Astroparticle physics - exercise session

Menu

- Application of Heitler model for indirect detection of gamma rays
- Proving diffusive nature of the CR motion in perturbed B-field
- Classical evidence for diffusive motion from chemical information: a simple model
- Making sense of proton/electron ratio in cosmic rays?
- Some challenge in searches for exotics in secondary species to be aware of

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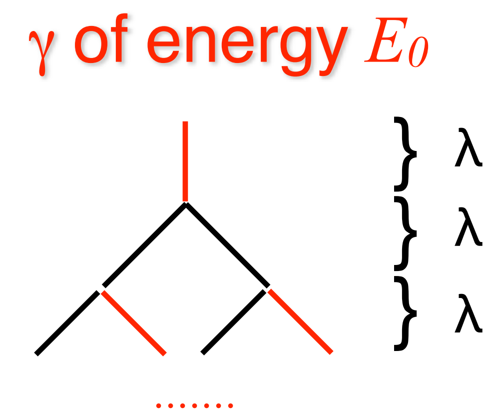
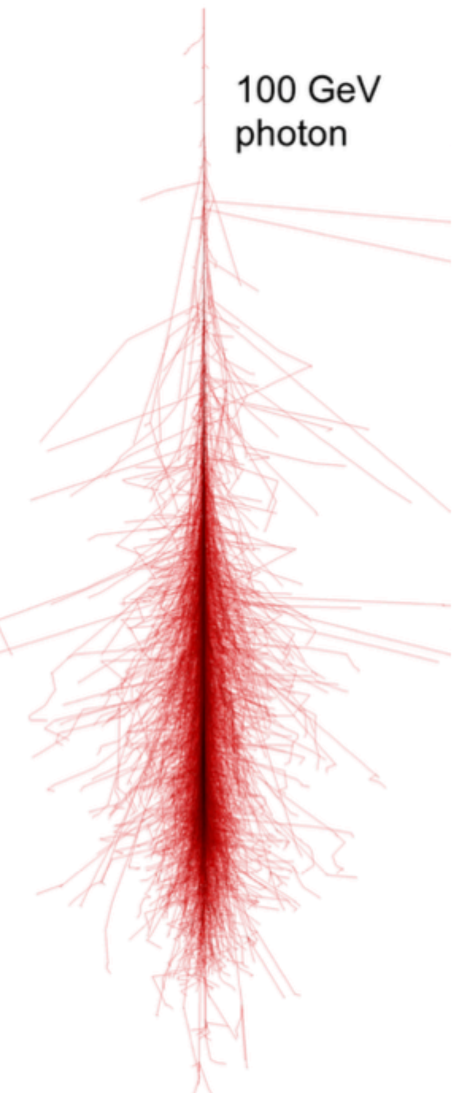
I. Heitler model for e.m. cascades

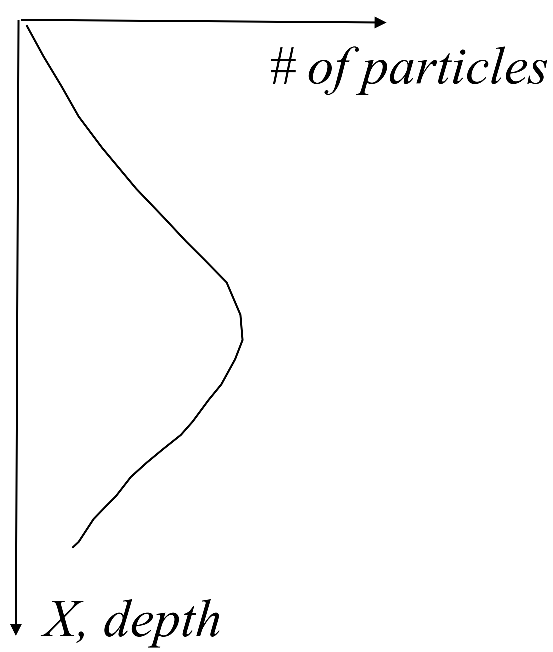
Ind. variable: **grammage** $$X = \int \rho(\ell) \mathrm{d}\ell$$

Assume a primary γ, impinging on the atmosphere, generating a pair after a **characteristic grammage λ (g/cm²);** each lepton in turn generates a γ via bremssthralung after about the same λ.

Critical energy E_c below which particles lose energy without radiating new particles (e.g. ionization, etc.)







2ⁿ particles will be present in the shower after n interactions

$$n = X/\lambda$$
$$N(X) = 2^n = 2^{X/\lambda}$$
$$\langle E \rangle = E_0/2^{X/\lambda}$$

The shower maximum is reached at: $$N_{\max} = E_0/E_c \qquad X_{\max} = \lambda \log_2(E_0/E_c)$$

Since $\lambda \approx 35$ g/cm² (see PDG, E-losses in matter) and are $E_c \approx 80$ MeV "atmospheric constants", once calibrated the method can provide an estimate of primary energy

Exercise

➡ Use an isothermal (exponential) model for the (upper) atmosphere

$$\rho(h) \approx \rho_0 \, \exp(-h/h_0) \qquad h_0 \simeq 6.4\,\mathrm{km} \qquad \rho_0\, h_0 \simeq 1300\,\mathrm{g/cm}^2$$

$$X(\ell,\theta) = \int_\ell^\infty dl\, \rho(h(l,\theta))$$

$$h(l,\theta) = \sqrt{R_\oplus^2 + 2lR_\oplus \cos\theta + l^2} \approx l\cos\theta + \frac{l^2}{2R_\oplus}\sin^2\theta$$

➡ Estimate what is the height in the atmosphere at which the first interaction of a vertically downgoing VHE photon takes place.

➡ Based on the Heitler model, how many particles are expected in a 100 GeV (1 TeV) gamma-ray induced shower? In the above model, what is the typical height of this maximum for a downgoing photon?

