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Topological Defects in Cosmology

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Outline of the lectures

- **First lecture:**

  - Introduction to solitons in field theory
  - Domain wall solution
  - Vortex solutions
  - Cosmic strings.

- **Second Lecture:**

  - Cosmological Formation of Defects
  - Topological defect dynamics
  - Observational signatures

Introduction to Solitons

- Many aspects of physics can be studied at a linear level.
- There are however interesting new phenomena that appear once we study a theory at a non-linear level.
- One interesting aspect of these non-linear interactions is the appearance of localized energy solutions, **soliton solutions (** ***solitary waves*** **)**.
- They have been studied in many branches of physics:

  - Condensed matter physics
  - Non-linear optics
  - Particle Physics
  - Cosmology
  - String Theory

Historical remarks

John Scott Russell first observed the formation of these “solitary waves” in a water channel in Scotland.

However it took some years before it was understood.

Eventually, this type of phenomena found its way into high energy physics applications.

The general idea

- We are interested in studying these soliton solutions and their implications in the hope that they are realized by Nature.
- We will be interested in studying models BSM that allow for some of the objects to be present in our universe.
- This could lead either to the detection of these objects confirming the model or to constrain those models BSM.
- The typical energy scale of these objects would be too high to leave any imprint on accelerators, so we need to use cosmology as our probe for those energies.

Kink solution

We will consider a simple scalar field theory in (1+1)

$$S = \int dx^2 \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} \left(\phi^2 - \eta^2 \right)^2 \right]$$

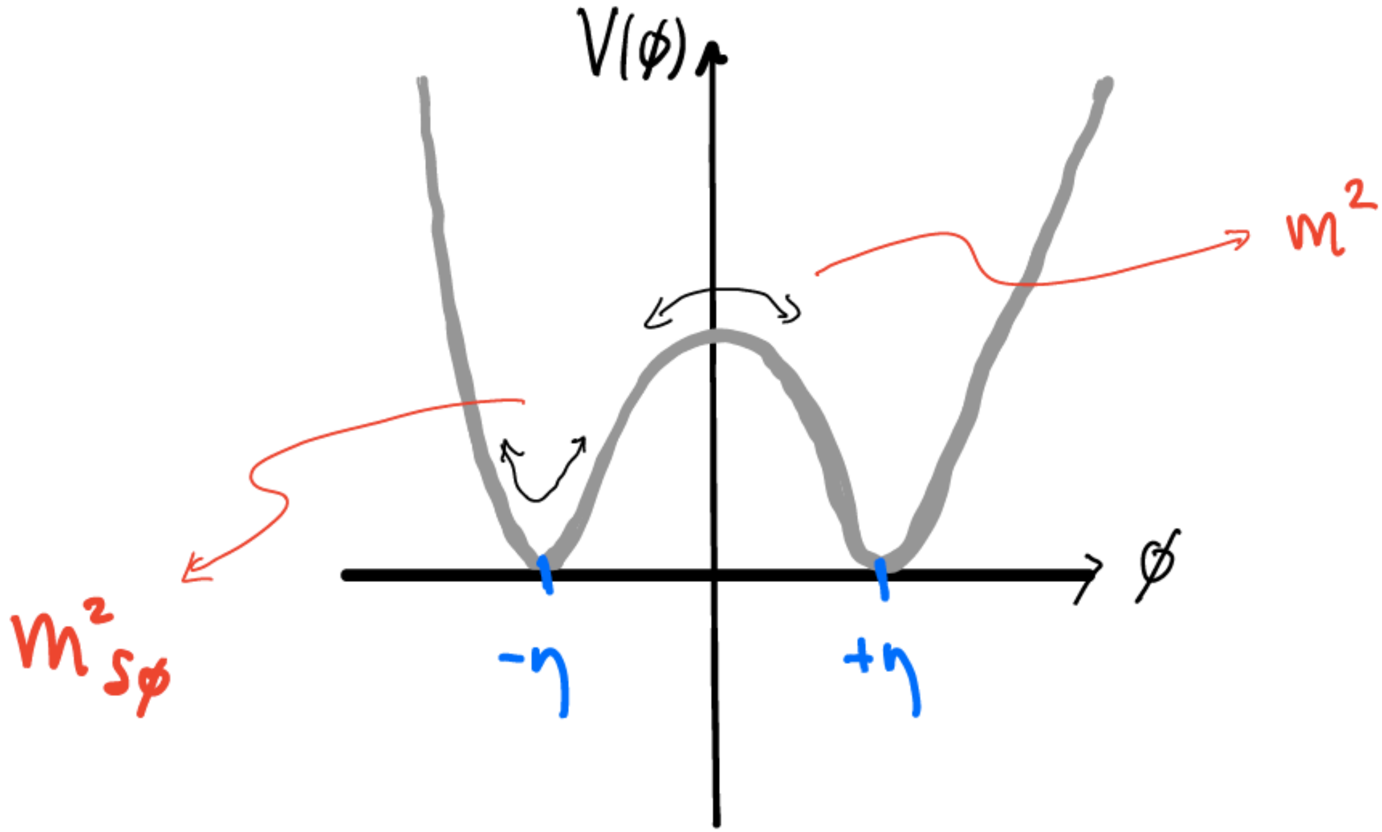
The theory has two distinct vacua: $\phi = \pm \eta$

