \bar{d}_R^i i\not{D} d_R^i + \bar{\ell}_L^i i\not{D} \ell_L^i + \bar{e}_R^i i\not{D} e_R^i$$

This Lagrangian is invariant under unitary rotations of the fields

$$\mathcal{G}_{\rm ferm} = U(3)^5 = U(1)^5 \otimes \mathcal{G}_q \otimes \mathcal{G}_l$$

where

$$\mathcal{G}_q = SU(3)_{q_L} \otimes SU(3)_{u_R} \otimes SU(3)_{d_R} \qquad \mathcal{G}_l = SU(3)_{\ell_L} \otimes SU(3)_{e_R}$$

This flavor group will be broken by the Higgs interactions (see later)

Mass terms are fordbidden by the global symmetries of the SM. E.g.,

$$-m\bar{q}_L u_R + \text{h.c.}$$

is not invariant under $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$.

THE SM MATTER CONTENT

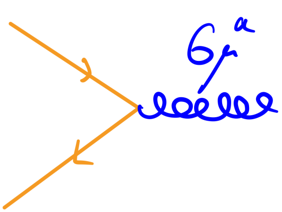
The covariant derivatives include the fermion interactions with the different gauge bosons of the SM

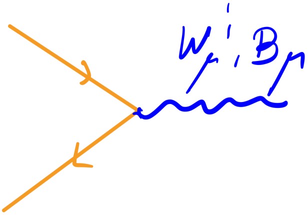
$$\begin{aligned}
\mathcal{L}_{\text{ferm}} = & \bar{q}^i_L i\gamma^\mu \Big[\partial_\mu - ig_s \frac{\lambda^a}{2} G^a_\mu - ig\frac{\sigma^i}{2} W^i_\mu - ig'\Big(\frac{1}{6}\Big) B_\mu\Big] q^i_L \\
& + \bar{u}^i_R i\gamma^\mu \Big[\partial_\mu - ig_s \frac{\lambda^a}{2} G^a_\mu - ig'(\frac{2}{3}) B_\mu\Big] u^i_R \\
& + \bar{d}^i_R i\gamma^\mu \Big[\partial_\mu - ig_s \frac{\lambda^a}{2} G^a_\mu - ig'(-\frac{1}{3}) B_\mu\Big] u^i_R \\
& + \bar{\ell}^i_L i\gamma^\mu \Big[\partial_\mu - ig\frac{\sigma^i}{2} W^i_\mu - ig'\Big(-\frac{1}{2}\Big) B_\mu\Big] \ell^i_L \\
& + \bar{e}^i_R i\gamma^\mu \Big[\partial_\mu - ig'(-1) B_\mu\Big] e^i_R
\end{aligned}$$

where $\sigma^i,\ i = 1, 2, 3$ are the Pauli matrices and $\lambda^a,\ a = 1, 2, \ldots, 8$ are the Gell-Mann ones (or any other equivalent representation of the corresponding algebras!).

THE SM MATTER CONTENT

The covariant derivatives include the fermion interactions with the different gauge bosons of the SM





RENORMALIZATION

RENORMALIZABILITY

['t Hooft and Veltman 1972]

Each operator of our Lagrangian contributes to the action (for a process with energies $E \ll \Lambda$):

$$\delta S_{(i,k)} \sim \mathcal{C}_{(i,k)} \left(\frac{E}{\Lambda}\right)^{k-4} ,$$

with k being the operator dimension. Operators are then classified by their importance at low energies ($E \to 0$):

k	low-energy behavior	classical renormalizability	name
< 4	grows	super-renormalizable	*relevant*
$= 4$	constant	renormalizable	*marginal*
> 4	decreases	non-renormalizable	*irrelevant*

The SM by definition only includes operators with $k \leq 4$. Including terms with $k > 4$ leads to the SM Effective Field Theory (SMEFT).

RENORMALIZABILITY

['t Hooft and Veltman 1972]

The condition of renormalizability allows us to compute the counter-term for a given parameter only once at a given order in perturbation theory.

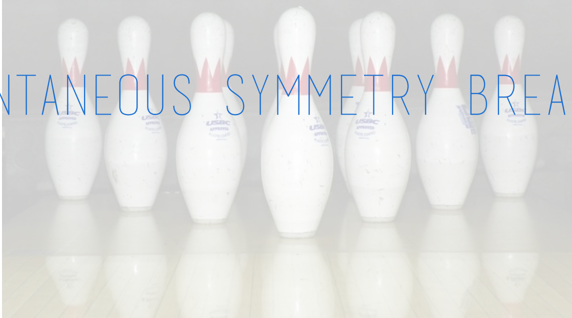
It is useful to remember that

$$[\psi] = \frac{3}{2}, \qquad [\phi] = 1, \qquad [\partial_\mu] = 1, \qquad [A_\mu] = 1$$

Then, for instance, terms like the ones below will be forbidden

$$\left(\bar{q}_L^i \gamma_\mu q_L^j\right)\left(u_R^k \gamma^\mu u_R^l\right), \qquad \left(\bar{q}_L^i \gamma_\mu \frac{\sigma^i}{2} q_L^j\right)\left(q_L^k \gamma^\mu \frac{\sigma^i}{2} q_L^l\right) \qquad f^{abc} G_\mu^{a\,\nu} G_\nu^{b\,\rho} G_\rho^{c\,\mu}.$$

SPONTANEOUS SYMMETRY BREAKING



SPONTANEOUS SYMMETRY BREAKING

SPONTANEOUS SYMMETRY BREAKING

If the symmetry group of the SM is not broken, fermions and gauge bosons will be massless. A way out is given by spontaneous symmetry breaking (SSB).

Our experimental knowledge tell us that the breaking pattern has to be

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \mapsto SU(3)_C \otimes U(1)_Q$$

SPONTANEOUS SYMMETRY BREAKING

Imagine a simple theory of a scalar field ϕ with a potential $V(\phi)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)$$

The generating functional $\mathcal{Z}[J]$ will read

$$\mathcal{Z}[J] = e^{i\mathcal{W}[J]} = \mathcal{N} \int \mathcal{D}\phi \exp\Big\{ i \int d^4x \left[\mathcal{L}(\phi) + J(x)\phi(x) \right] \Big\}$$

We define the vacuum expectation value (vev) of $\phi(x)$ in presence of J

$$\begin{aligned} \bar{\phi}(x) &= \frac{\delta \mathcal{W}[J]}{\delta J(x)} = \frac{1}{\mathcal{Z}[J]} \int \mathcal{D}\phi \, \phi(x) \exp\Big\{ i \int d^4x \left[\mathcal{L}(\phi) + J(x)\phi(x) \right] \Big\} \\ &= \left[\frac{\langle 0|\phi(x)|0\rangle}{\langle 0|0\rangle} \right]_J \end{aligned}$$

which is the 'conjugated' of J. We assumme that we can invert the relationship between $\bar{\phi}$ and J, i.e., $\bar{\phi} = \bar{\phi}[J]$ and $J = J[\bar{\phi}]$.

The Effective Potential

Using $J = J[\bar{\phi}]$, we can thus define the Legendre transform of $\mathcal{W}[J]$

$$\Gamma[\bar{\phi}] = \mathcal{W}[J] - \int d^4x\, J(x)\bar{\phi}(x)$$

One can see that,

$$\begin{aligned}\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)} &= \int d^4y\, \frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\bar{\phi}(x)} - \int d^4y \left\{\frac{\delta J(y)}{\delta\bar{\phi}(x)}\bar{\phi}(y) + J(y)\frac{\delta\bar{\phi}(y)}{\delta\bar{\phi}(x)}\right\} \\ &= \int d^4y\, \bar{\phi}(y)\frac{\delta J(y)}{\delta\bar{\phi}(x)} - \int d^4y \left\{\frac{\delta J(y)}{\delta\bar{\phi}(x)}\bar{\phi}(y) + J(y)\delta(y-x)\right\} = -J(x)\end{aligned}$$

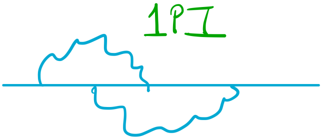
and thus

$$J(x) = -\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)}$$

It can be shown that, while $\mathcal{W}[J]$ is the generating functional of the connected diagrams, $\Gamma[\bar{\phi}]$ generates 1PI-diagrams.

THE EFFECTIVE POTENTIAL

1 particle irreducible (1PI) diagrams are those which can not be separated in two by cutting a single line



Diagrams which are not 1PI can be generated via 1PI ones, thus Γ is enough to generate the S matrix elements. Moreover

$$\Gamma[\bar{\phi}] = S[\bar{\phi}] + \frac{i\hbar}{2}\mathrm{Tr}\left[\log \left.\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\bar{\phi}}\right] + \mathcal{O}(\hbar^2).$$

The effective potential

In the case of $J=0$ we know that $\delta\Gamma[\bar{\phi}]/\delta\bar{\phi}=0$. We will have SSB if

$$\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)}\bigg|_{\bar{\phi}(x)=\langle\phi(x)\rangle\neq 0}=0$$

for a non-vanishing configuration $\bar{\phi}(x)|_{J=0}=\langle\phi(x)\rangle$. In general, if we expand in derivatives,

$$\Gamma[\bar{\phi}]=\int d^4x\left[-V_{\text{eff}}(\bar{\phi})+\frac{1}{2}(\partial_\mu\bar{\phi})(\partial^\mu\bar{\phi})Z(\bar{\phi})+\ldots\right]$$

and assuming a translationally invariant vev, i.e., $\langle\phi(x)\rangle=\langle\phi\rangle$, this condition is translated to

$$\frac{dV_{\text{eff}}(\bar{\phi})}{d\bar{\phi}}\bigg|_{\bar{\phi}=\langle\phi\rangle}=0.$$

THE EFFECTIVE POTENTIAL

It can be seen that

$$V_{\text{eff}}(\bar{\phi})\big|_{\bar{\phi}=\langle\phi\rangle} = \mathcal{E}_0$$

where $\mathcal{E}_0$ is the energy density of the ground state $|0\rangle$, i.e.,

$$\mathcal{E}_0 = \langle 0|\mathcal{H}|0\rangle.$$

It can also be shown that

$$V_{\text{eff}}(\bar{\phi}) = V(\bar{\phi}) + \frac{\hbar}{2}\int \frac{d^4 p_E}{(2\pi)^4}\log\Big(p_E^2 + V''(\bar{\phi})\Big) + \mathcal{O}(\hbar^2)$$

This is the so-called Coleman-Weinberg potential [Coleman and E. J. Weinberg 1973]

EXAMPLE I

Consider a real scalar field with

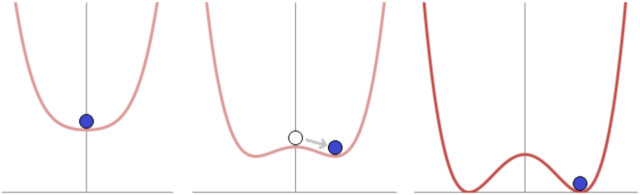
$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4, \qquad \lambda > 0$$

invariant under $\phi \mapsto -\phi$. Then

$$\mathcal{H} = \frac{1}{2}\left[(\partial_0\phi)^2 + (\nabla\phi)^2\right] + V(\phi)$$

For $\mu^2 > 0$ the minimum of the potential happens for $\langle\phi\rangle = 0$, while for $\mu^2 < 0$ we have

$$\langle\phi\rangle = v = \pm\sqrt{\frac{-\mu^2}{\lambda}}$$



EXAMPLE I

In order to have a quantum field with no vev (such that $a|0\rangle = 0$), we do

$$\phi(x) = v + \eta(x) \qquad \langle \eta \rangle = 0$$

At the quantum level,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \lambda v^2 \eta^2 - \lambda v \eta^3 - \frac{\lambda}{4}\eta^4$$

which is not invariant under $\eta \mapsto -\eta$.