Let us use the key model constants $\lambda \approx 35$ g/cm^2 and $E_c \approx 80$ MeV

Exercise

➡ Estimate what is the height in the atmosphere at which the first interaction of a downgoing VHE photon takes place.

i.e. deduce h_1 such that the crossed grammage is **λ**

$$\lambda = \int_{h_1}^{\infty} \mathrm{d}h \rho(h) \Rightarrow h_1 = h_0 \ln\left(\frac{h_0 \rho_0}{\lambda}\right) \simeq 23\,\mathrm{km}$$

➡ Based on the Heitler model, how many particles are expected in a 100 GeV (1 TeV) gamma-ray induced shower? In the above model, what is the typical height of this maximum for a downgoing photon?

$$N_{\max} \simeq \frac{E_\gamma}{E_c} = \frac{100 \div 1000\,\mathrm{GeV}}{80\,\mathrm{MeV}} = 1.25 \times 10^{3 \div 4}$$

$$X_{\max} \simeq \lambda \log_2 N_{\max} = 35\,\mathrm{g/cm^2} \log_2(1.25 \times 10^{3 \div 4}) \simeq 360 \div 476\,\mathrm{g/cm^2}$$

$$h_{\max} = h_0 \ln\left(\frac{1300}{360 \div 476}\right) = 6.4 \div 8.2\,\mathrm{km}$$

Note: IACTs placed on mountains (like MAGIC & CTA-north in La Palma, ~2.2 km) makes them, among other things (atmospheric quality, etc.), somewhat closer to the maximum of the shower

II. Emergence of diffusive motion

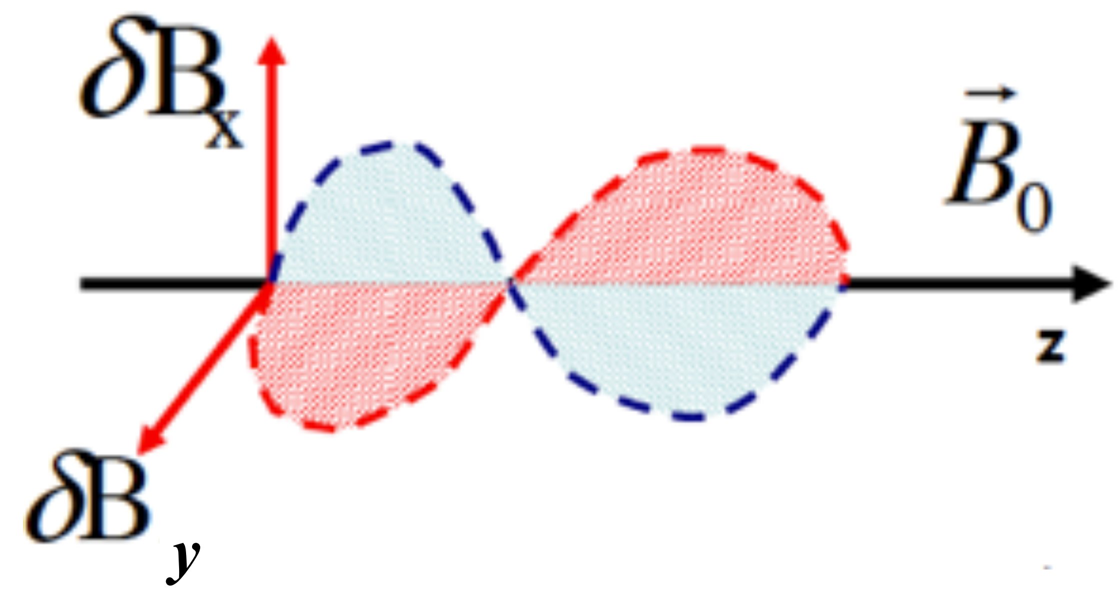
Adding **small**, **stochastic**, **static** perturbations to the B-field, orthogonal to its regular value:

$$|\delta \mathbf{B}| << |\mathbf{B}_0| \,, \quad \delta \mathbf{B} \perp \mathbf{B}_0$$

$$\frac{d\mathbf{p}}{dt} = \frac{q\,\mathbf{v}}{c} \times (\mathbf{B}_0 + \delta\mathbf{B})$$

NB: perturbations chosen so that only p_z evolution non-trivial; previous solution for the *x*-*y* components of ***p*** still valid in a "perturbation theory" spirit

$$p_z = p\,\mu$$



Averaging over fluctuations with random phase ψ, one can prove it shows **diffusive features**

Zero average
$$\left\langle \frac{\mathrm{d}\mu}{\mathrm{d}t} \right\rangle_\psi = 0$$

resonant variance, Linear in time
$$\frac{\mathrm{d}\left\langle \Delta\mu^2 \right\rangle_\psi}{\mathrm{d}t} \to \pi C^2 \delta(w) = \pi\,(1-\mu^2)\Omega \frac{|\delta\mathbf{B}|^2}{B_0^2} k_{\mathrm{res}} \delta\,(k - k_{\mathrm{res}})$$

$$\Delta t \gg \Omega^{-1}$$

gyrophase average

$$k_{\mathrm{res}} \equiv \frac{\Omega}{v\mu}$$

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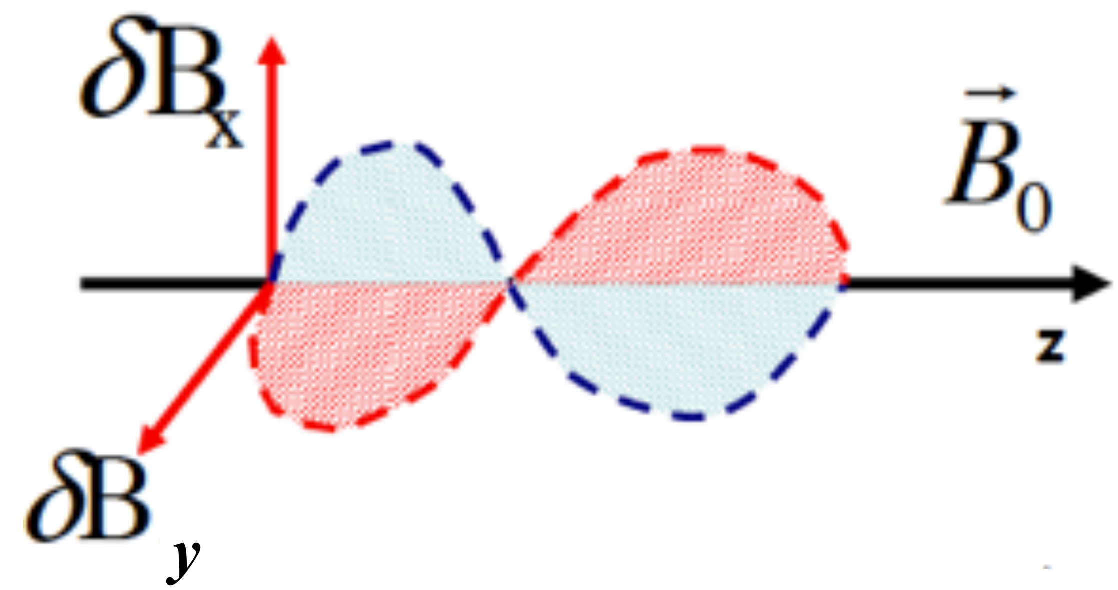

