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The Standard Model of Particle Physics
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SM Tutorial

Part 1. First tutorial

Problem 1. Check that the vectorial representation of the Lorentz group

$$(J^{\mu\nu})^{\alpha}{}_{\beta} = i(g^{\mu\alpha}\delta^{\nu}_{\beta} - g^{\nu\alpha}\delta^{\mu}_{\beta})$$

satisfies the Lorentz algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}).$$

Problem 2. Prove that, if we define the covariant derivative as

$$D_\mu = \partial_\mu - igT^a A^a_\mu = \partial_\mu - ig\hat{A}_\mu, \qquad \hat{A}_\mu = A^a_\mu T^a,$$

the term $i\bar{\psi}\not{D}\psi = i\bar{\psi}\gamma^\mu D_\mu\psi$ is invariant under gauge transformations

$$\psi \to U\psi, \quad U = \exp\Big(-iT^a\theta^a(x)\Big),$$
$$\hat{A}_\mu \to U\hat{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger.$$

Problem 3. Check that the gauge transformation

$$\hat{A}_\mu \to U\hat{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger, \qquad U = \exp\Big(-iT^a\theta(x)\Big),$$

translates for infinitesimal gauge transformations as

$$A^a_\mu \to A^a_\mu - f^{abc}A^b_\mu\theta - \frac{1}{g}\partial_\mu\theta^a$$

with $[T^a, T^b] = if^{abc}T^c$ and $\hat{A}_\mu = A^a_\mu T^a$.

Problem 4. Check that

$$\hat{A}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - ig\Big[\hat{A}_\mu, \hat{A}_\nu\Big]$$

implies that

$$A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$$

where $\hat{A}_{\mu\nu} = A^a_{\mu\nu}T^a$.

Problem 5. Check that the gauge transformation

$$\hat{A}_\mu \to U\hat{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger, \quad U = \exp\Big(-igT^a\theta(x)\Big),$$

implies that

$$\hat{A}_{\mu\nu} \to U\hat{A}_{\mu\nu}U^\dagger.$$

Demonstrate also that this implies that

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2}\text{Tr}\Big(\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\Big)$$

is gauge invariant.

Problem 6. Show that the Faddeev-Popov determinant reads

$$M_{ab} = \frac{\delta F[A^a_\mu]}{\delta\theta^b} = -\frac{1}{g}(\delta^{ab}\Box - gf^{abc}A^c_\mu\partial^\mu)$$

using that under a gauge transformation $U = \exp(-iT^a\theta^a(x))$

$$A^a_\mu \mapsto A^a_\mu - f^{abc}A^b_\mu\theta^c - \frac{1}{g}\partial_\mu\theta^a.$$

Finally, shows that redefining the ghost fields and integrating by parts we obtain

$$\mathcal{L}_{\text{FP}} = (\partial^\mu\bar{\eta}^a)(D^{\text{Adj}}_\mu)^{ab}\eta^b, \qquad D^{\text{Adj}}_\mu = \partial_\mu - igT^{\text{Adj}}_c A^c_\mu, \qquad (T^{\text{Adj}}_c)_{ab} = -if^{abc}.$$

Why are ghost irrelevants for abelian theories?

Problem 7. Show that the full gauge Lagrangian for a Yang-Mills theory

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} = -\frac{1}{4}A^a_{\mu\nu}A^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + (\partial^\mu\bar{\eta}^a)(\partial_\mu - gf^{abc}\eta^b A^c_\mu).$$

is invariant under BRS transformations,

$$A^a_\mu \to A^a_\mu + \delta A^a_\mu, \qquad \delta A^a_\mu = -\frac{1}{g}\theta(D^{\text{Adj}}_\mu)^{ab}\eta^b,$$

$$\eta^a \to \eta^a + \delta\eta^a, \qquad \eta^a = \frac{1}{2}\theta f^{abc}\eta^b\eta^c,$$

$$\bar{\eta}^a \to \bar{\eta}^a + \delta\bar{\eta}^a, \qquad \bar{\eta}^a = \frac{1}{g}\theta\frac{1}{\xi}\partial^\nu A^a_\nu,$$

where θ is a Grassmann variable $\{\theta,\theta\} = \{\theta,\eta^a\} = \{\theta,\eta^a\} = \{\eta^a,\eta^b\} = 0$. For that:

1) Check that $\mathcal{L}_{\text{YM}}$ is invariant under A^a_μ BRS transformations since

$$A^a_\mu \to A^a_\mu - \frac{1}{g}\theta(\partial_\mu - gf^{abc}\eta^b A^c_\mu)$$

can be written as a gauge transformation of gauge parameter $\omega^a = \theta\eta^a$.

2) Assuming that $(D^{\text{Adj}}_\mu)^{ab}\eta^b$ is invariant under BRS transformations, check that the BRS transformations of A^a_μ and $\bar{\eta}^a$ precisely make the Faddeed-Poppov term invariant.

3) Using the Jacobi identity

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0$$

check that

$$(D^{\text{Adj}}_\mu)^{ab}\eta^b = \partial_\mu\eta^b - gf^{abc}\eta^b A^c_\mu$$

is indeed invariant.

Problem 8. Consider the Proca Lagrangian of a massive abelian vector field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu, \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

show that the propagator takes the following form

$$\tilde{D}_{\mu\nu}(k) = \frac{i}{k^2 - M^2 + i0^+}\Big[-g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\Big].$$

Hint: Write the quadratic piece of the lagrangian as

$$\mathcal{L} \supset \frac{1}{2}A_\nu\Pi^{\mu\nu}(k)A_\mu$$

and define the propagator as

$$\tilde{D}_{\mu\nu} = i(\hat{\Pi}^{\mu\nu}(k))^{-1}$$

where $\hat{\Pi}^{\mu\nu}$ is the Fourier transformed of $\Pi^{\mu\nu}$ and $^{-1}$ denotes the inverse operator.

Problem 9. Compute the mass dimension of the following operators

$$(a)\ \varphi^4, \quad (b)\ \varphi^6, \quad (c)\ \varphi\bar{\psi}\psi, \quad (d)\ \varphi^3\bar{\psi}\psi, \quad (e)\ A_{\mu\nu}A^{\mu\nu},$$

$$(f)\ \Big(\bar{q}_L^i\gamma_\mu q_L^j\Big)\Big(u_R^k\gamma^\mu u_R^l\Big), \quad (g)\ \Big(\bar{q}_L^i\gamma_\mu \frac{\sigma^i}{2} q_L^j\Big)\Big(q_L^k\gamma^\mu \frac{\sigma^i}{2} q_L^l\Big), \quad (h)\ f^{abc}G_\mu^{a\,\nu}G_\nu^{b\,\rho}G_\rho^{c\,\mu},$$

where φ is a scalar singlet, ψ is a gauge-singlet Dirac fermion, $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with A_μ an abelian gauge boson, while $G^a_{\mu\nu}$ is the gluon field tensor and q_L, u_R are the usual quark chiral fields of the SM.

Problem 10. Demonstrate that

$$G^a_{\mu,k} = \langle 0|T\{j^a_\mu(x)\varphi_k(y)|0\rangle = \theta(x^0-y^0)\langle 0|j^a_\mu(x)\varphi_k(y)|0\rangle + \theta(y^0-x^0)\langle 0|\varphi_k(y)j^a_\mu(x)|0\rangle$$

implies that

$$\partial_x^\mu G_{\mu,k}(x-y) = \delta(x^0-y^0)\langle 0|[j^a_0(x),\varphi_k(y)]|0\rangle.$$

Problem 11. Let us consider a real scalar $SU(2)$ triplet Φ with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi^T)(\partial^\mu\Phi) - \frac{1}{2}\mu^2(\Phi^T\Phi) - \frac{\lambda}{4}(\Phi^T\Phi)^2.$$