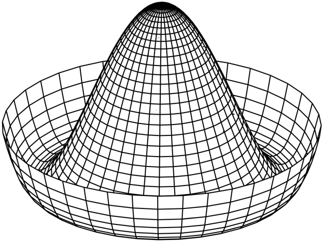
Even if $\mathcal{L}$ feature some symmetry, it can happen that the parameters are such that the ground state of the Hamiltonian is not symmetric $\Leftrightarrow$ SSB

EXAMPLE II

Let us consider now a complex scalar field $\phi(x)$ with Lagrangian

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2|\phi|^2 - \lambda|\phi|^4$$

invariant under $U(1)$ rotations $\phi \mapsto e^{i\alpha}\phi$. If we assume $\lambda > 0, \mu^2 < 0$



and we obtain

$$\langle 0|\phi|0\rangle = \frac{v}{\sqrt{2}}, \qquad |v| = \sqrt{\frac{-\mu^2}{\lambda}}$$

EXAMPLE II

If we take $v > 0$, we can write

$$\phi(x) = \frac{1}{\sqrt{2}}\Big[v + \eta(x) + i\chi(x)\Big], \qquad \langle 0|\eta|0\rangle = \langle 0|\chi|0\rangle = 0$$

and

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \lambda v^2 \eta^2 - \lambda v \eta(\eta^2 + \chi^2) \\ - \frac{\lambda}{4}(\eta^2 + \chi^2)^2 + \frac{\lambda}{4} v^4, \qquad m_\eta = \sqrt{2\lambda} v, \qquad m_\chi = 0.\end{aligned}$$

$U(1)$ is no longer respected and one scalar remains massless $\Rightarrow$

Goldstone theorem [Goldstone 1961; Nambu 1960]

GOLDSTONE THEOREM

Consider a theory with n real scalar fields, ϕ_i, $i = 1, \ldots, n$ featuring some global continuous symmetry G, generated by charges Q^a. Under G

$$\phi_i \to \phi_i' \approx \phi_i - i\Theta^a T^a_{ij}\phi_j \Longleftrightarrow \left[Q^a(t), \phi_i(\mathbf{x}, t)\right] = -T^a_{ij}\phi_j(\mathbf{x}, t)$$

where

$$j^a_\mu(x) = -i\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_i)} T^a_{ij}\phi_j = -i(\partial_\mu \phi_i)T^a_{ij}\phi_j, \quad Q^a(t) = \int d^3\mathbf{x}\, j_0(\mathbf{x}, t).$$

We also assume that $\langle 0|\phi_i|0\rangle = v_i \neq 0,\ i = 1, \ldots, n$, and consider

$$G^a_{\mu,k}(x-y) = \langle 0|T\{j^a_\mu(x)\phi_k(y)\}|0\rangle = \theta(x^0 - y^0)\langle 0|j^a_\mu(x)\phi_k(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi_k(y)j^a_\mu(x)|0\rangle$$

such that

$$\partial^\mu_x G^a_{\mu,k}(x-y) = \delta(x^0 - y^0)\langle 0|\left[j^a_0(x), \phi_k(y)\right]|0\rangle.$$

On the other hand, assuming translational invariance,

$$\left[j^a_0(\mathbf{x}, t), \phi_k(\mathbf{y}, t)\right] = -T^a_{kj}\phi_j(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{y})$$

$$\Rightarrow \partial^\mu_x G^a_{\mu,k}(x-y) = -\delta(x-y)T^a_{kj}\langle 0|\phi_j(0)|0\rangle$$

GOLDSTONE THEOREM

If we introduce its Fourier transform

$$G^a_{\mu,k}(x-y) = \int \frac{d^4p}{(2\pi)^4} \exp\Big(-ip_\mu(x-y)^\mu\Big)\tilde{G}^a_{\mu,k}(p),$$

we obtain

$$ip^\mu \tilde{G}^a_{\mu,k}(p) = T^a_{kj}\langle 0|\phi_j(0)|0\rangle.$$

Lorentz invariance imposes that $\tilde{G}^a_{\mu,k}(p) = p_\mu F^a_k(p^2)$ and thus

$$F^a_k(p^2) = -iT^a_{kj}\langle 0|\phi_j(0)|0\rangle \frac{1}{p^2}$$

Therefore, $\langle 0|\phi_j(0)|0\rangle = v_j \neq 0 \Leftrightarrow$ we have poles at $p^2 = 0$.

There is one massless boson for each 'broken' generator, i.e., $Q_a|0\rangle \neq 0$.

In the case of gauge symmetries, gauge-fixing requires the specification of some four-vector n_μ and $G^a_{\mu,k}(p) \neq p_\mu F^a_k(p^2)$. Loophole!

Goldstone Theorem

For a global symmetry given by the Lie group G, one should have

$$\Big[Q_a(t), H\Big] = 0, \qquad a = 1, \dots, n, \qquad H = \int d^3\mathbf{x}\, \mathcal{H}(\mathbf{x}, t).$$

By definition

$$H|0\rangle = 0 \Rightarrow H(Q_a|0\rangle) = Q_a H|0\rangle = 0, \qquad a = 1, \dots, n.$$

Therefore, $Q_a|0\rangle$ is also a vacuum state. There are two possibilites:

1. $Q_a|0\rangle = 0, \forall a$. There is just one vacuum.
2. $\exists\, A \subset \{1, \dots, n\} \,|\, \forall a' \in A,\, Q'_a|0\rangle \neq 0$. Then

   $$e^{-iQ_{a'}\Theta^{a'}}|0\rangle$$

   are degenerate minimum, and the excitations between them cost no energy $\Rightarrow$ Goldstone bosons!

THE GAUGE CASE: EXAMPLE II REVISITED

Let us consider a complex scalar field charged under some $U(1)$ (sQED),

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2|\phi|^2 - \lambda|\phi|^4, \qquad D_\mu = \partial_\mu - ieA_\mu$$

the gauged version of Example II. The Lagrangian is invariand under

$$\phi(x) \to \phi'(x) = e^{-i\theta(x)}\phi(x), \qquad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\theta(x).$$

In the case of $\mu^2 < 0, \lambda > 0$ we can again write

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \eta(x) + i\chi(x)], \qquad \mu^2 = -\lambda v^2,$$

so we obtain

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)^2 + \frac{1}{2}(\partial_\mu\chi)^2 - \lambda v^2\eta - \lambda v\eta(\eta^2+\chi^2) - \frac{\lambda}{4}(\eta^2+\chi^2)^2$$
$$+\frac{1}{4}\lambda v^4 - evA_\mu\partial^\mu\chi + eA_\mu(\chi\overleftrightarrow{\partial^\mu}\eta) + \frac{e^2v^2}{2}A_\mu A^\mu + \frac{e^2}{2}A_\mu A^\mu[\eta^2+\chi^2+2v\eta]$$

THE GAUGE CASE: EXAMPLE II REVISITED

Some highlights:

- The boson A_μ becomes massive, $m_A = |ev|$.
- The scalar η gets a mass $m_\eta = \sqrt{2\lambda}v$.
- The scalar χ is massless but has a kinetic mixing with A_μ, $A_\mu \partial^\mu \chi$.

In order to remove the mixing we add the following gauge-fixing term

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi}\Big(\partial_\mu A^\mu + \xi m_A \chi\Big)^2$$

Indeed,

$$\begin{aligned}\mathcal{L} + \mathcal{L}_{\text{GF}} \overset{\text{IBP}}{=} & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m_A^2}{2}A_\mu A^\mu - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \\ & + \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \frac{\xi m_A^2}{2}\chi^2 + \ldots\end{aligned}$$

χ gets a gauge-dependent mass, $m_\chi = \sqrt{\xi} m_A$. It is an *unphysical* field.

The gauge case: Example II revisited

This can be made simpler by using a smarter parametrization of the complex field

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \eta(x)]e^{i\frac{a(x)}{v}}.$$

We can then gauge away the exponential by doing the following gauge transformation

$$\phi(x) \to \phi'(x) = e^{-ia(x)/v}\phi(x) = \frac{1}{\sqrt{2}}[v + \eta(x)].$$

This is the unitary gauge ($\Leftrightarrow \xi \to \infty$), where

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 \\ &+ \frac{1}{4}\lambda v^4 + \frac{e^2v^2}{2}A_\mu A^\mu + \frac{e^2}{2}A_\mu A^\mu[\eta^2 + 2v\eta]\end{aligned}$$

Goldstone bosons are decoupled and no need to take them into account in loop calculations. However, gauge propagators are more complicated.

Brout - Englert - Higgs mechanism

> [...] has it ever occurred to you, that, [...] this could be a-a-a-a lot more, uh, uh, uh, uh, uh, uh, complex, I mean, it's not just, it might not be just such a simple... uh, you know?
>
> The Dude to The Big Lebowski

Theorem (Brout-Englert-Higgs mechanism)

The gauge bosons associated with the spontaneously broken generators become massive, the corresponding would-be Goldstone bosons are unphysical and can be absorbed, the remaining massive scalars (Higgs bosons) are physical.

[Englert and Brout 1964; Guralnik, Hagen, and Kibble 1964; Higgs 1964]

THE HIGGS MECHANISM IN THE SM

We introduce a complex scalar

$$\Phi \sim (\mathbf{1}, \mathbf{2}, \frac{1}{2}), \qquad \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \qquad \langle 0|\phi|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

with the following Lagrangian

$$\mathcal{L}_{\boldsymbol{\Phi}} = |D_\mu \Phi|^2 - \mu^2 |\Phi|^2 - \lambda |\Phi|^4, \qquad D_\mu \Phi = \Big(\partial_\mu - igW_\mu^i \frac{\sigma^i}{2} - ig' B_\mu \frac{1}{2}\Big)\Phi.$$

We want to break $SU(2)_L \otimes U(1)_Y \to U(1)_Q$ so we define $Q = T_L^3 + Y$, where $Y\varphi = y_\varphi \varphi, \ \forall \varphi$. Then

$$Q|0\rangle = \Big[T_L^3 + Y\Big]|0\rangle = \Big[\frac{\sigma^3}{2} + \frac{\mathbf{1}}{2}\Big]|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}|0\rangle = Q \begin{pmatrix} 0 \\ v \end{pmatrix} = 0$$

The term $|D_\mu \Phi|^2$ will generate masses for $W_\mu^{1,2}$ and one linear combination of W_μ^3 and B_μ.

The Higgs mechanism in the SM

We can write

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\phi^+(x) \\ v + h(x) + i\chi(x) \end{pmatrix}$$

It is useful to define the following linear combinations

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c_W & -s_W \\ s_W & c_W \end{pmatrix} \cdot \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \qquad s_W = \sin\theta_W, \quad c_W = \cos\theta_W$$

where $\tan\theta_W = g'/g$. We also define

$$W^\pm = \frac{1}{\sqrt{2}}[W^1_\mu \mp iW^2_\mu], \qquad T^\pm_L = \frac{1}{2\sqrt{2}}[\sigma^1 \pm i\sigma^2].$$

Then, defining $e = gs_W = g'c_W = gg'/\sqrt{g^2 + g'^2}$, we can write

$$D_\mu\Phi = \left[\partial_\mu - igW^\pm_\mu T^\pm_L - i\frac{g}{c_W} Z_\mu (T^3_L - s^2_W Q) - ieA_\mu Q\right]\Phi.$$

THE HIGGS MECHANISM IN THE SM

Then, taking into account that

$$T_L^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad T_L^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we obtain

$$\begin{aligned} D_\mu \Phi(x) = &\frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\partial_\mu \phi^+(x) \\ \partial_\mu h(x) + i\partial_\mu \chi(x) \end{pmatrix} - \frac{ig}{2} \begin{pmatrix} W_\mu^+(x)[v + h(x) + i\chi(x)] \\ \sqrt{2} W_\mu^-(x) \phi^+(x) \end{pmatrix} \\ &- \frac{ig}{\sqrt{2} c_W} Z_\mu \begin{pmatrix} (\frac{1}{2} - s_W^2)\sqrt{2}\phi^+(x) \\ -\frac{1}{2}[v + h(x) + i\chi(x)] \end{pmatrix} - ieA_\mu \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\phi^+(x) \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\mathcal{L}_{\boldsymbol{\Phi}} \supset \frac{g^2 v^2}{4} W_\mu^+ W^{-\mu} + \frac{1}{2} \frac{g^2 v^2}{4 c_W^2} Z_\mu Z^\mu \mp \frac{igv}{2} \partial_\mu \phi^\mp W^{\pm\mu} + \frac{gv}{2 c_W} \partial_\mu \chi Z^\mu + \ldots \,.$$

Then $m_W = gv/2$, $m_Z = m_W / c_W$ and

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi_\gamma} (\partial_\mu A^\mu)^2 - \frac{1}{\xi_W} |\partial_\mu W^{+\mu} - i\xi_W m_W \phi^+|^2 - \frac{1}{2\xi_Z} (\partial_\mu Z^\mu - \xi_Z m_Z \chi)^2$$