the evolution the "pitch angle" obeys to is

we plug in the unperturbed solution for x-y coordinates...

$$\frac{d\mu}{dt} = \frac{q\,v_\perp^0}{c\,p}\left[\cos(\Omega\,t)\delta B_y - \sin(\Omega\,t)\delta B_x\right] =$$

$$= \frac{q\sqrt{1-\mu^2}\,|\delta\mathbf{B}|}{c\,m\,\gamma}\left[\cos(\Omega\,t)\cos(k\,z+\psi) - \sin(\Omega\,t)\sin(k\,z+\psi)\right]$$

We used ultrarelativistic approximation for the CR, and the magnetic field perturbation as "slow", hence practically seen as stationary by the CR

II. Emergence of diffusive motion

$$\frac{d\mu}{dt} = \frac{q\sqrt{1-\mu^2}\,|\delta \mathbf{B}|}{c\,m\,\gamma}\left[\cos(\Omega - k\,v\mu)t + \psi\right]$$

$$\left\langle \frac{d\mu}{dt} \right\rangle_{\psi} = 0\,,$$

$$\langle \Delta\mu^2 \rangle_{\Delta t} = \left(\frac{q\sqrt{1-\mu^2}\,|\delta \mathbf{B}|}{c\,m\,\gamma} \right)^2 \int^{\Delta t} \cos\left[(\Omega - k\,v\mu)t'' + \psi\right] dt \int^{\Delta t} \cos\left[(\Omega - k\,v\mu)t' + \psi\right] dt'$$

By using prosthaphaeresis formula (high school was useful, after all...)

$$\cos(A)\cos(B) = \frac{\cos(A-B) + \cos(A+B)}{2}$$

averaging over the random phase ψ , the (A+B) term vanishes and the only one left is

$$\langle \Delta\mu^2 \rangle_{\psi} = \frac{1}{2}\left(\frac{q\sqrt{1-\mu^2}\,|\delta \mathbf{B}|}{c\,m\,\gamma} \right)^2 \int dt' \int dt'' \cos[(\Omega - k\,v\mu)(t'' - t')]$$

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We can now differentiate wrt to time and use Euler formula

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

as well as the Dirac representation

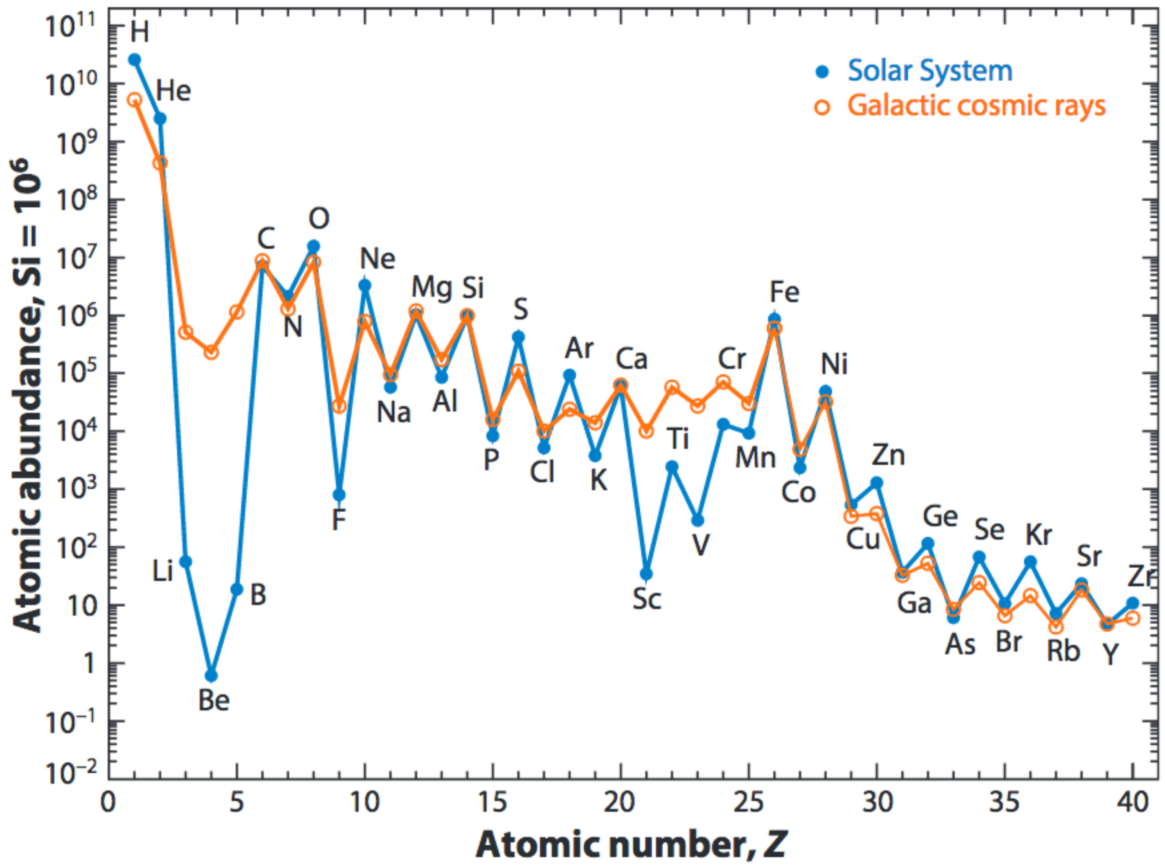
$$\int_{-\infty}^{+\infty} e^{ikx} dk = 2\pi\delta(x)$$