1) Show that such Lagrangian is invariant under global $SU(2)$ rotations

$$\Phi \to U\Phi = \exp\Big(-iT_a^{adj}\theta_a\Big)\Phi, \qquad (T_a^{adj})_{bc} = -i\epsilon^{abc},$$

with ϵ^{abc} the fully antisymmetric rank 3 tensor.
2) Show that if $\mu^2 < 0$ and $\lambda > 0$ the potential has a minimum for $\langle 0|\Phi^T\Phi|0\rangle = v^2$ with $v^2 = -\mu^2/\lambda$.
3) Write the Lagrangian as a function of φ and η, where

$$\Phi = \begin{pmatrix}\varphi_1\\ \varphi_2\\ v+\eta\end{pmatrix}, \qquad \varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2).$$

4) Write the values for m_φ and m_η.
5) How many Nambu-Goldstone bosons are present? How can you explain that?

Problem 12. Consider sQED with a gauge-fixing term

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) - \mu^2|\varphi|^2 - \lambda|\varphi|^4 - \frac{1}{2\xi}(\partial_\mu A^\mu + \xi m_A\xi)^2$$

$$D_\mu = \partial_\mu - ieA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad m_A = ev, \quad e, v, \lambda, (-\mu^2) \in \mathbb{R}^+.$$

Defining

$$\varphi = \frac{1}{\sqrt{2}}(v + \eta + i\chi),$$

1) write the Lagrangian as a function of η, χ and A_μ and find all their masses.
2) Show that the propagators for η, χ and A_μ are, respectively,

$$\tilde{D}^\eta(k) = \frac{i}{k^2 - m_\eta^2 + i0^+}, \qquad \tilde{D}^\chi = \frac{i}{k^2 - \xi m_A^2 + i0^+},$$

$$\tilde{D}_{\mu\nu}(k) = \frac{i}{k^2 - m_\eta^2 + i0^+}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi m_A^2}\right].$$

Part 2. **Second tutorial**

Problem 13. Check that, the covariant derivative for a colorless scalar or fermion field Ψ of hypercharge Y and electric charge $Q = T^3_L + Y$

$$D_\mu \Psi = \left[\partial_\mu - igW^i_\mu \frac{\sigma^i}{2} - ig' Y B_\mu\right]\Psi$$

can be writen as

$$D_\mu \Psi = \left[\partial_\mu - igW^\pm_\mu T^\pm_L - i\frac{g}{c_W} Z_\mu \left(T^3_L - s^2_W Q\right) - ieA_\mu Q\right]\Psi.$$

where

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c_W & -s_W \\ s_W & c_W \end{pmatrix} \cdot \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \qquad s_W = \sin\theta_W, \quad c_W = \cos\theta_W \quad \tan\theta_W = g'/g$$

and

$$W^\pm = \frac{1}{\sqrt{2}}\left[W^1_\mu \mp iW^2_\mu\right], \qquad T^\pm_L = \frac{1}{2\sqrt{2}}\left[\sigma^1 \pm i\sigma^2\right].$$

Problem 14. Demonstrate that

$$\mathcal{L}_\Phi = |D_\mu \Phi|^2 - \mu^2|\Phi|^2 - \lambda|\Phi|^4, \qquad D_\mu \Phi = \left(\partial_\mu - igW^i_\mu \frac{\sigma^i}{2} - ig' B_\mu \frac{1}{2}\right)\Phi$$

is equal to

$$\mathcal{L}_\Sigma = \frac{1}{2}\mathrm{Tr}\left((\mathcal{D}_\mu\Sigma)^\dagger(\mathcal{D}\Sigma)\right) - \frac{1}{2}\mu^2\mathrm{Tr}\left(\Sigma^\dagger\Sigma\right) + \frac{\lambda}{4}\left[\mathrm{Tr}\left(\Sigma^\dagger\Sigma\right)\right]^2,$$

where

$$\Sigma = (\tilde{\Phi}\,\Phi), \qquad \mathcal{D}_\mu\Sigma = \partial_\mu\Sigma - igW^i_\mu\frac{\sigma^i}{2}\Sigma + ig'\Sigma\frac{\sigma^3}{2}B_\mu.$$