THE HIGGS MECHANISM IN THE SM

Then, the propagators for the electroweak (EW) bosons become

$$\tilde{D}^{\gamma}_{\mu\nu}(k) = \frac{i}{k^2 + i\varepsilon}\Big[- g_{\mu\nu} + (1-\xi_A)\frac{k_\mu k_\nu}{k^2}\Big]$$

$$\tilde{D}^{Z}_{\mu\nu}(k) = \frac{i}{k^2 - m_Z^2 + i\varepsilon}\Big[- g_{\mu\nu} + (1-\xi_Z)\frac{k_\mu k_\nu}{k^2 - \xi_Z m_Z^2}\Big],$$

$$\tilde{D}^{W}_{\mu\nu}(k) = \frac{i}{k^2 - m_W^2 + i\varepsilon}\Big[- g_{\mu\nu} + (1-\xi_Z)\frac{k_\mu k_\nu}{k^2 - \xi_W m_W^2}\Big],$$

$$\tilde{D}^{\chi}(k) = \frac{i}{k^2 - \xi_Z m_Z^2 + i\varepsilon}$$

$$\tilde{D}^{\phi}(k) = \frac{i}{k^2 - \xi_W m_W^2 + i\varepsilon}.$$

t'Hooft-Feynman gauge: $\xi_A = \xi_Z = \xi_W = 1$

Unitary gauge: $\xi_Z = \xi_W \to \infty$

THE HIGGS MECHANISM IN THE SM

Let's see what happens to ghosts. The gauge-fixing term (for the EW part of the SM) could be writen as

$$\mathcal{L}_{\rm GF} = -\frac{1}{2\xi_\gamma}F_\gamma^2 - \frac{1}{\xi_W}|F_+|^2 - \frac{1}{2\xi_Z}F_Z^2 = -\frac{1}{2\xi_\gamma}F_\gamma^2 - \frac{1}{2\xi_W}(F_{W_1}^2 + F_{W_2}^2) - \frac{1}{2\xi_Z}F_Z^2$$

where ($\phi^+ = 1/\sqrt{2}(\phi_1 - i\phi_2)$)

$$\begin{aligned} F_{W_1} &= \partial_\mu W^{1\mu} - \xi_W m_W \phi_2, & F_{W_2} &= \partial_\mu W^{2\mu} + \xi_W m_W \phi_1 \\ F_Z &= \partial_\mu Z^\mu - \xi_Z m_Z \chi & F_\gamma &= \partial_\mu A^\mu. \end{aligned}$$

Defining analoguous linear combinations for the ghost fields,

$$c_\gamma = s_W c_w^3 + c_W c_b, \quad c_Z = c_W c_w^3 - s_W c_b, \qquad c_\pm = \frac{1}{\sqrt{2}}[c_w^1 \mp i c_w^2]$$

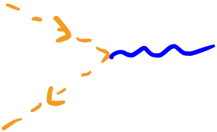
we can write (where $c_{1,2,3} = c_w^{1,2,3}$, $c_4 = c_b$, $U(\theta) = \exp(-iT_L^i\theta_i)$ and $U(\theta_4) = \exp(-iY\theta_4)$)

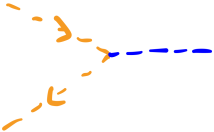
$$\mathcal{L}_{\rm ghost}|_{\rm EW} = \sum_{i=1}^{4}\left[\bar{c}_+\frac{\delta F_+}{\delta\theta_i} + \bar{c}_-\frac{\delta F_-}{\delta\theta_i} + \bar{c}_\gamma\frac{\delta F_\gamma}{\delta\theta_i} + \bar{c}_Z\frac{\delta F_Z}{\delta\theta_i}\right]c_i$$

THE HIGGS MECHANISM IN THE SM

At the end of the day, we obtain

$$
\begin{aligned}
\mathcal{L}_{\mathrm{FP}} = & (\partial_\mu \bar{c}_\gamma)(\partial^\mu c_\gamma) + (\partial_\mu \bar{c}_Z)(\partial^\mu c_Z) + (\partial_\mu \bar{c}_+)(\partial^\mu c_+) + (\partial_\mu \bar{c}_-)(\partial^\mu c^-) \\
& - \xi_Z m_Z^2 \bar{c}_Z c_Z - \xi_W m_W^2 \bar{c}_+ \bar{c}_+ - \xi_W m_W^2 \bar{c}_- c_- \\
& + \text{ interactions with EW gauge bosons} + \text{ interactions with } \Phi(x)
\end{aligned}
$$





with propagators

$$
\tilde{D}_{c_\gamma}(k) = \frac{i}{k^2 + i\varepsilon}, \ \tilde{D}_{c_Z}(k) = \frac{i}{k^2 - \xi_Z m_Z^2 + i\varepsilon}, \ \tilde{D}_{c_\pm}(k) = \frac{i}{k^2 - \xi_W m_W^2 + i\varepsilon}
$$

CUSTODIAL SYMMETRY

One can notice that the Higgs potential $V(\Phi)$ is invariant under $SO(4)$ rotations, broken after EWSB to $SO(3)$, since the Higgs could get its vev in any of its four real degrees of freedom.

Since $SU(2) \otimes SU(2)$ is the double cover of $SO(4)$, we can also describe this breaking as $SU(2)_L \times SU(2)_R \to SU(2)_V$. For this, it can useful to to write

$$\Sigma = \begin{pmatrix} \tilde{\Phi} & \Phi \end{pmatrix}, \quad \text{where} \quad \tilde{\Phi} = i\sigma^2 \Phi^*$$

obtaining

$$\mathcal{L}_\Phi = \frac{1}{2}\mathrm{Tr}\left((\mathcal{D}_\mu \Sigma)^\dagger (\mathcal{D}\Sigma)\right) - \frac{1}{2}\mu^2 \mathrm{Tr}\left(\Sigma^\dagger \Sigma\right) + \frac{\lambda}{4}\left[\mathrm{Tr}\left(\Sigma^\dagger \Sigma\right)\right]^2,$$

where

$$\mathcal{D}_\mu \Sigma = \partial_\mu \Sigma - igW_\mu^i \frac{\sigma^i}{2}\Sigma + ig'\Sigma\frac{\sigma^3}{2}B_\mu.$$

CUSTODIAL SYMMETRY

One can see, that indeed, the Higgs potential is invariant under a $SU(2)_L \otimes SU(2)_R$ symetry – dubbed custodial symmetry – under which

$$\Sigma \to U_L \, \Sigma \, U_R^\dagger.$$

After EWSB,

$$\langle \Sigma \rangle = \frac{1}{2} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$$

so that $SU(2)_L \otimes SU(2)_R \to SU(2)_V$ as anticipated.

This is however not a symmetry of the entire SM Lagrangian. Indeed, already $\mathcal{D}\Sigma$ breaks such symmetry, since $SU(2)_R$ will mix components with different hypercharge. The difference in quark masses will also violate $SU(2)_R$ as well.

One thus expect large radiative corrections coming from top loops (since the breaking of custodial symmetry should be proportional to $m_t - m_b$)

$$\rho = \frac{m_W^2}{m_Z^2 c_W^2} \approx 1 + \frac{3}{8} \frac{G_F}{\sqrt{2}} \frac{m_t^2}{\pi^2}.$$

RECAP

RECAP

Right now, our SM Lagrangian consists of the following terms

$$\mathcal{L}_{\text{SM}} \supset \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} + \mathcal{L}_{\text{ferm}} + \mathcal{L}_{\Phi}$$

These terms provide:

1. Massive gauge bosons W_μ, Z_μ with their longitudinal dof
2. Massless gauge bosons G^a_μ, A_μ
3. Ghosts required for loop calculations
4. Chiral fermions $q^i_L, u^i_R, d^i_R, \ell^i_L, e^i_R$ interacting with SM gauge bosons
5. An additional scalar dof, the Higgs $h(x)$

However, at the moment, fermions are still massless!

GIVING FERMIONS A MASS

We can give masses to fermions through the so-called Yukawa Lagrangian

$$\mathcal{L}_{\text{Yuk}} = -(\mathbf{Y}_u)_{ij}\bar{q}_L^i\tilde{\Phi}u_R^j - (\mathbf{Y}_d)_{ij}\bar{q}_L^i\Phi d_R^j - (\mathbf{Y}_e)_{ij}\bar{\ell}_L^i\Phi e_R^j + \text{h.c.}$$

After EWSB, in the unitary gauge, we obtain

$$\mathcal{L}_{\text{Yuk}} = -\frac{1}{\sqrt{2}}(v+h)\Big[(\mathbf{Y}_u)_{ij}\bar{u}_L^i u_R^j + (\mathbf{Y}_d)_{ij}\bar{d}_L^i d_R^j + (\mathbf{Y}_e)_{ij}\bar{e}_L^i e_R^j + \text{h.c.}\Big].$$

We can define mass matrices

$$\mathcal{M}_u = \frac{v}{\sqrt{2}}\mathbf{Y}_u, \qquad \mathcal{M}_d = \frac{v}{\sqrt{2}}\mathbf{Y}_d, \qquad \mathcal{M}_e = \frac{v}{\sqrt{2}}\mathbf{Y}_e.$$

- Note that neutrino are massless in the SM. Neutrino masses constitute physics beyond the SM.

GIVING FERMIONS A MASS

We can diagonalize fermion masses through a singular value decomposition

$$u_L = \mathcal{U}_u\, u'_L, \quad d_L = \mathcal{U}_d\, d'_L, \quad u_R = \mathcal{V}_u\, u'_R, \quad d_R = \mathcal{V}_d\, d'_R,$$
$$e_L = \mathcal{U}_e\, e'_L, \quad e_R = \mathcal{V}_e\, e'_R,$$

with $\mathcal{U}_{u,d,e}$ and $\mathcal{V}_{u,d,e}$ unitary satisfying

$$\mathcal{U}_u^\dagger \mathcal{M}_u \mathcal{V}_u = \lambda_u, \quad \mathcal{U}_d^\dagger \mathcal{M}_d \mathcal{V}_d = \lambda_d, \quad \mathcal{U}_e^\dagger \mathcal{M}_e \mathcal{V}_e = \lambda_e,$$

with $\lambda_{u,d,e}$ diagonal. These rotations do not affect fermion kinetic terms nor the neutral currents because they are family universal

$$\bar{\psi}\, i\gamma^\mu D_\mu \bar{\psi} \supset \bar{\psi}_L i\gamma^\mu \Big[-i\frac{g}{c_W} Z_\mu (T^3_L - s^2_W Q_\psi) - ieA_\mu Q_\psi \Big] \psi_L + (L \leftrightarrow R)$$

Neutral currrents do not change flavour

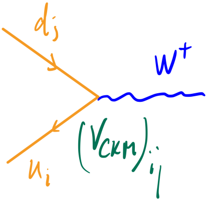
CHARGED CURRENTS

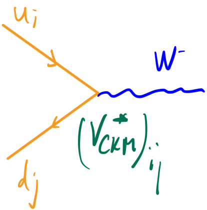
Charged currents however are different

$$\bar{q}_L i\gamma^\mu D_\mu q_L \supset -igW_\mu^\pm \bar{q}_L T_L^\pm q_L = \frac{g}{\sqrt{2}} W_\mu^+ \bar{u}_L \gamma^\mu d_L + \text{h.c.}$$
$$= \frac{g}{2\sqrt{2}} W_\mu^+ \bar{u}' \left(\mathcal{U}_u^\dagger \mathcal{U}_d\right) \gamma^\mu (1-\gamma_5) d_L' + \text{h.c.}$$

The matrix $\mathbf{V}_{\text{CKM}} = \mathcal{U}_u^\dagger \mathcal{U}_d$ is the so-called Cabibbo-Kobayashi-Maskawa (CKM) matrix. – btw, henceforth we will drop the primes.

[Cabibbo 1963; Kobayashi and Maskawa 1973]





CHARGED CURRENTS

In the lepton sector we will have something similar

$$\bar{\ell}_L i\gamma^\mu D_\mu \ell_L \supset -igW_\mu^\pm \bar{\ell}_L T_L^\pm \ell_L = \frac{g}{\sqrt{2}} W_\mu^+ \bar{\nu}_L \gamma^\mu e_L + \text{h.c.}$$

$$\to \frac{g}{2\sqrt{2}} W_\mu^+ \bar{\nu}\, \mathcal{U}_e \gamma^\mu (1-\gamma_5) e_L + \text{h.c.}$$

However, since ν are massless (in the SM), we could rotate ν_L to make the interaction diagonal

$$\nu_L \to \mathcal{U}_e \nu_L.$$

This is a consequence of the

$$\mathcal{G}_l = SU(3)_{\ell_L} \otimes SU(3)_{e_R}$$

global symmetry of $\mathcal{L}_{\text{ferm}}$.

THE CKM MATRIX

In general, a $n \times n$ unitary matrix has n^2 real parameters. However, some phases can be rotated away, leading to $(n-1)^2$ real parameters,

$$n(n-1)/2 \quad \text{moduli} \quad \text{and} \quad (n-1)(n-2)/2 \quad \text{phases.}$$

The standard parametrization of the CMK matrix gives

$$\mathbf{V}_{\mathrm{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where $c_{ij} = \cos\theta_{ij} > 0,\ s_{ij} = \sin\theta_{ij} > 0$.

δ is the only source of CP violation in the SM (modulo $\mathcal{L}_\theta$).