to find that, in the limit $\Delta t \gg \Omega^{-1}$, the above Eq. writes in fact as a **resonant** diffusion

$$\left\langle \frac{\Delta\mu^2}{\Delta t} \right\rangle_\psi = \frac{\pi\, q^2 v (1-\mu^2)\,|\delta\mathbf{B}|}{c^2 p^2 \mu} \delta\left(k - k_{\rm res}\right)$$

$$k = k_{\rm res} \equiv \frac{\Omega}{v\mu}$$

III. Classical evidence for diffusive propagation



Very large Li-Be-B to e.g. C-O ratios, e.g. B/C~0.25
(Similarly, for sub-Fe/Fe)

Why interesting?

Inconsistent with the Galaxy crossing timescale:

✦ a CR crossing **ballistically** the thin disc would accumulates a grammage *$X \sim m_p\, n_H\, h/\cos\theta \sim 10^{-3}$ g/cm^2* ***(h/cosθ**$=300$ pc$\sim 10^{21}$ cm as Galactic disc thickness, $n_H \sim 1$ cm^{-3} as ISM density)*

✦ simple model can show that the observed ratio requires a much larger grammage, hence a much larger residence time (motion is far from ballistic!)

Simple modelling

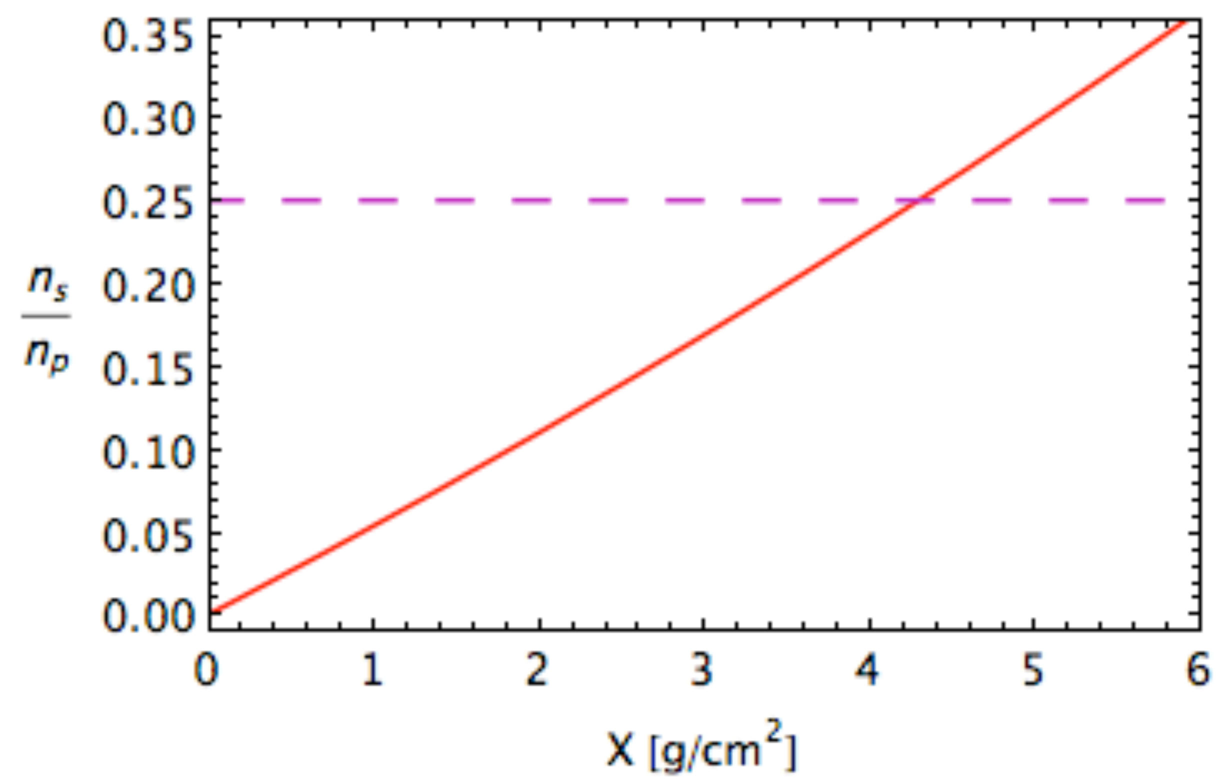
Consider two CR species: primaries with number density n_p and secondaries with number density n_s. If the two are coupled by the spallation process $p \rightarrow s+\ldots$, then

$$\frac{dn_p}{dX} = -\frac{n_p}{\lambda_p},$$
$$\frac{dn_s}{dX} = -\frac{n_s}{\lambda_s} + \frac{p_{\rm sp} n_p}{\lambda_p},$$

$$X = \int dl\, \rho(l)$$ Grammage (amount of traversed matter)

$$\lambda_i = m/\sigma_i$$ interaction lengths (in g/cm^2), CNO ~6.7 LiBeB~10

$$p_{\rm sp} = \sigma_{\rm sp}/\sigma_{\rm tot}$$ spallation probability for CNO ~0.35



$$\frac{n_s}{n_p} = \frac{p_{\rm sp}\lambda_s}{\lambda_s - \lambda_p}\left[\exp\left(\frac{X}{\lambda_p} - \frac{X}{\lambda_s}\right) - 1\right]$$

From the ratio S/P being ~0.25, one deduces X~4.3 g/cm^2

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Inconsistent with the Galaxy crossing timescale:

✦ a CR crossing **ballistically** the thin disc would accumulate $X_{ball} \sim m_p\, n_H\, h/\cos\theta \sim 10^{-3}$ g/cm^2 ($h/\cos\theta = 300$ pc $\sim 10^{21}$ cm as Galactic disc thickness, $n_H \sim 1$ cm^{-3} as ISM density)

✦ simple model can show that the observed ratio requires a much larger grammage, hence a much larger residence time (motion is far from ballistic!)

