Exercises for QFT & EFTs tutorials

Exercise 1. In the lecture we have discussed the dimensional analysis for scalars in dimensional regularization. Following the same reasoning obtain the dimensional scaling of fermions and vector bosons.

Exercise 2. Compute the one loop relation between the pole bottom quark mass and the $\overline{\rm MS}$ renormalization scheme one evaluated at $\mu = m_b$.

Exercise 3 (Scaleless integrals)**.** Scaleless integrals vanish in dimensional regularization as we have seen in the lectures.

a. Proof that one can express

$$\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^4} = I_1 - I_2,$$

$$I_1 \equiv \int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k^2-M^2)}, \quad I_2 \equiv M^2 \int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^4(k^2-M^2))}$$

and compute both integrals in dimensional regularization.

b. Is each of the integral divergent in the UV or in the IR? What does it mean in terms of renormalization?

You may use the formula

$$I_{nm} \equiv \tilde{\mu}^{4-D}\int \frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2)^n(k^2-M^2)^m}$$
$$= \frac{i(-M^2)^{2-\alpha-\beta}}{(4\pi)^2}\left(\frac{M^2}{4\pi\tilde{\mu}^2}\right)^{\frac{D-4}{2}} \frac{\Gamma\left(\frac{D}{2}-\alpha\right)\Gamma\left(\alpha+\beta-\frac{D}{2}\right)}{\Gamma(\beta)\Gamma\left(\frac{D}{2}\right)}$$

Exercise 4 (Fermi Theory)**.** Consider the muon decay into electron $\mu \to e\overline{\nu}_e\nu_\mu$. The Fermi theory for weak interactions in the lepton sector is given at leading order by the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\rm QED} - \frac{4G_F}{\sqrt{2}}\left(\bar{e}\gamma^\mu P_L\nu_e\right)\left(\overline{\nu}_\mu\gamma_\mu P_L\mu\right),$$

where $P_L = (1-\gamma_5)/2$.

a. Compute the leading order muon decay rate in the Fermi theory

b. Compare the result with the experimental value for the muon mean lifetime $\tau_\mu = 2.1970 \cdot 10^{-6}$ s to obtain the value of G_F.

c. Compute the muon decay process in the Standard Model in the low energy limit and find an expression for G_F at leading order in terms of SM parameters. Compare the result with the one obtained before. Is the agreement as expected?

Exercise 5 (Euler-Heissenberg Lagrangian)**.** We consider the photon-photon scattering at energies much below the electron mass. The theory for this process is described by the Euler-Heissenberg Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{\Lambda^4}\left[c_1\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + c_2\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right]$$

a. Thinking of the lowest order diagram in QED that contributes to the $\gamma\gamma$ scattering, what is a good estimate for the cut-off scale Λ?

b. Based on the QED diagram and dimensional analysis, make an estimate for the amplitude of the QED diagram in the low-energy limit in terms of the coupling constant α and the electron mass m_e and the photon energy ω.

c. What is then a reasonable estimate for the scattering cross section?

d. Knowing that the true matching onto QED gives as a result $c_1 = \alpha^2/90$, $c_2 = 7\alpha^2/360$ so that $\sigma = \alpha^4\omega^6/(16\pi m_e^8)15568/10125$. How good was your naive estimate based on power counting and dimensional analysis?

e. How different would this scaling be if we did not impose gauge symmetry?

Exercise 6. **a.** Compute the integral

$$I = \mu^{4-D}\int \frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-M^2)(k^2-m^2)}.$$

b. Compute the integral

$$I_F = \mu^{4-D}\int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m^2}\frac{-1}{M^2}\left(1+\frac{k^2}{M^2}+\frac{k^4}{M^4}+\dots\right)$$

c. Compute the integral I using the expansion by regions.