THE LAST MISSING PIECE

For $SU(n)$ gauge theories, with $n \geq 2$, there are gauge configurations that do not vanish fast enough to be ignored. In Euclidean space, keeping only the gauge part of the QCD lagrangian, we obtain

$$\mathcal{S}_E = -\frac{1}{2g_s^2} \int d^4\hat{x}\, \mathrm{Tr}(\hat{\mathcal{G}}_{\mu\nu}\hat{\mathcal{G}}_{\mu\nu}),$$

where

$$\hat{\mathcal{G}}_\mu \equiv -ig_s \hat{G}^a_\mu T^a, \qquad \hat{\mathcal{G}}_{\mu\nu} \equiv -ig_s \hat{G}^a_{\mu\nu} T^a,$$

and

$$x_0 = -i\hat{x}_4, \qquad x^i = \hat{x}_i, \qquad x^2 = -\hat{x}^2, \qquad \hat{x}_\mu = (\hat{x}_0, \hat{x}_i), \qquad \hat{x}^\nu = \delta^{\mu\nu}\hat{x}_\nu.$$

$$A_0 = i\hat{A}_4, \qquad A^i = -\hat{A}_i, \qquad \hat{A}_\mu = (\hat{A}_0, \hat{A}_i), \qquad \hat{A}^\mu = \hat{A}_\nu \delta^{\mu\nu}.$$

with $x^\mu = (x_0, x^i)$ and $A^\mu = (A_0, A^i)$ any Lorentz vector. Henceforth, we will drop the hat, besides for $\hat{x}$.

THE LAST MISSING PIECE

There are non-trivial gauge configurations having $|\mathcal{S}_E| < \infty$ and therefore $\lim_{|\hat{x}|\to\infty} \mathcal{G}_{\mu\nu} = 0$. This implies

$$\lim_{|\hat{x}|\to\infty} \mathcal{G}_\mu(\hat{x}) = \mathcal{G}_\mu(\hat{x}) = -(\partial_\mu U(\hat{x}))U(\hat{x})^{-1} = U(\hat{x})\partial_\mu U(\hat{x})^{-1}$$

For $SU(2)$, they are (with $r = |\hat{x}|$) [Belavin et al. 1975]

$$\mathcal{G}_\mu(\hat{x}) = -\frac{r^2}{r^2+\lambda^2}(\partial_\mu U)U^{-1}, \qquad U = \frac{\hat{x}_4 + i\hat{x}_a\tau^a}{r}, \qquad \lambda \in \mathbb{R}^+.$$

and satisfy

$$\mathcal{S}_E = -\frac{1}{2g_s^2}\int d^4\hat{x}\,\mathrm{Tr}(\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}) = -\frac{1}{2g_s^2}\int d^4\hat{x}\,\mathrm{Tr}(\mathcal{G}_{\mu\nu}\tilde{\mathcal{G}}^{\mu\nu}) = \frac{8\pi^2 n[G]}{g_s^2},$$

with

$$n[G] = -\frac{1}{16\pi^2}\int d^4\hat{x}\,\mathrm{Tr}(\mathcal{G}_{\mu\nu}\widetilde{\mathcal{G}}_{\mu\nu}) = \frac{1}{24\pi^2}\oint_{S_3^\infty} d\sigma_\mu\varepsilon_{\mu\nu\rho\lambda}\,\mathrm{Tr}(\mathcal{G}_\nu\mathcal{G}_\rho\mathcal{G}_\lambda)$$

$$= -\frac{1}{24\pi^2}\oint_{S_3^\infty} d\sigma_\mu\varepsilon_{\mu\nu\rho\lambda}\mathrm{Tr}\Big((\partial_\nu U)U^{-1}(\partial_\rho U)U^{-1}(\partial_\lambda U)U^{-1}\Big).$$

THE LAST MISSING PIECE. INSTANTONS

The funtion $n[G]$ is called the winding number. The aforementioned solution has $n[G] = 1$ and it is called instanton. Therefore,

$$\mathcal{S}_E = \frac{8\pi^2}{g_s^2}.$$

It can be proven that $n[G_1 G_2] = n[G_1] + n[G_2]$. It is a topological charge that classifies different homotopy classes in $\pi_3(S^3) = \mathbb{Z}$, since these configurations are maps

$$S^3 \to SU(2) \cong S^3.$$

Bott's theorem tell us that an arbitrary mapping $S^3 \to G$ can be deformed into a mapping $S^3 \to SU(2)$ and this are also solutions of the QCD gauge group $SU(3)$.

THE QCD VACUA

The Euclidean QCD partition function reads

$$\mathcal{Z}[t] = \langle \Omega | e^{-Ht} | \Omega \rangle$$

where $|\Omega\rangle$ is the vacuum of the theory. We assume the existence of infinite vacuum states, $|n\rangle$, with winding numbers $n \in \mathbb{Z}$. Moreover, there exist a correspondence

$$G_{\mu[n]} \longleftrightarrow U_n, \qquad U_n \text{ acting on the fock space}.$$

For $n = 1$,

$$U_1 |n\rangle = |n+1\rangle.$$

A gauge-invariant vacuum state should have contributions from *all* classes, so it makes sense to define a coherent superposition

$$|\theta\rangle = \sum_{n \in \mathbb{Z}} e^{in\theta} |n\rangle$$

where θ is an arbitrary parameter. Then, this vacuum is gauge-invariant up to an overall phase

$$U_1 |\theta\rangle = e^{-i\theta} |\theta\rangle.$$

The QCD vacua

Consider now a gauge invariant operator B, $[B, U_1] = 0$, then

$$0 = \langle\theta|[B, U_1]|\theta'\rangle = (e^{-i\theta'} - e^{-i\theta})\langle\theta|B|\theta'\rangle$$

and $\langle\theta|B|\theta'\rangle = 0$ if $\theta \neq \theta'$. Therefore, each $|\theta\rangle$ is the vacuum of a separate sector of states, unconnected by any gauge-invariant operator. In particular,

$$\langle\theta|e^{-Ht}|\theta'\rangle = 2\pi\delta(\theta - \theta')e^{-E_\theta t}$$

and

$$\mathcal{Z}(t) = \langle\theta|e^{-Ht}|\theta\rangle = \sum_{n,m\in\mathbb{Z}} e^{-in\theta}\langle n+m|e^{-Ht}|m\rangle \underset{t\to\infty}{\to}$$

$$\sum_{n\in\mathbb{Z}} e^{-in\theta}\int[\mathcal{D}G_\mu]_{(n)}e^{-\mathcal{S}_E} = \int[\mathcal{D}G_\mu]\exp\Big(-\mathcal{S}_E - \frac{ig_s^2\theta}{32\pi^2}\int \mathrm{d}^d\hat{x}\, G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}\Big)$$

where $[\mathcal{D}G_\mu]_{(n)}$ implies that we only integrate over gauge configurations with winding number n.

THE QCD VACUA

Therefore, this effect can be parametrized by adding a term to the Euclidean action

$$\frac{ig_s^2\theta}{32\pi^2}\int \mathrm{d}^d\hat{x}\, G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}$$

which correspond in Minkowski space, to

$$\mathcal{L}_\theta = \theta\frac{g_s^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{a\mu\nu}, \qquad \widetilde{G}^{a\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G^a_{\rho\sigma}.$$

Chiral rotations can also induce a similar term, so at the end of the day, the physical combination is

$$\bar{\theta} = \theta + \mathrm{argdet}\mathcal{M}$$

where $\mathcal{M}$ is the mass matrix after combining up and down-type quark matrices in one.

The strong CP problem

Such a term, in particular, induces an electric dipole moment of the neutron of the size of $d_n = C_{\text{EDM}} e\bar{\theta}$, where $C_{\text{EDM}} = 2.4(1.0) \times 10^{-16}\,\text{cm}$.

Experimental bounds on the latter leads to $|d_n| < 1.8 \times 10^{-26} e\,\text{cm}$ and

$$|\bar{\theta}| \lesssim 10^{-10}.$$

This is what is known as the strong CP problem.

In the $SU(2)_L \otimes U(1)_Y \to U(1)_Q$ case, due to the Higgs sector, the parameter is unphysical and we can rotate it away.

THE WHOLE THING

THE WHOLE SM LAGRANGIAN

Now, we have the whole SM Lagrangian

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} + \mathcal{L}_{\text{ferm}} + \mathcal{L}_{\Phi} + \mathcal{L}_{\text{Yuk}} + \mathcal{L}_{\theta}$$

SOME EW PHENO

INPUT PARAMETERS

The input parameters of the EW sector are:

$$g,\ g',\ v,\ \lambda,\ m_f\,[\times 9],\ \text{CKM physical parameters}\,[\times 4]$$

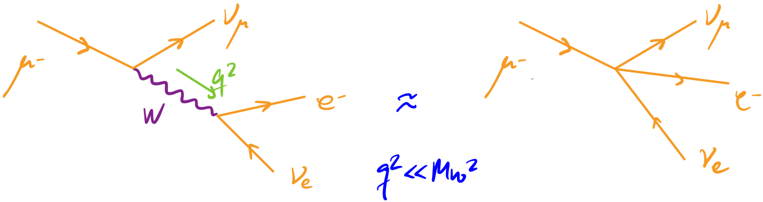
We can trade the first four parameters by

$$\alpha = \frac{e^2}{4\pi} = \frac{g^2 s_W^2}{4\pi}, \quad m_W = \frac{1}{2} g v, \quad m_Z = \frac{m_W}{c_W}, \quad m_H = \sqrt{2\lambda} v,$$

or α, m_Z, G_F, m_H.

THE MUON DECAY

The muon decay is very well measured experimentally



$$\frac{g^2}{m_W^2 - q^2} \approx \frac{g^2}{m_W^2} = 4\sqrt{2}G_F, \qquad \frac{1}{\tau_\mu} = \Gamma(\mu \to \nu_\mu e^- \nu_e) \approx \frac{G_F^2 m_\mu^5}{192\pi^2},$$

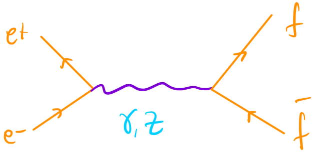
leading to

$$v = (\sqrt{2}G_F)^{-\frac{1}{2}} = 246\,\mathrm{GeV}.$$

This relation will (most likely) change for new physics models!

$e^+e^- \to f\bar{f}$

MEASURED AT PEP, PETRA, TRISTAN, ..., LEP1, SLD



$$\frac{d\sigma}{d\Omega} = N_c^f \frac{\alpha^2}{4s} \beta_f \Big\{ \left[1 + \cos^2\theta + (1-\beta_f)^2 \sin^2\theta \right] G_1(s) + 2(\beta_f^2 - 1) G_2(s) + 2\beta_f \cos\theta \, G_3(s) \Big\}$$

$$G_1(s) = Q_e^2 Q_f^2 + 2 Q_e Q_f v_e v_f \mathrm{Re}(\xi_Z(s)) + (v_e^2 + a_e^2) |\xi_Z(s)|^2,$$

$$G_2(s) = (v_e^2 + v_f^2) a_f^2 |\xi_Z(s)|^2,$$

$$G_3(s) = 2 Q_e Q_f a_e a_f \mathrm{Re}(\xi_Z(s)) + 4 v_e v_f a_e a_f |\xi_Z(s)|^2$$

where

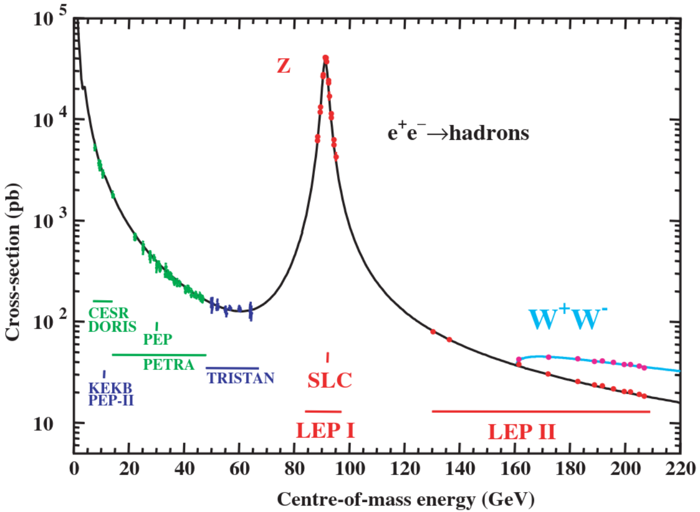
$$\mathcal{L}_Z = e \bar{f} \gamma^\mu (v_f - a_f \gamma_5) f Z_\mu, \quad \xi_Z(s) = \frac{s}{s - m_Z^2 + i m_Z \Gamma_Z},$$

$$\beta_f = \sqrt{1 - 4m_f^2/s}$$

$e^+e^- \to f\bar{f}$

If we integrate over the whole solid angle we obtain

$$\sigma(s) = N_c^f \frac{2\pi\alpha^2}{3s}\beta_f\left[(3-\beta_f)G_1(s) - 3(1-\beta_f^2)G_2(s)\right]$$



Z POLE OBSERVABLES

At the Z pole we can neglect the $\gamma - Z$ interference. Then (neglecting m_f)

$$\sigma_{\text{had}} = 12\pi \frac{\Gamma(e^+e^-)\Gamma(had)}{m_Z^2 \Gamma_Z^2}, \quad \Gamma(Z \to f\bar{f}) = N_c^f \frac{\alpha m_Z}{3}(v_f^2 + a_f^2)$$

$$R_b = \frac{\Gamma(b\bar{b})}{\Gamma(had)}, \quad R_c = \frac{\Gamma(c\bar{c})}{\Gamma(had)}, \quad R_\ell = \frac{\Gamma(had)}{\Gamma(\ell^+\ell^-)}.$$

Some asymmetries can be very usefull

$$A_{\text{FB}}^f = \frac{\sigma(\cos\theta > 0) - \sigma(\cos\theta < 0)}{\sigma(\cos\theta > 0) + \sigma(\cos\theta < 0)} = \frac{3}{4} A_f \frac{A_e + P_e}{1 + P_e A_e},$$

$$A_{\text{LR}} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = A_e P_e,$$

where P_e is the initial electron polarization and

$$A_f = \frac{2 v_f a_f}{v_f^2 + a_f^2}.$$

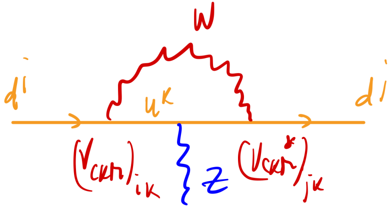
Some other observables

- W pair production (LEP2), W production (Tevatron/LHC)
- Top quark production
- Higgs production and decay
- Higgs pair production, ...