$X_{obs} \sim 4.3$ g/cm^2 >> $X_{ball} \sim 10^{-3}$

The residence time of cosmic rays in the galaxy follows as

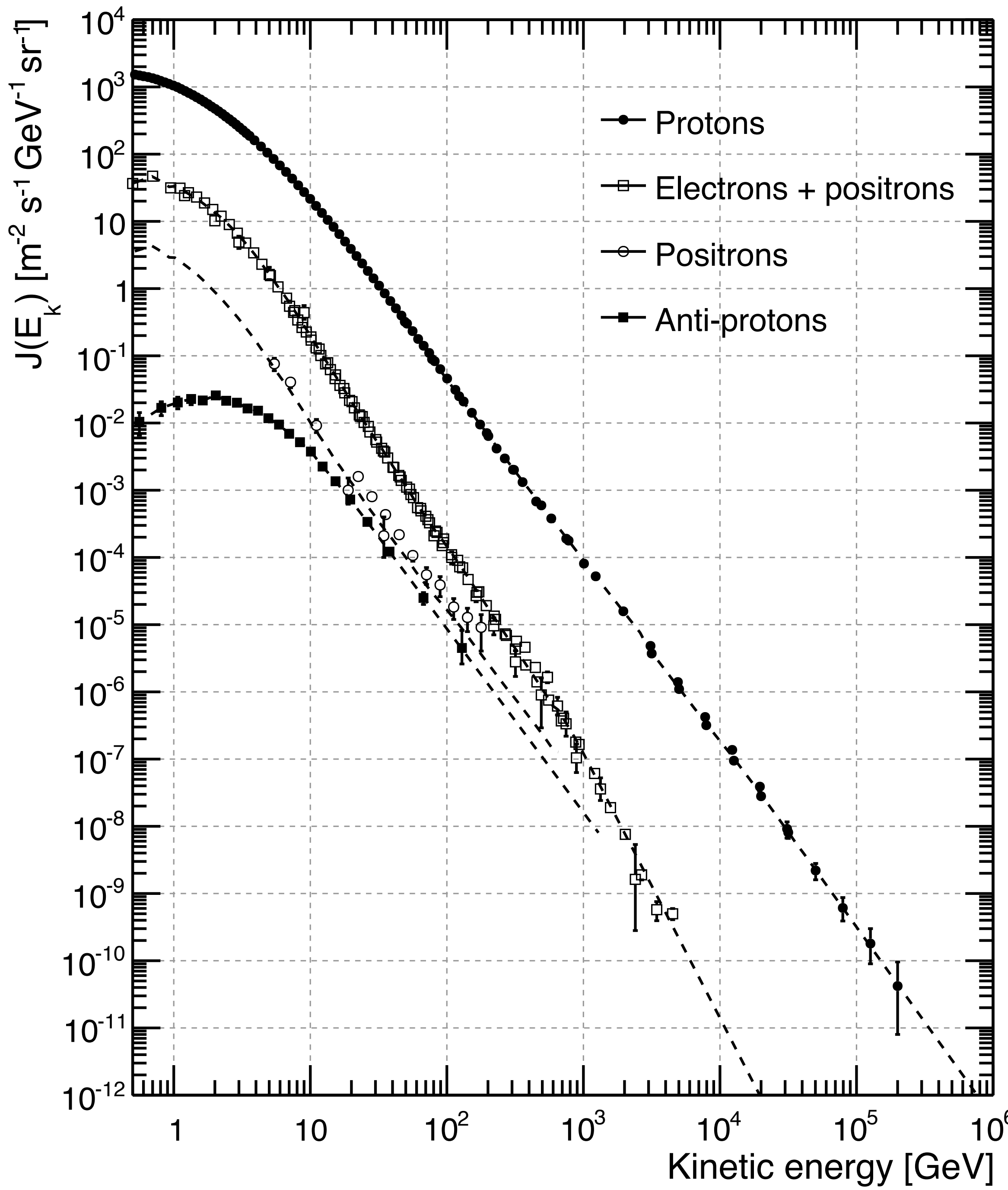
$t_{prop} \sim (X/10^{-3}\ g/cm^2)(h/c\cos\theta) \sim 10^7$ yr.

A similar timescale of ~ 10^7 yr follows from the relative abundances of radioactive isotopes, like $^{10}Be/^{9}Be$ ($\tau_{10Be} \sim 1.5 \times 10^6$ yr)

The comparison of the isotopic ratio at the production in spallation events in the Lab wrt what measured in CR provides a measure of their "age".

(Another manifestation of 'astroparticle' physics)

IV. Making sense of O(1%) p/e ratio (wrt kin energy) in CR?



Lemma: Rigidity is what matters in acceleration!

EOM in presence of electromagnetic fields only

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\left(\vec{v} \times \mathbf{B} + \mathbf{E}\right)$$

+ identity

$$\vec{p} \cdot \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}p^2}{\mathrm{d}t}$$

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E-field needed to change momentum modulus!