Exercise 7. Compute the 1-loop photon polarization $\Pi(q^2)$ induced by a fermion of mass M and charge Q.

a. Use the "physical" point $\mu^2 = -q^2$ to subtract the UV divergence. Compute the renormalized self-energy and the β function. Demonstrate that decoupling occurs when $M^2 \gg \mu^2, q^2$.

b. Repeat the same calculation in the MS scheme. Show that there is no decoupling when $M \to \infty$ and perturbation theory breaks down in that limit.

The β function is the anomalous dimension associated to the charge and it is usually defined as

$$\mu \frac{d\alpha}{d\mu} = \beta(\alpha) \quad \textit{with} \quad \beta(\alpha) = -2\alpha \sum_{n=0}^{\infty} \left(\frac{\alpha}{4\pi}\right)^{n+1} \beta_n$$

Exercise 8. Consider the theory of QCD with n_f number of quark flavors $\mathcal{L}_{\rm QCD}^{(n_f)}$. In the case where there are $n_f - 1$ light quark flavors and one heavy quark of mass M, one can develop an EFT described by $\mathcal{L}_{\rm QCD}^{(n_f-1)}$ plus operators suppressed by powers of $1/M$ which we may neglect in this exercise.

The one loop QCD β function, $\beta_0 = 11/3C_A - 4/T_F n_f$ depends on the number of light flavors, and thus the coupling constant in each of the theories will be different and the can be related order by order by an equation

$$\alpha_s^{(n_f)}(\mu) = \alpha_s^{(n_f-1)}(\mu) \left[1 + \sum_k c_k(L) \left(\frac{\alpha_s^{(n_f-1)}(\mu)}{4\pi}\right)^k\right]$$

where $L = \ln(\mu/M)$.

a. Knowing that $c_0(0) = 0$ compute $c_0(L)$.

b. Compute in both theories the momentum dependence of α_s as a function of β_0 and $\beta_1 = 34/3C_A^2 - 20/3C_A T_F n_f - 4C_F T_F n_f$.

c. Make a plot of α_s between 3 and 100 GeV.

Exercise 9 (Non-relativistic QED)**.** NRQED is the low energy theory for QED suitable for leptons with large masses compared to their typical momentum scales.

a. Obtain the NRQED Lagrangian up to order $1/m^2$ following the steps in the lecture.

b. Obtain the leading contribution to the Wilson coefficients by matching onto the QED vertex function at low energies.

Exercise 10 (pNRQED for hydrogen)**.** pNRQED is a low energy theory of NRQED where the dynamical degrees of freedom are only potential fermions $(E, |\mathbf{P}|) \sim (M\alpha^2, M\alpha)$, where M is the typical fermion mass, and ultrasoft photons $p_\gamma \sim M\alpha^2$ and it is suited for computations of two heavy fermion bound states $F\bar{F}$. Its Lagrangian can be written in a compact way as

$$L_{\rm pNRQED} = \int d^3\mathbf{r} d^3\mathbf{R} dt S^\dagger(\mathbf{r}, \mathbf{R}, t) \left\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) + e\mathbf{r} \cdot E(\mathbf{R}, t) \right\} S(\mathbf{r}, \mathbf{R}, t)$$
$$- \int d^3\mathbf{R} \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where S is the field representing the hydrogen, $\mathbf{R}$ the center of mass coordinate and $\mathbf{r}$ the relative distance. V stands for the potential and admits an expansion in powers of $1/m_e$

$$V = V^{(0)} + \frac{V^{(1)}}{m_e} + \frac{V^{(2)}}{m_e^2} + \dots$$

We know that the Lamb shift in hydrogen $2S_{1/2} - 2P_{1/2}$ is an effect of order $\mathcal{O}(m_e\alpha^4)$.

a. To what order in the expansion in α do we need to compute each $V^{(i)}$ in order to obtain the leading order Lamb shift?

b. Compute the $V^{(1)}$ potential up to order α^0 by matching onto NRQED.

c. Compute the $V^{(2)}$ potential up to order α by matching onto NRQED.

d. Compute the leading order Lamb shift in hydrogen from pNRQED.