GIM MECHANISM

[Glashow, Iliopoulos, and Maiani 1970]

We just saw that there is no flavour-changing-neutral currents at tree-level. We are also protected at the loop level,



It is proportional to

$$\sum_k (\mathbf{V}_{\mathrm{CKM}})_{ki} (\mathbf{V}^*_{\mathrm{CKM}})_{kj} F(m_{u_k})$$

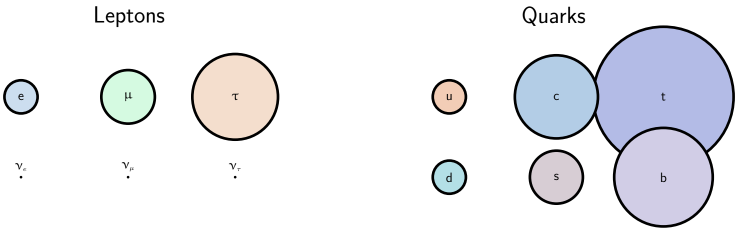
In the limit of identical masses this sum goes to zero. In general, it is going to be suppressed $\Rightarrow$ GIM mechanism

HIERARCHICAL MASSES AND MIXING ANGLES

Fermion masses display huge hierarchies

m_u [GeV]	m_d [GeV]	m_s [GeV]	m_c [GeV]	m_b [GeV]	m_t [GeV]
$2.16 \cdot 10^{-3}$	$4.67 \cdot 10^{-3}$	93.4	1.27	4.18	172.69

m_e [GeV]	m_μ [GeV]	m_τ [GeV]
$0.511 \cdot 10^{-3}$	0.105	1.78



HIERARCHICAL MASSES AND MIXING ANGLES

The same happens with the entries in the CKM matrix

$$\mathbf{V}_{\mathrm{CKM}} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\varrho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\varrho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

with $\lambda = |(\mathbf{V}_{\mathrm{CKM}})_{us}| \approx 0.22$ and $A, \varrho, \eta = \mathcal{O}(1)$.

It also helps with flavour

$$\sum_k (\mathbf{V}_{\mathrm{CKM}})_{ki} (\mathbf{V}^*_{\mathrm{CKM}})_{kj} F(m_{u_k}) = (\mathbf{V}_{\mathrm{CKM}})_{ui} (\mathbf{V}^*_{\mathrm{CKM}})_{uj} F(m_u)$$

$$+ (\mathbf{V}_{\mathrm{CKM}})_{ci} (\mathbf{V}^*_{\mathrm{CKM}})_{cj} F(m_c) + (\mathbf{V}_{\mathrm{CKM}})_{ti} (\mathbf{V}^*_{\mathrm{CKM}})_{tj} F(m_t)$$

$$\sim \left[(\mathbf{V}_{\mathrm{CKM}})_{ui} (\mathbf{V}^*_{\mathrm{CKM}})_{uj} + (\mathbf{V}_{\mathrm{CKM}})_{ci} (\mathbf{V}^*_{\mathrm{CKM}})_{cj}\right] F(0) + (\mathbf{V}_{\mathrm{CKM}})_{ti} (\mathbf{V}^*_{\mathrm{CKM}})_{tj} F(m_t)$$

$$\sim (\mathbf{V}_{\mathrm{CKM}})_{ti} (\mathbf{V}^*_{\mathrm{CKM}})_{tj} \Big[F(m_t) - F(0)\Big]$$

THE UNITARITY TRIANGLE

The unitarity of the CKM matrix implies in particular (henceforth, we drop the CKM subscript for simplicity)

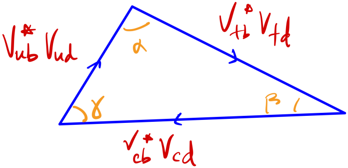
$$\mathbf{V}_{ud}\mathbf{V}_{ub}^* + \mathbf{V}_{cd}\mathbf{V}_{cb}^* + \mathbf{V}_{td}\mathbf{V}_{tb}^* = 0$$

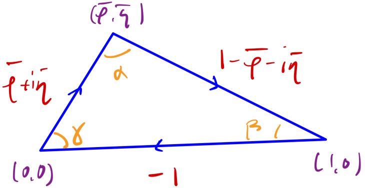
or

$$\frac{\mathbf{V}_{ud}\mathbf{V}_{ub}^*}{\mathbf{V}_{cd}\mathbf{V}_{cb}^*} + \frac{\mathbf{V}_{td}\mathbf{V}_{tb}^*}{\mathbf{V}_{cd}\mathbf{V}_{cb}^*} + 1 = 0 \Leftrightarrow [\bar\varrho + i\bar\eta] + [(1-\bar\varrho) - i\bar\eta] - 1 = 0$$

where

$$\bar\varrho + i\bar\eta = -\frac{\mathbf{V}_{ub}^*\mathbf{V}_{ud}}{\mathbf{V}_{cb}^*\mathbf{V}_{cd}}, \quad \bar\varrho = \varrho\Big(1 - \frac{\lambda^2}{2}\Big) + \mathcal{O}(\lambda^4), \quad \bar\eta = \eta\Big(1 - \frac{\lambda^2}{2}\Big) + \mathcal{O}(\lambda^4)$$





THE UNITARITY TRIANGLE

A global fit leads to

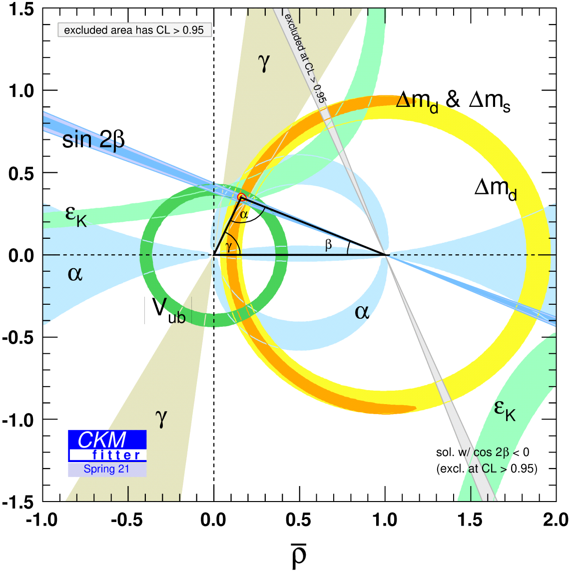
$$\lambda = 0.22500 \pm 0.00067, \qquad A = 0.826^{+0.018}_{-0.015},$$
$$\bar{\varrho} = 0.159 \pm 0.010, \qquad \bar{\eta} = 0.348 \pm 0.010,$$

and

$$\alpha + \beta + \gamma = (173 \pm 6)^\circ, \quad J = \Im\left(\mathbf{V}_{us}\mathbf{V}_{cb}\mathbf{V}^*_{ub}\mathbf{V}^*_{cs}\right) = \left(3.08^{+0.15}_{-0.13}\right) \times 10^{-5},$$

with J being twice the area of all the unitarity triangles.

THE UNITARITY TRIANGLE



THE CHIRAL ANOMALY

Let us start with QED

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A_\mu \exp\Big[i\int d^4x\Big(-\frac{1}{4}F_{\mu\nu}^2 + i\bar{\psi}\not{D}\psi\Big)\Big]$$

The integrand is unvariant under $\psi \to e^{i\alpha}\psi$ and $\psi \to e^{i\beta\gamma_5}\psi$ with A_μ unchanged. However, under a local axial transformation the path-integral measure is no longer invariant (A_μ is also invariant)

$$\mathcal{D}\bar{\psi}\,\mathcal{D}\psi \to |\mathcal{J}|^{-2}\mathcal{D}\bar{\psi}\,\mathcal{D}\psi, \qquad \mathcal{J} = \det\left(e^{i\beta(x)\gamma_5}\right) = \exp\operatorname{tr}\log\left(e^{i\beta(x)\gamma_5}\right)$$

Moreoever,

$$\mathcal{J} = \exp\left(i\int d^4x\beta(x)\mathrm{Tr}[\gamma_5]\right) \to \infty$$

Regulating the integral leads to [Fujikawa 1979]

$$\mathcal{J} = \exp\left[-i\int d^4x\Big(\beta(x)\frac{e^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(x)F_{\alpha\beta}(x)\Big)\right]$$

THE CHIRAL ANOMALY

Therefore,

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A_\mu \exp\Big[i\int d^4x\mathcal{L}_{\text{QED}}\Big] \to$$
$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A_\mu \exp\Big[i\int d^4x\mathcal{L}_{\text{QED}} - J^5_\mu \partial^\mu \beta(x) + \beta\frac{e^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\alpha\beta}\Big]$$

and

$$\partial_\mu J^{5\mu} = -\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

The current is no longer conserved!

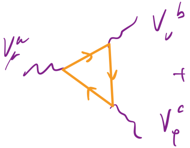
THE CHIRAL ANOMALY

In the SM

$$\partial_\mu J_Y^{5\mu} = \Big(\sum_{LH} Y_{LH} - \sum_{RH} Y_{RH}\Big)\frac{g'^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}B_{\mu\nu}B_{\alpha\beta} \qquad : U(1)_Y^3$$

which vanishes for the hypercharge assignments of the SM! For the non-abelian part, something similar happens but involve

$$\mathbf{Tr}\Big[T_R^a\{T_R^b, T_R^c\}\Big] = A(R)d^{abc}, \qquad \mathbf{Tr}\Big[T_L^a\{T_L^b, T_L^c\}\Big] = A(L)d^{abc},$$



+ crossed

$$\mathcal{A}^{abc} = \mathrm{Tr}(\{T_L^a, T_L^b\}, T_L^c) - \mathrm{Tr}(\{T_R^a, T_R^b\}, T_R^c)$$

All these anomalies cancel in the SM, but just for the case of 3 generations!!

VIOLATION OF BARYON NUMBER

Instanton transitions violates B and L number in three units. Since the tunneling rate of the transition is proportional to $\exp(-\mathcal{S}_{\mathrm{E}})$, it leads to

$$\Gamma \sim e^{-\mathcal{S}_{\mathrm{E}}(n=1)} = e^{-\frac{8\pi^2}{g_s^2}} \sim 10^{-173}.$$

At high temperature (finite T QFT), thermal fluctuations dominate (sphalerons). Lattice calculations provide the best approximation to the rate of sphaleron transitions in the symmetric phase (before EWSB)

$$\Gamma = (18 \pm 3)\alpha_W^5 T^4 \approx (8.0 \pm 1.3) \times 10^{-7} T^4, \qquad \alpha_W = g^2/(4\pi).$$

After EWSB, these transitions become exponentially suppressed

$$\Gamma \sim A(\alpha_W T)^4 \left(\frac{E_{\mathrm{sph}}}{T}\right)^7 \exp\Big(-E_{\mathrm{sphal}}(T)/T\Big).$$

At hight temperatures, sphaleron transitions provide sizable violation of B-number. Enough for baryogenesis!

Baryogenesis in the SM

Sakharov conditions for successful baryogenesis require

1. B violation. ✓
2. Loss of thermal equilibrium. ✗
3. C, CP violation. ✗

CP violation is given by δ and $\bar{\theta}$ but both are constrained to be too small to give a sizable violation of CP.