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$$\frac{\mathrm{d}p^2}{\mathrm{d}t} = 2q\,\vec{p} \cdot \vec{E}$$

Divide by q^2

$$\frac{\mathrm{d}\vec{\mathcal{R}} \cdot \vec{\mathcal{R}}}{\mathrm{d}t} = 2\vec{\mathcal{R}} \cdot \mathbf{E}$$

Equivalent to say that any electromagnetic acceleration process 'orders' particle distribution functions with respect to their rigidity'

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In any process/environment in which both *e* and *p* (proxy for all nuclei) are accelerated from the 'same pool':

Neglecting losses ('collisional effects') they should attain the same momentum spectrum

Implication

Let's parameterise this distribution as a power-law (inspired by acceleration theory…) $\phi_e = \kappa_e p^{-s}$

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Impose charge neutrality in the pool from which they are accelerated

(i.e. integrals from non-relativistic *kinetic energy* ~ T_0, T_0<< m_e, e..g 1 keV)

$$* \int_{T_0}^{\infty} \mathrm{d}T \phi_e(p(T)) \frac{\mathrm{d}p}{\mathrm{d}T} = \int_{T_0}^{\infty} \mathrm{d}T \phi_p(p(T)) \frac{\mathrm{d}p}{\mathrm{d}T} \equiv n_0$$

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Re-express fluxes in terms of kinetic energies

$$J_i(T) \equiv \phi_i(p(T)) \frac{\mathrm{d}p}{\mathrm{d}T} = \kappa_i (T + m_i) \left[2T\, m_i + T^2 \right]^{-(s+1)/2}$$

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From which it follows

$$\frac{J_e(T)}{J_p(T)} = \left(\frac{m_e}{m_p}\right)^{(s-1)/2} \frac{T + m_e}{T + m_p} \left[\frac{2m_e + T}{2m_p + T}\right]^{-(s+1)/2} \underset{\text{relativistic limit}}{\rightarrow} \left(\frac{m_e}{m_p}\right)^{(s-1)/2}$$

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Some Limitations:

- power-law not valid to non-relativistic energy?
- Losses not accounted for
- Plasma assumed pure p-e (e.g. no positrons)

Yet right ballpark for s~2-3, notably around ~10 GeV where some effects mitigated…

V. Some challenge in searches for exotics in secondary species to be aware of

Reminder on Secondaries

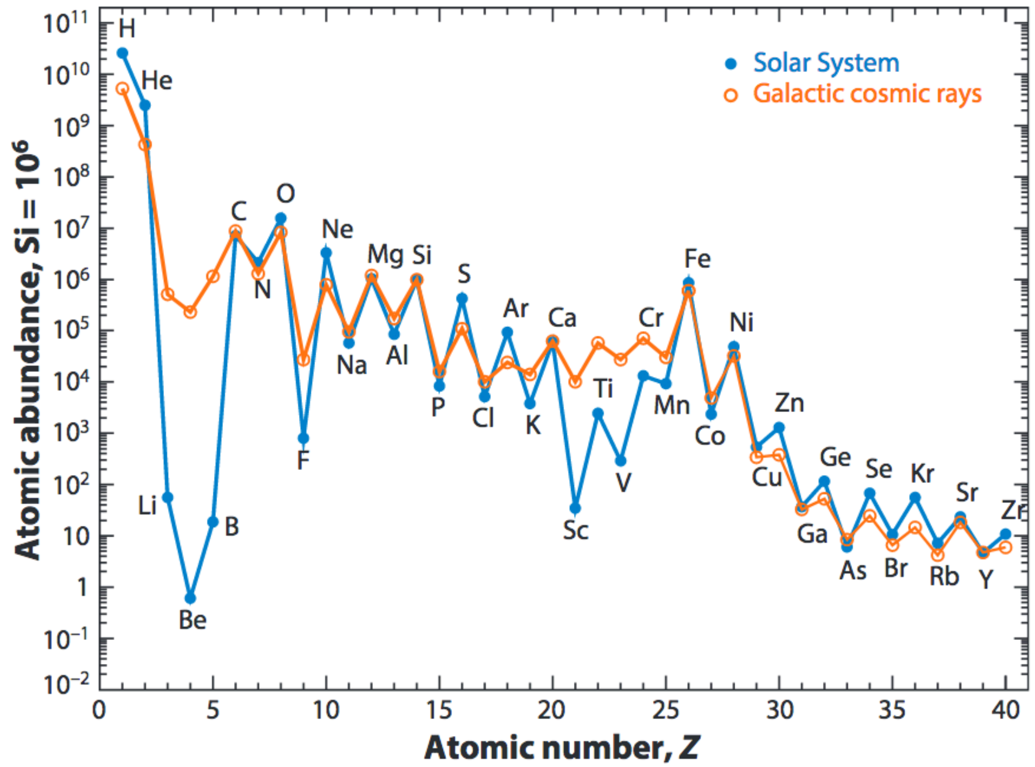
Similarly as above

$$-\frac{\partial}{\partial z}\left(D\frac{\partial \phi^S}{\partial z}\right) \simeq 2h\,\Gamma_{P\to S}\,\phi^P\,\delta(z) - 2h\,\Gamma_S\,\phi^S\delta(z)$$