Problem 15. Show that

1)

$$\Gamma(Z \to f\bar{f}) = N^f_c \frac{\alpha m_Z}{3}(v^2_f + a^2_f)$$

2)

$$A_{\mathrm{FB}} = \frac{\sigma(\cos\theta > 0) - \sigma(\cos\theta < 0)}{\sigma(\cos\theta > 0) + \sigma(\cos\theta < 0)} = \frac{3}{4}A_f, \qquad A_f = \frac{2v_f a_f}{v^2_f + a^2_f}.$$

For that, remember that for a 1 to 2 decay

$$\frac{d\Gamma(i \to 1, 2)}{d\Omega} = \frac{1}{32\pi^2}\frac{|\mathbf{p}|}{M^2}|\mathcal{M}|^2$$

with

$$|\mathbf{p}| = |\mathbf{p}_1| = |\mathbf{p}_2| = \frac{\{[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]\}^{1/2}}{2M},$$

where M is the mass of the decaying particle while m_1 and m_2 are the masses of the decay products.

Problem 16. Show that

$$\Gamma(h \to f\bar{f}) = N^f_c \frac{G_F m_h}{4\pi\sqrt{2}} m^2_f \left(1 - \frac{4m^2_f}{m^2_h}\right).$$

Problem 17. Consider the leading derivative term in ChPT with two flavors,

$$\mathcal{L} = \frac{f^2}{4}\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U), \qquad U = \exp\left(2i\pi^i T^i/f\right), \qquad T^i = \frac{\sigma^i}{2},$$

where

$$\Pi = \pi^i T^i = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{2}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{2}\pi^0 \end{pmatrix}$$

and we have defined

$$\pi^\pm = \frac{1}{\sqrt{2}}\left[\pi^1 \mp i\pi^2\right], \qquad \pi^0 = \pi^3.$$

Write down the Lagrangian that one gets after expanding the Lagrangian above up to four fields. It could be useful to use that

$$\partial_\mu\left[\exp\left(A(x)\right)\right] = \left\{\partial_\mu A(x) + \frac{1}{2!}\Big[A(x), \partial_\mu A(x)\Big] + \frac{1}{3!}\Big[A(x), \Big[A(x), \partial_\mu A(x)\Big]\Big] + \mathcal{O}(A^4)\right\}\exp(A(x))$$

where $A(x)$ is x_μ-dependent matrix.

Problem 18. Compute the pion masses after adding a term

$$\Delta\mathcal{L} = \frac{f^2}{4}\mathrm{Tr}(U^\dagger\chi + \chi U)$$

where $\chi = 2B_0\,\mathrm{diag}(m_u, m_d)$ to the Lagrangian of the previous problem.

Problem 19. Show that when $k \ll p_1, p_2$ the amplitude

$$i\mathcal{M}_{q\bar{q}g} = \bar{u}(p_1) i g_s T^a \not{\epsilon}^* \frac{i}{\not{p}_1 + \not{k}} ieQ_q\gamma_\mu v(p_2) - \bar{u}(p_1) ieQ_q\gamma_\mu \frac{i}{\not{p}_2 + \not{k}} ig_s T^a \not{\epsilon}^* v(p_2)$$

can be written as

$$i\mathcal{M}_{q\bar{q}g} \approx \bar{u}(p_1) ieQ_q\gamma_\mu T^a v(p_2) g_s\left(\frac{p_1\cdot\epsilon^*}{p_1\cdot k} - \frac{p_2\cdot\epsilon^*}{p_2\cdot k}\right).$$

References

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