Sphalerons are in thermal equilibrium for $T \in [132, 10^{12}]$ GeV. One can show that if the SM undergoes a sufficiently strong first order phase transition (EW baryogenesis), this could provide the required departure from thermal equilibrium. However, this is not the case.

So, the SM can not account for the baryon asymmetry of the universe.

RUNNING IN THE SM

Running in the SM

All divergences are local, and can be thus absorbed by counterterms. For instance, in the case of massless QED

$$\mathcal{L}_{\text{QED}}^{\text{1loop}} = -\frac{1}{4} F_{\mu\nu}^{(0)} F^{(0)\mu\nu} + \bar{\psi}^{(0)} i\gamma^\mu D_\mu^{(0)} \psi^{(0)} = -\frac{1}{4} Z_A F_{\mu\nu} F^{\mu\nu} + Z_\psi i\bar{\psi}\gamma^\mu \partial_\mu \psi$$
$$- \mu^{-\varepsilon} \sqrt{Z_A} Z_e Z_\psi e \bar{\psi} \gamma^\mu \psi A_\mu, \quad Z_{\psi,A} = 1 + \mathcal{O}(1/\varepsilon), \quad D = 4 - 2\varepsilon,$$

where $Z_e = Z_A^{-1/2}$ to all orders due to the Ward-Takahashi identity.
Since $e_0 = e\mu^{-\varepsilon} Z_e$, we have

$$e = e_0 \mu^\varepsilon Z_e^{-1} = e_0 \mu^\varepsilon \sqrt{Z_A}.$$

In particular, in the $\overline{\text{MS}}$ scheme

$$\mu \frac{d}{d\mu} e_0 = \mu \frac{d}{d\mu} [\mu^\varepsilon e Z_e] = 0 \Rightarrow \beta(e) \equiv \mu \frac{d}{d\mu} e = -\varepsilon e + \frac{e^3}{12\pi^2}$$

Couplings run!

QCD

THE QCD LAGRANGIAN

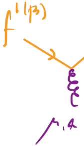
We have already seen the QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{2\xi^a}(\partial^\mu G^a_\mu)^2 + (\partial^\mu \bar{c}^a_g)(\partial_\mu c^a_g - g_s f^{abc} c^b_g G^b_\mu)$$
$$+ \bar{\psi}_i \left[i\gamma^\mu D_\mu - m_i\right]\psi_i + \bar{\theta}\frac{g_s^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{a\mu\nu}$$

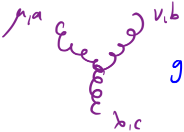
If we forget about the EW part the covariant derivatives look like

$$D_\mu = \partial_\mu - i g_s T^a G^a_\mu = \partial_\mu - i\frac{\lambda^a}{2}G^a_\mu, \quad a = 1, \ldots, 8$$

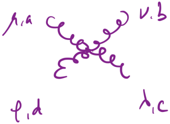
where λ^a are the Gell-Mann matrices (3×3 matrices).



$i g_s (T_a)_{\alpha\beta} \gamma^\mu \delta_{ff'}$



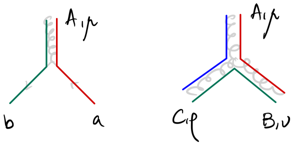
$g_s f^{abc} \left[g_{\mu\nu}(k_1-k_2)_\lambda + g_{\nu\lambda}(k_2-k_3)_\mu + g_{\lambda\mu}(k_3-k_1)_\nu\right]$



$-i g_s^2 \left[f^{abc} f^{cde} (g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) + \text{permutations}\right]$

COLOR ALGEBRA

In the limit of large N_c (yes, it is only 3!) we can represent



Since $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$ and $\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}$, we can build color singlest as

$$q\bar{q}' \sim \mathbf{1} \in \mathbf{3} \otimes \bar{\mathbf{3}} \Rightarrow \text{mesons} \quad \frac{1}{\sqrt{3}}\delta^{\alpha\beta}|q^{(\alpha)}\bar{q}'^{(\beta)}\rangle$$

$$qq'q'' \sim \mathbf{1} \in \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \Rightarrow \text{baryons} \quad \frac{1}{\sqrt{6}}\epsilon^{\alpha\beta\gamma}|q^{(\alpha)}q'^{(\beta)}q''^{(\gamma)}\rangle$$

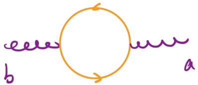
We can not build qq' invariants (but we can do tetraquarks and pentaquarks!).

COLOR FACTORS

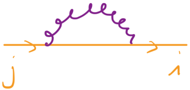
We can define useful color factors for a given $\mathfrak{su}(N)$ representation, $T_R^a,\ a = 1, \dots, N^2 - 1$:

$$\mathrm{Tr}[T_R^a T_R^b] = T_R \delta^{ab}, \qquad (T_R^a T_R^a)_{ij} = \delta_{ij} C_R$$

where T_R is the index of the irrep and C_R the quadratic Casimir. In particular for the fundamental (F) and the adjoint (A)



$\sim \mathrm{Tr}[T_F^a, T_F^b] = T_F \delta^{ab}$



$\sim (T_F^a T_F^a)_{ij} = \delta_{ij} C_F$

$\sim (T_A^a T_A^a)_{ij} = \delta_{ij} C_A$

One can show that $T_F = 1/2$, $C_F = (N^2 - 1)/2N$ and $C_A = N$.

THE QCD RUNNING

When renormalizing such theory non-abelian theory with $SU(N)$ and n_f flavors (at 1-loop) we obtain

$$\beta(g_s) = -\varepsilon g_s - \frac{g_s^3}{16\pi^2}\left[\frac{11}{3}C_A - \frac{4}{3}n_f T_F\right]$$

Then, for $N = 3$ and $\alpha_s = g_s^2/4\pi$ we obtain (at $\varepsilon = 0$)

$$\beta(\alpha_s) = \mu\frac{d}{d\mu}\alpha_s = -\frac{\alpha_s^2}{2\pi}\beta_0, \qquad \beta_0 = 11 - \frac{2n_f}{3}$$

As long as $n_f < 17$ we obtain at one loop that [Gross and Wilczek 1973]

$$\beta_0 > 0 \quad \text{and} \quad \alpha_s(\mu) \text{ decreases with energy!}$$

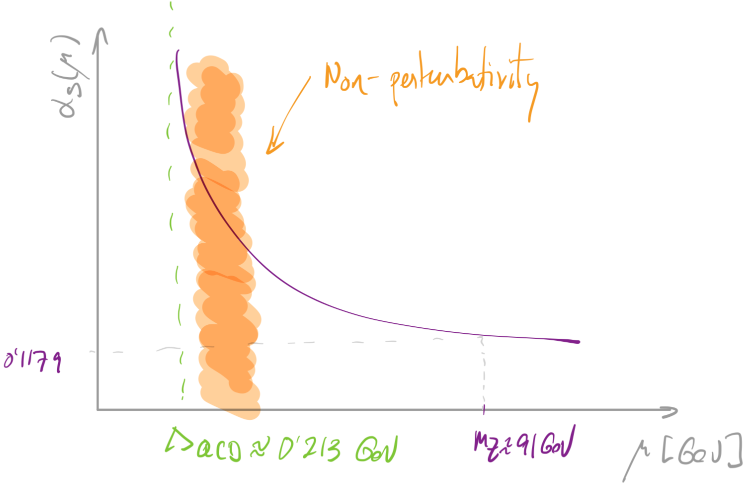
Solving the equation at one-loop results in

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0}\left[\log\left(\frac{\mu}{\Lambda_{\mathrm{QCD}}}\right)\right]^{-1}$$

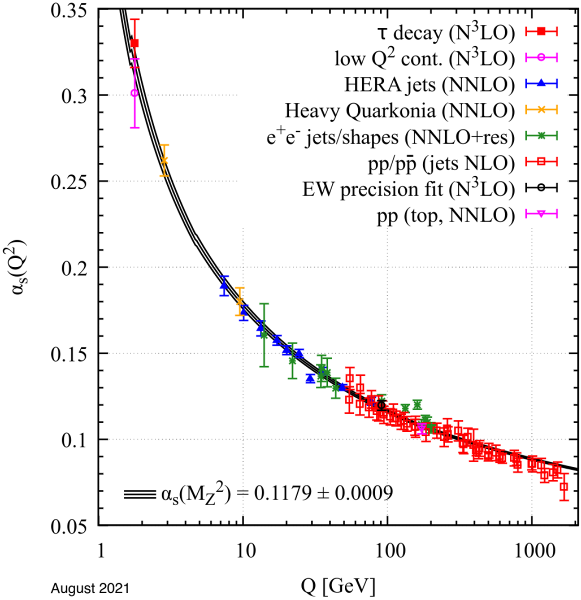
where Λ_{QCD} is the position of the QCD Landau pole. The coupling constant gets weaker at high energy $\Rightarrow$ asymptotic freedom!

ASYMPTOTIC FREEDOM

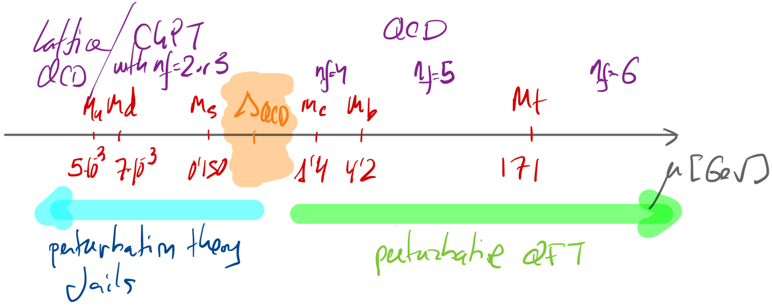
At higher orders, we get something similar



ASYMPTOTIC FREEDOM



DIFFERENT REGIMES OF QCD



- At high scales: coupling becomes small, quarks and gluons are almost free and the strong interactions become weak
- At low scales: coupling becomes large, quarks and gluons interact strongly and perturbation theory fails

QCD AT LOW ENERGIES

At energies $\lesssim \Lambda_{\rm QCD}$ perturbation theory fails. We can try to solve QCD numerically: lattice QCD. We can also try to take advante of the symmetries of the QCD Lagrangian at these scales.

For the case of $n_f = 3$ active flavors, neglecting quark masses, we can write

$$\mathcal{L}^0_{\rm QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu\, a} + i\bar{q}_L \gamma^\mu D_\mu q_L + i\bar{q}_R \gamma^\mu D_\mu q_R, \qquad q^T = (u, d, s),$$

which is invariant under a global $SU(3)_L \otimes SU(3)_R$

$$q_L \to g_L q_L, \qquad q_R \to g_R q_R, \qquad g_{L,R} \in SU(3)_{L,R}.$$

This symmetry will be spontaneously broken by the vacuum condensate (and explicitly by the different quark masses)

$$\langle 0 | \bar{q}_i q_j | 0 \rangle \propto \delta_{ij} \Lambda^3_{\rm QCD}$$

making $SU(3)_L \otimes SU(3)_R \to SU(3)_V$ and delivering thus $3^2 - 1 = 8$ (pseudo)Goldstone bosons.