Now the source term is due to primary collisions, yielding

$$\phi_0^S(p) = \phi_0^P(p)\Gamma_{P\to S}\tau_{\rm eff,S}(p)$$

where $\tau_{\rm eff,S}^{-1}(p) = \tau_d^{-1}(p) + \Gamma_S$



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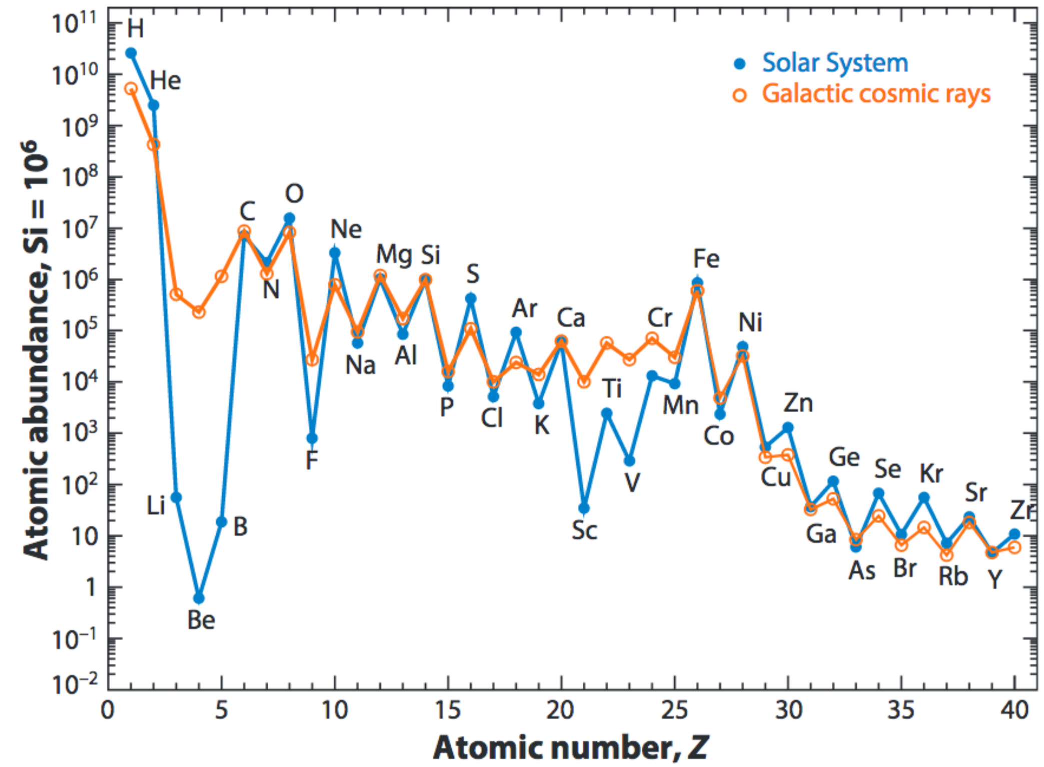
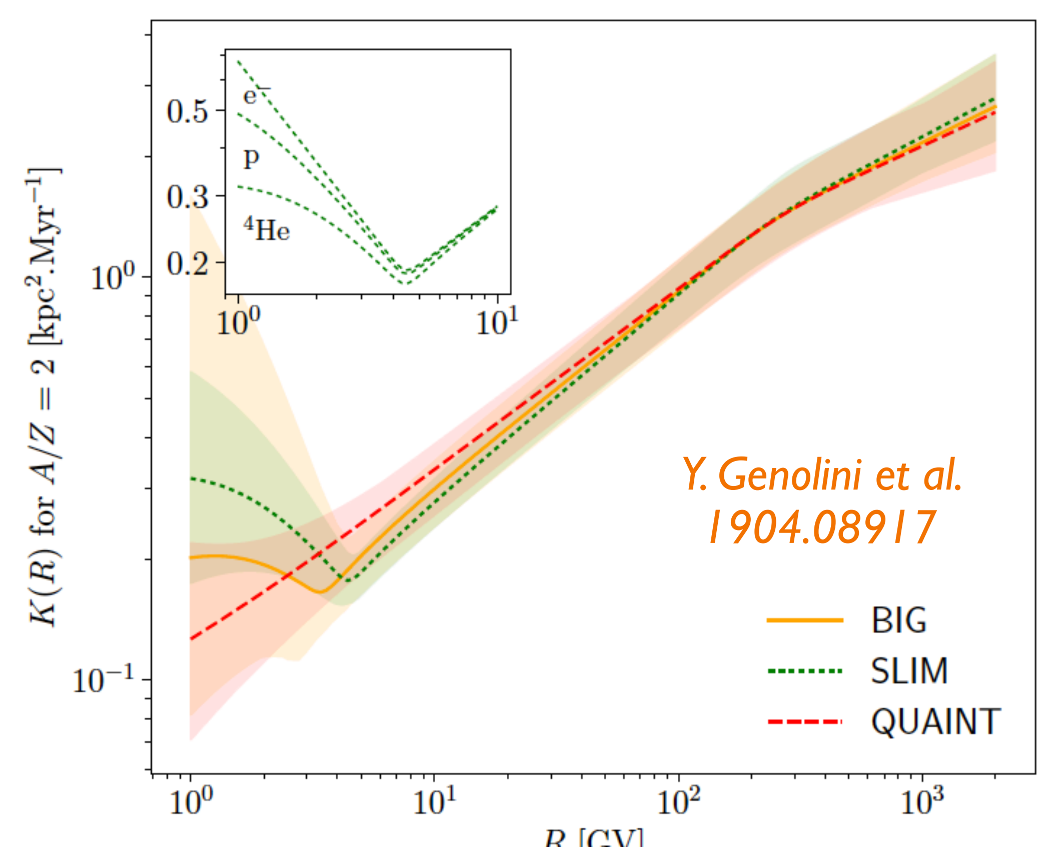
where $\tau^{-1}_{\rm eff,S}(p) = \tau_d^{-1}(p) + \Gamma_S$

S/P ratio used to determine the diffusion timescale, breaking the degeneracy of injection and propagation effects

However, uncertainties in x-sec introduce an important systematic error, especially at low rigidities!



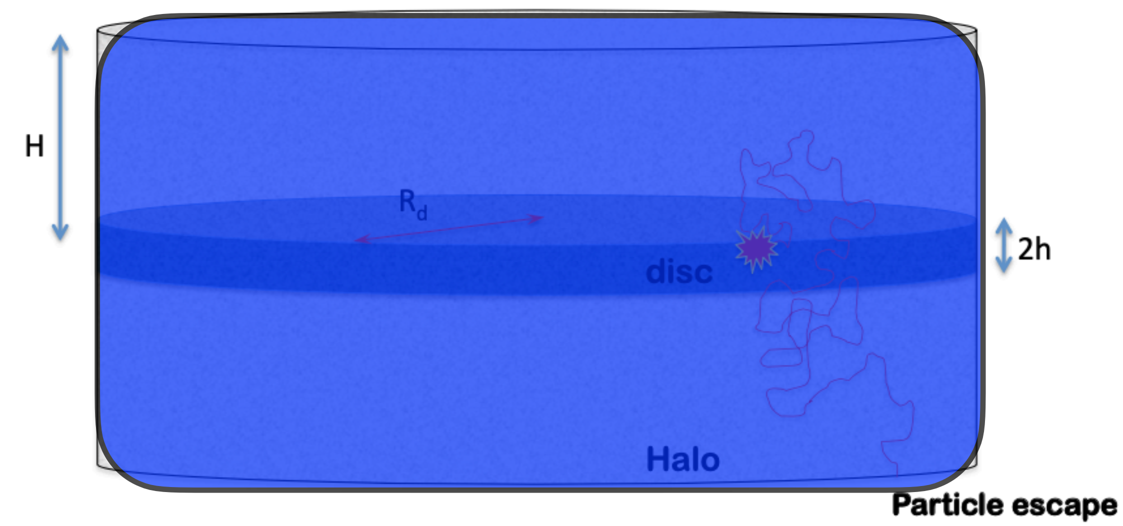


What if an exotic source injects it in the whole volume?

$$-\frac{\partial}{\partial z}\left(K\frac{\partial \phi_X}{\partial z}\right) = q_X(p) - 2h\,\Gamma_\sigma\,\phi_X\delta(z)\,.$$

its solution has the form

$$\frac{1}{\tau_{\rm eff}(p)}\phi_X^0 = \frac{H}{h}q_X(p)$$



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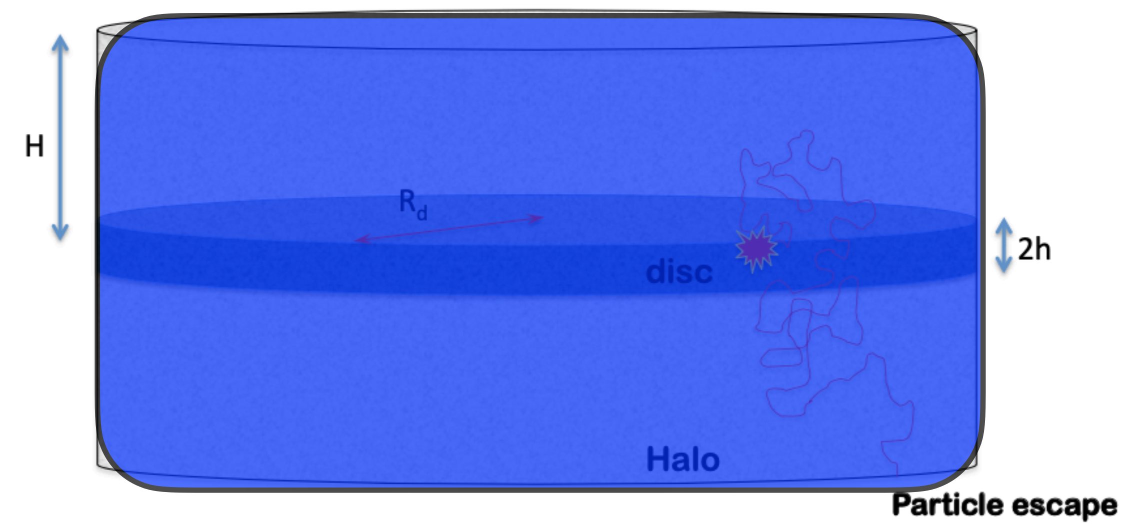

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Basically, because besides depending from parameters that can be inferred e.g. from constraints e.g. on secodaries/primaries [$\tau_{\rm eff}$], also on (additional combination) of parameter(s) [Here H/h], which are much more poorly known!

Part of the game consist in devising new observables providing such additional handle on *astrophysical parameters* in order to sharpen sensitivity to BSM physics!

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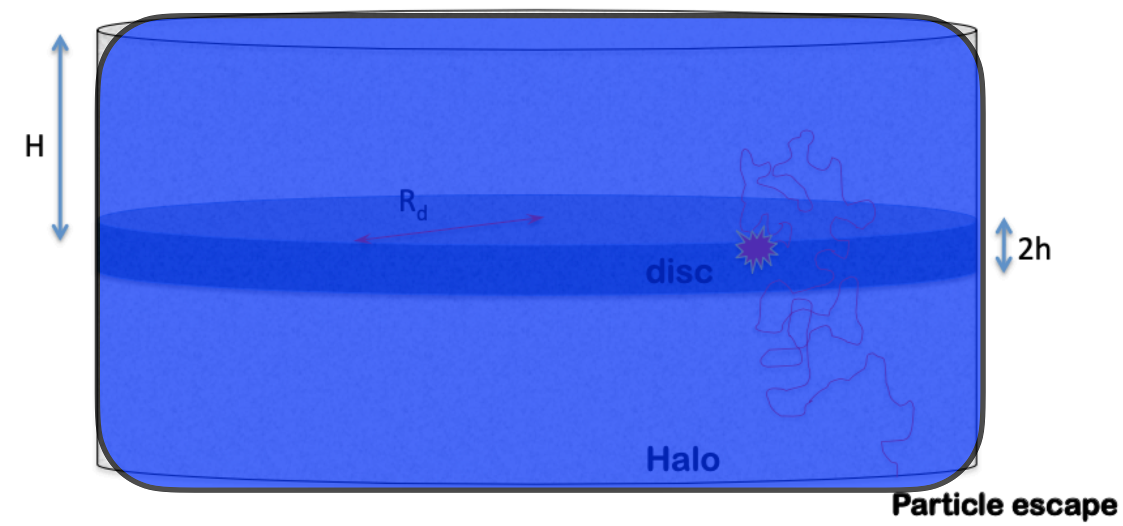

The solution satisfying the correct boundary conditions writes

$$\phi(z,p) = \phi_0(p)\left(1-\frac{|z|}{H}\right) + \frac{q(p)}{2\,K}\left(H\,|z| - z^2\right)\,. \quad *$$

Again, by infinitesimal integration around zero it holds

$$-2K(p)\left.\frac{\partial\,\phi}{\partial z}\right|_0 + \frac{h}{\tau_\sigma(p)}\phi_0 = 0\,.$$

But by differentiating Eq. * we know that

$$-2K\left.\frac{\partial\phi(z,p)}{\partial z}\right|_0 = \phi_0(p)\left(\frac{2K}{H}\right) - H\,q(p)\,.$$

from which

$$-\frac{h}{\tau_\sigma(p)}\phi_0 = \phi_0(p)\left(\frac{2K}{H}\right) - H\,q(p)$$

This yields the claimed form of solution