CHIRAL PERTURBATION THEORY

We can write a non-linear realization of this symmetry breaking through the Goldstone matrix

$$U(\pi^a) = \exp(2i\pi^a T^a/f), \qquad \mathrm{Tr}(T^a \cdot T^b) = \frac{1}{2}\delta^{ab},$$

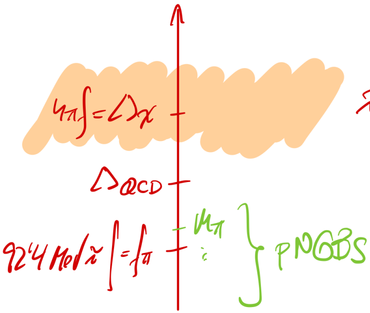
transforming as $U \to g_R U g_L^\dagger$, where

$$\pi^a T^a = \pi^a \frac{\lambda^a}{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}.$$

We can write the most general Lagrangian which is compatible with the symmetries of QCD organized in increasing powers of derivatives. At lowest order

$$\mathcal{L}_{\mathrm{ChPT}} = \frac{f^2}{4}\mathrm{Tr}\Big(\partial_\mu U^\dagger \partial^\mu U\Big)$$

CHIRAL PERTURBATION THEORY



ChPT

$$\frac{f^2}{4}\mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right)$$

$$+ \dots$$

Chiral Perturbation Theory: External Sources

The ChPT effective Lagrangian becomes much more powerful if we asumme that QCD is coupled to some external classical fields:

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}^0_{\text{QCD}} + \bar{q}\gamma^\mu(v_\mu + a_\mu\gamma_5)q - \bar{q}(s - i\gamma_5 p)q$$

where v_μ, a_μ, s, p are in principle 3×3 hermitian matrices. This Lagrangian is invariant under the following set of *local* $SU(3)_L \otimes SU(3)_R$ transformations

$$\begin{aligned}
&q_L \to g_L(x)q_L, \quad q_R \to g_R(x)q_R, \quad s + ip \to g_R(x)(s+ip)g_L(x)^\dagger, \\
&\ell_\mu \to g_L(x)\ell_\mu g_L(x)^\dagger + ig_L(x)\partial_\mu g_L(x)^\dagger, \\
&r_\mu \to g_R(x)r_\mu g_R(x)^\dagger + ig_R(x)\partial_\mu g_R(x)^\dagger,
\end{aligned}$$

where

$$r_\mu = v_\mu + a_\mu, \qquad \ell_\mu = v_\mu - a_\mu.$$

Chiral perturbation theory: external sources

The way to incorporate the external sources is through the covariant derivative

$$D_\mu U = \partial_\mu U - i r_\mu U + i U \ell_\mu, \qquad D_\mu U^\dagger = \partial_\mu U^\dagger + i U^\dagger r_\mu - i \ell_\mu U^\dagger.$$

Then, to lowest order,

$$\mathcal{L}_2 = \frac{f^2}{4} \mathrm{Tr}\left(D_\mu U^\dagger D^\mu U \right) + \frac{f^2}{4} \mathrm{Tr}\left(U^\dagger \chi + \chi^\dagger U \right), \quad \chi = 2B_0 (s + ip).$$

In the case of QCD,

$$s = \mathcal{M} + \ldots = \mathrm{diag}(m_u, m_d, m_s) + \ldots, \quad p = 0,$$

$$r_\mu = e\mathcal{Q} A_\mu + \ldots, \quad \mathcal{Q} = \frac{1}{3}\mathrm{diag}(2, -1, -1).$$

$$\ell_\mu = e\mathcal{Q} A_\mu + \frac{e}{\sqrt{2} s_W}\left(W_\mu^+ T^+ + \mathrm{h.c.} \right) + \ldots, \quad T^+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Chiral Perturbation Theory: Meson Masses

We obtain for the different mesons:

$$m_{\pi^\pm}^2 = 2\hat{m}B_0, \qquad m_{\pi^0}^2 = 2\hat{m}B_0 - \delta + \mathcal{O}(\delta^2),$$
$$m_{K^\pm}^2 = (m_u + m_s)B_0, \qquad m_{K^0}^2 = (m_d + m_s)B_0,$$
$$m_{\eta_8}^2 = \frac{2}{3}(\hat{m} + 2m_s)B_0 + \delta + \mathcal{O}(\delta^2)$$

where

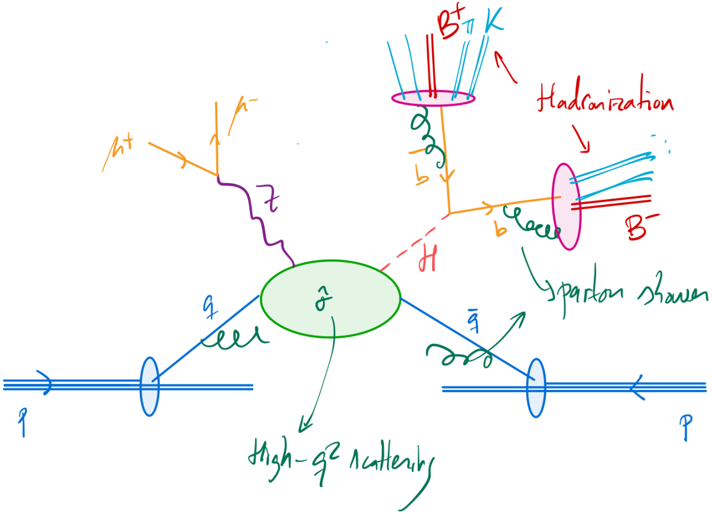
$$\hat{m} = \frac{1}{2}(m_u + m_d), \qquad \delta = \frac{B_0}{4}\frac{(m_u - m_d)^2}{(m_s - \hat{m})}.$$

However, we are just describing the lightest mesons, which are pNGBs

$\Rightarrow$ Resonances and baryons are missing

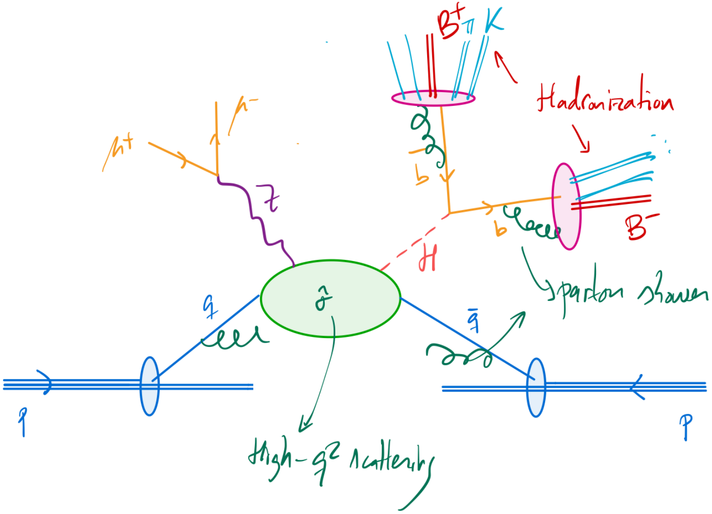
QCD AT HIGH ENERGIES

At high energies, we can rely on perturbation theory. We will concentrate on (hadron) colliders. However, even in this case, stuff is complicated.



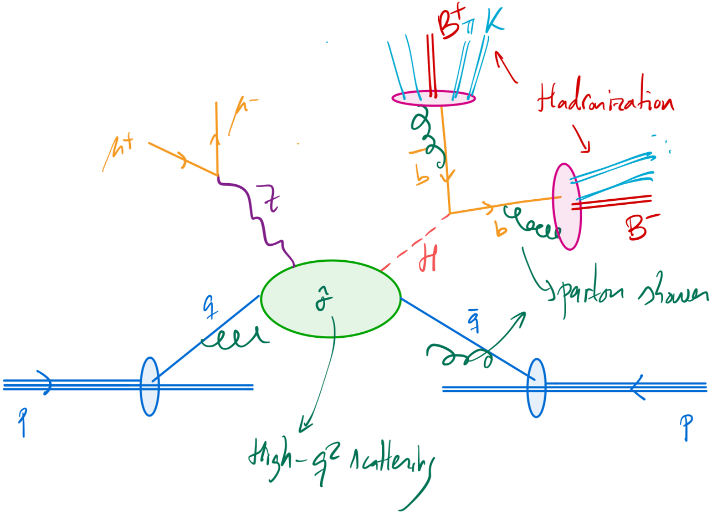
QCD AT HIGH ENERGIES

At colliders, very different energy scales are at work. Neither perturbation theory nor lattice QCD can give a full solution for QCD at colliders.



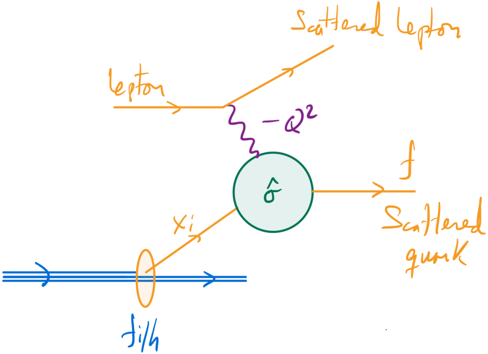
QCD AT HIGH ENERGIES

The common approach relies on perturbative QCD $\oplus$ non-perturbative modelling/factorization.



FACTORIZATION

Factorization in Deep Inelastic Scattering (DIS) [Collins and Soper 1987]



The cross-section can be written in factorised form:

$$\sigma^{\ell h} = \sum_i \sum_f \int dx_i \int d\Phi_f \, f_{i/h}(x_i, Q_F^2) \frac{d\hat{\sigma}^{\ell i \to f}(x_i, \Phi_f, Q_F^2)}{dx_i d\Phi_f}$$

FACTORIZATION

We assume that an analogous factorization works for hadron collisions

$$\frac{d\sigma}{dX} = \sum_{a,b}\sum_{f}\int_{\hat{X}_f} f_a(x_a, Q_i^2) f_b(x_b, Q_i^2) \times \frac{d\hat{\sigma}_{ab\to f}(x_a, x_b, f, Q_i^2, Q_f^2)}{d\hat{X}_f}$$
$$\times D(\hat{X}_f \to X, Q_i^2, Q_f^2)$$

⋆ $f_a(x_a, Q_i^2)$ Parton Distribution Function (PDF). It gives the probability of finding a quark/gluon a inside the incoming hadron, carrying a fraction x_i of the incoming momentum. Determined experimentally.

⋆ $d\hat{\sigma}/d\hat{X}_f$ Differential Partonic Hard Scattering. Computed in perturbation theory.

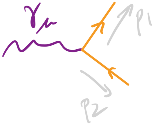
⋆ $D(\hat{X}_f \to X, Q_i^2, Q_f^2)$ Fragmentation Functions. Connect high-scale processes with final state hadrons. Phenomenological models.

Stuff is even more complicated in experiments like ATLAS and CMS since we do not identify hadrons but 'jets' from the activity in the hadronic calorimeter.

FACTORIZATION

To understand better factorization we study e^+e^- collisions with hadronic final state.

Start with $\gamma^* \to q\bar{q}$



$$i\mathcal{M}_{\bar{q}q} = \bar{u}(p_1)\, ieQ_q\, v(p_2)$$

Now let's radiate a gluon

$$i\mathcal{M}_{\bar{q}qg} = \bar{u}(p_1)\, ig_s T^a \not{\epsilon}^* \frac{i}{\not{p}_1 + \not{k}}\, ieQ_q \gamma_\mu v(p_2)$$

$$- \bar{u}(p_1)\, ieQ_q \gamma_\mu \frac{i}{\not{p}_2 + \not{k}}\, ig_s T^a \not{\epsilon}^* v(p_2)$$

FACTORIZATION

$$i\mathcal{M}_{\bar{q}qg} = \bar{u}(p_1)\, i g_s T^a \not{\epsilon}^* \frac{i}{\not{p}_1 + \not{k}}\, ieQ_q \gamma_\mu v(p_2) - \bar{u}(p_1)\, ieQ_q \gamma_\mu \frac{i}{\not{p}_2 + \not{k}}\, i g_s T^a \not{\epsilon}^* v(p_2)$$

After some algebra, if we make the gluon soft, i.e., $k \ll p_1, p_2$, we obtain

$$i\mathcal{M}_{q\bar{q}g} \approx \bar{u}(p_1) ieQ_q \gamma_\mu T^a v(p_2) g_s \left(\frac{p_1 \cdot \epsilon^*}{p_1 \cdot k} - \frac{p_2 \cdot \epsilon^*}{p_2 \cdot k} \right)$$

Then

$$\sum_{a,\epsilon} |\mathcal{M}_{q\bar{q}g}|^2 = \sum_{a,\epsilon} \left| \bar{u}(p_1) eQ_q \gamma_\mu T^a v(p_2) g_s \left(\frac{p_1 \cdot \epsilon^*}{p_1 \cdot k} - \frac{p_2 \cdot \epsilon^*}{p_2 \cdot k} \right) \right|^2$$

$$= -|\mathcal{M}_{q\bar{q}}|^2 C_F g_s^2 \left(\frac{p_1}{p_1 \cdot k} - \frac{p_2}{p_2 \cdot k} \right)^2 = |\mathcal{M}_{q\bar{q}}|^2 C_F g_s^2 \frac{2 p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)}$$

FACTORIZATION

If we include phase space, we obtain

$$d\Phi_{q\bar{q}g}|\mathcal{M}_{q\bar{q}g}|^2 \approx (d\Phi_{q\bar{q}}|\mathcal{M}_{q\bar{q}}|^2)\frac{d^3\mathbf{k}}{2E_{\mathbf{k}}(2\pi)^3}C_F g_s^2 \frac{2p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}$$
$$\approx (d\Phi_{q\bar{q}}|\mathcal{M}_{q\bar{q}}|^2)d\mathcal{S}$$

where the soft-gluon emission piece can be written as (with $\theta = (\widehat{\mathbf{k},\mathbf{p}_1})$)

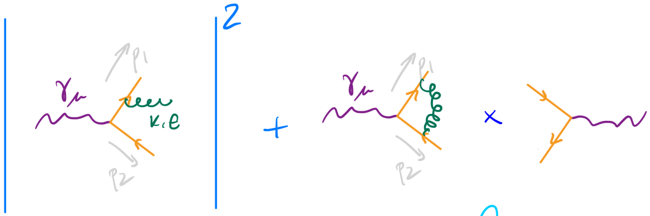
$$d\mathcal{S} = EdEd\cos\theta\frac{d\phi}{2\pi}\frac{2\alpha_s C_F}{\pi}\frac{2p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)} = \frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\sin\theta}\frac{d\phi}{2\pi}$$

which diverges for $E = 0$ (soft or IR) and for $\sin\theta \to 0$ (collinear).

So we have seen an example of factorization of soft and/or collinear divergences, which is indeed a very general feature of QCD.

SOFT/COLLINEAR DIVERGENCES

Kinoshita-Lee-Nauenberg theorem tells that if we sum over all the allowed states, the result should be finite!



Real + virtual = finite

Thus, if we regularize the divergences with dimensional regularization (dimreg), with $D = 4 - 2\varepsilon$, $1/\varepsilon$ terms of the real part should be cancelled by $1/\varepsilon$ terms of the virtual part

$$\sigma_{\text{tot}} = \sigma_{q\bar{q}} \left(1 + \frac{3}{4} \frac{\alpha_s C_F}{\pi} + \mathcal{O}(\alpha_s^2) \right)$$

Total cross-section

- Corrections to σ_{tot} come from hard large-angle gluons, as well as from large virtualities: physics at short distance.
- Since soft gluons are emitted on long timescales relative to the collision scale they can not influence the cross section.
- Hadronization also happens on a long timescale and is thus factorized.

INFRARED AND COLLINEAR-SAFE OBSERVABLES

We can also avoid the problem of IR divergences if we define IR and collinear-safe obervables $\mathcal{O}$. Then, if

$$d\sigma^{\text{Born}} = \mathcal{B}(\Phi_B)d\Phi_B, \qquad d\sigma^{\text{NLO}} = \left[\mathcal{B}(\Phi_B) + \mathcal{V}(\Phi_B)\right]d\Phi_B + \mathcal{R}(\Phi_R)d\Phi_R$$

with $d\Phi_R = d\Phi_B d\Phi_{\text{rad}} = d\Phi_B dE d\cos\theta d\phi$,

$$\begin{aligned}\langle\mathcal{O}\rangle = \int \Big[\mathcal{B}(\Phi_B) + \mathcal{V}(\Phi_B) + \int \mathcal{C}(\Phi_R)d\Phi_{\text{rad}}\Big]\,\mathcal{O}(\Phi_B)d\Phi_B \\ + \Big[\mathcal{R}(\Phi_R)\mathcal{O}(\Phi_R) - \mathcal{C}(\Phi_R)\mathcal{O}(\Phi_B)\Big]d\Phi_R\end{aligned}$$

with $\mathcal{C}(\Phi_R) \to \mathcal{R}(\Phi_R)$ in the soft/collinear limit. Both, the □ and the □ pieces are independently finite.

SUDAKOV FORM FACTOR

Factorization of soft/collinear emissions allows us to write

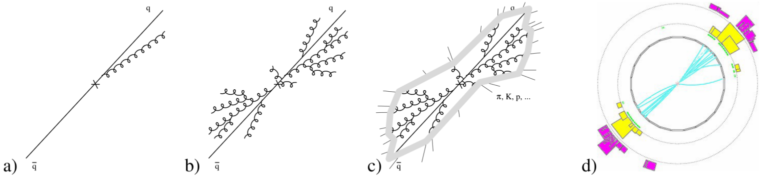
$$d\sigma_{n+1} = d\sigma_n(\Phi_n)\mathcal{P}(\Phi_{\text{rad}})d\Phi_{\text{rad}}, \qquad \mathcal{P}(\Phi_{\text{rad}})d\Phi_{\text{rad}} \approx \frac{\alpha_s(q)}{\pi}\frac{dq}{q}P(z,\phi)dz\frac{d\phi}{2\pi}$$

Then, we can introduce the so-called Sudakov form factor

$$\Delta_S(q_1, q_2) = \exp\left\{-\int_{q_1}^{q_2}\frac{\alpha_s(q)}{\pi}\frac{dq}{q}\int_{z_0}^{1}P(z)dz\right\}$$

which gives the probability of no emission between the scales q_1 and q_2.
This is the principle of monte-carlo parton-showers!

SHOWERING AND HADRONIZATION



Taken from G. P. Salam 2020.

REFERENCES I

't Hooft, Gerard and M. J. G. Veltman (1972). "Regularization and Renormalization of Gauge Fields". In: *Nucl. Phys. B* 44, pp. 189–213. DOI: 10.1016/0550-3213(72)90279-9 (cit. on pp. 40, 41).

Altarelli, Guido (2017). *Collider Physics within the Standard Model: A Primer*. Ed. by James Wells. Vol. 937. Springer. DOI: 10.1007/978-3-319-51920-3 (cit. on p. 5).

Becchi, C., A. Rouet, and R. Stora (1976). "Renormalization of Gauge Theories". In: *Annals Phys.* 98, pp. 287–321. DOI: 10.1016/0003-4916(76)90156-1 (cit. on p. 30).

Belavin, A. A. et al. (1975). "Pseudoparticle Solutions of the Yang-Mills Equations". In: *Phys. Lett. B* 59. Ed. by J. C. Taylor, pp. 85–87. DOI: 10.1016/0370-2693(75)90163-X (cit. on p. 77).

REFERENCES II

- Branco, Gustavo C., Luis Lavoura, and Joao P. Silva (1999). *CP Violation*. Clarendon Press (cit. on p. 5).
- Cabibbo, Nicola (1963). "Unitary Symmetry and Leptonic Decays". In: *Phys. Rev. Lett.* 10, pp. 531–533. DOI: 10.1103/PhysRevLett.10.531 (cit. on p. 73).
- Cheng, Ta-Pei and Ling-Fong Li (1984). *Gauge Theory of Elementary Particle Physics*. Oxford, UK: Oxford University Press (cit. on p. 4).
- Coleman, Sidney R. and Erick J. Weinberg (1973). "Radiative Corrections as the Origin of Spontaneous Symmetry Breaking". In: *Phys. Rev. D* 7, pp. 1888–1910. DOI: 10.1103/PhysRevD.7.1888 (cit. on p. 49).

References III

Collins, John C. and Davison E. Soper (1987). "The Theorems of Perturbative QCD". In: *Ann. Rev. Nucl. Part. Sci.* 37, pp. 383–409. DOI: 10.1146/annurev.ns.37.120187.002123 (cit. on p. 122).

Donoghue, John F., Eugene Golowich, and Barry R. Holstein (2014). *Dynamics of the Standard Model : Second edition*. Cambridge University Press. DOI: 10.1017/CBO9780511803512 (cit. on p. 4).

Ellis, R. Keith, W. James Stirling, and B. R. Webber (1996). *QCD and collider physics*. Cambridge University Press. DOI: 10.1017/CBO9780511628788 (cit. on p. 5).

Englert, F. and R. Brout (1964). "Broken Symmetry and the Mass of Gauge Vector Mesons". In: *Phys. Rev. Lett.* 13. Ed. by J. C. Taylor, pp. 321–323. DOI: 10.1103/PhysRevLett.13.321 (cit. on p. 60).

Faddeev, L. D. and V. N. Popov (1967). “Feynman Diagrams for the Yang-Mills Field”. In: *Phys. Lett. B* 25. Ed. by Jong-Ping Hsu and D. Fine, pp. 29–30. DOI: 10.1016/0370-2693(67)90067-6 (cit. on p. 28).

Fujikawa, Kazuo (1979). “Path Integral Measure for Gauge Invariant Fermion Theories”. In: *Phys. Rev. Lett.* 42, pp. 1195–1198. DOI: 10.1103/PhysRevLett.42.1195 (cit. on p. 98).

Glashow, S. L. (1961). “Partial Symmetries of Weak Interactions”. In: *Nucl. Phys.* 22, pp. 579–588. DOI: 10.1016/0029-5582(61)90469-2 (cit. on p. 7).

Glashow, S. L., J. Iliopoulos, and L. Maiani (1970). “Weak Interactions with Lepton-Hadron Symmetry”. In: *Phys. Rev. D* 2, pp. 1285–1292. DOI: 10.1103/PhysRevD.2.1285 (cit. on p. 92).

Goldstone, J. (1961). "Field Theories with Superconductor Solutions". In: *Nuovo Cim.* 19, pp. 154–164. DOI: 10.1007/BF02812722 (cit. on p. 53).

Gross, David J. and Frank Wilczek (1973). "Ultraviolet Behavior of Nonabelian Gauge Theories". In: *Phys. Rev. Lett.* 30. Ed. by J. C. Taylor, pp. 1343–1346. DOI: 10.1103/PhysRevLett.30.1343 (cit. on p. 109).

Guralnik, G. S., C. R. Hagen, and T. W. B. Kibble (1964). "Global Conservation Laws and Massless Particles". In: *Phys. Rev. Lett.* 13. Ed. by J. C. Taylor, pp. 585–587. DOI: 10.1103/PhysRevLett.13.585 (cit. on p. 60).

Higgs, Peter W. (1964). "Broken Symmetries and the Masses of Gauge Bosons". In: *Phys. Rev. Lett.* 13. Ed. by J. C. Taylor, pp. 508–509. DOI: 10.1103/PhysRevLett.13.508 (cit. on p. 60).

Illana, José Ignacio and Alejandro Jimenez Cano (2022). "Quantum field theory and the structure of the Standard Model". In: DOI: 10.22323/1.406.0314. arXiv: 2211.14636 [hep-ph] (cit. on p. 5).

Kobayashi, Makoto and Toshihide Maskawa (1973). "CP Violation in the Renormalizable Theory of Weak Interaction". In: *Prog. Theor. Phys.* 49, pp. 652–657. DOI: 10.1143/PTP.49.652 (cit. on p. 73).

Langacker, Paul (2017). *The Standard Model and Beyond*. Taylor & Francis. DOI: 10.1201/b22175 (cit. on p. 4).

Leader, E. and E. Predazzi (1996). *An Introduction to gauge theories and modern particle physics*. Vol. 1. Cambridge University Press. DOI: 10.1017/CBO9780511622595 (cit. on p. 4).

Lehmann, H., K. Symanzik, and W. Zimmermann (1955). "On the formulation of quantized field theories". In: *Nuovo Cim.* 1, pp. 205–225. DOI: 10.1007/BF02731765 (cit. on p. 23).

Nambu, Yoichiro (1960). "Quasiparticles and Gauge Invariance in the Theory of Superconductivity". In: *Phys. Rev.* 117. Ed. by J. C. Taylor, pp. 648–663. DOI: 10.1103/PhysRev.117.648 (cit. on p. 53).

Pich, A. (1995). "Chiral perturbation theory". In: DOI: 10.1088/0034-4885/58/6/001. arXiv: hep-ph/9502366 (cit. on p. 5).

Pich, Antonio (2012). "The Standard Model of Electroweak Interactions". In: arXiv: 1201.0537 [hep-ph] (cit. on p. 5).

Pokorski, S. (2000). *Gauge Field Theores*. Cambridge University Press. DOI: 10.1017/CBO9780511612343 (cit. on p. 4).

Ramond, Pierre (1999). *Journeys beyond the standard model*. Perseus Books (cit. on p. 4).

Salam, Abdus (1968). "Weak and Electromagnetic Interactions". In: *Conf. Proc. C* 680519, pp. 367–377. DOI: 10.1142/9789812795915_0034 (cit. on p. 7).

Salam, G. P. (2020). "Elements of QCD for hadron colliders". In: *CERN Yellow Rep. School Proc.* 5. Ed. by M. Mulders and J. Trân Thanh Vân, pp. 1–56. DOI: 10.23730/CYRSP-2020-005.1 (cit. on pp. 6, 131).

Schwartz, Matthew D. (2013). *Quantum Field Theory and the Standard Model*. Cambridge University Press. DOI: 10.1017/9781139540940 (cit. on p. 4).

REFERENCES IX

Skands, Peter (2013). "Introduction to QCD". In: *Theoretical Advanced Study Institute in Elementary Particle Physics: Searching for New Physics at Small and Large Scales*, pp. 341–420. DOI: 10.1142/9789814525220_0008. arXiv: 1207.2389 [hep-ph] (cit. on p. 6).

Slavnov, A. A. (1972). "Ward Identities in Gauge Theories". In: *Theor. Math. Phys.* 10, pp. 99–107. DOI: 10.1007/BF01090719 (cit. on p. 30).

Taylor, J. C. (1971). "Ward Identities and Charge Renormalization of the Yang-Mills Field". In: *Nucl. Phys. B* 33, pp. 436–444. DOI: 10.1016/0550-3213(71)90297-5 (cit. on p. 30).

Weinberg, Steven (1967). "A Model of Leptons". In: *Phys. Rev. Lett.* 19, pp. 1264–1266. DOI: 10.1103/PhysRevLett.19.1264 (cit. on p. 7).

Yang, Chen-Ning and Robert L. Mills (1954). "Conservation of Isotopic Spin and Isotopic Gauge Invariance". In: *Phys. Rev.* 96. Ed. by Jong-Ping Hsu and D. Fine, pp. 191–195. DOI: 10.1103/PhysRev.96.191 (cit. on p. 20).