Fluctuations around the vacuum have propagating degrees of freedom of mass:

$$m_{\delta_\phi} = \sqrt{2\lambda}\eta$$

We will now show that there are other type of configurations in this theory that represent localized concentrations of energy.

Kink solution



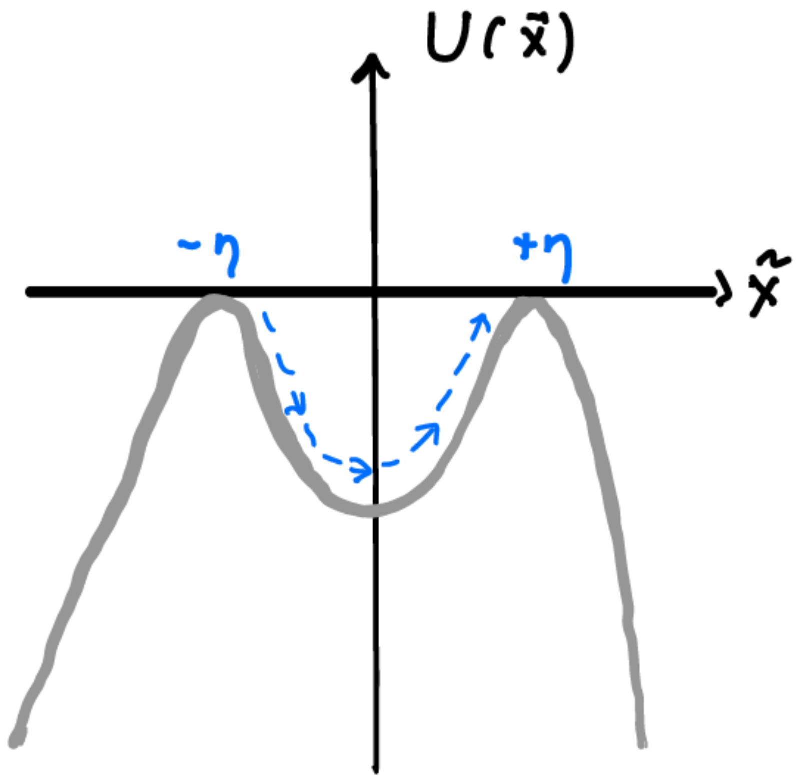
Kink solution

We are looking for a solution of the time independent eom, namely:

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial V(\phi)}{\partial \phi} = 0$$

This is analogous of a Newtonian equation with a potential

$$\left.\begin{array}{l} x \to \tilde{T} \\ \phi \to \tilde{X} \\ U(\tilde{X}) = -V(\phi) \end{array}\right\} \Longrightarrow \quad \frac{d^2 \tilde{X}}{d\tilde{T}^2} = -\frac{\partial \tilde{U}}{\partial \tilde{X}}$$



Energy conservation in this effective Newtonian problem

$$\tilde{E} = \frac{1}{2}\left(\frac{d\tilde{X}}{d\tilde{T}}\right)^2 + U(\tilde{X}) = 0$$

Similarly to what we do in Newtonian physics, we can integrate this equation. (Going back to the original formulation)

$$\frac{d\phi}{dx} = \pm\sqrt{2V(\phi)}$$

Which can be integrated to give in our case:

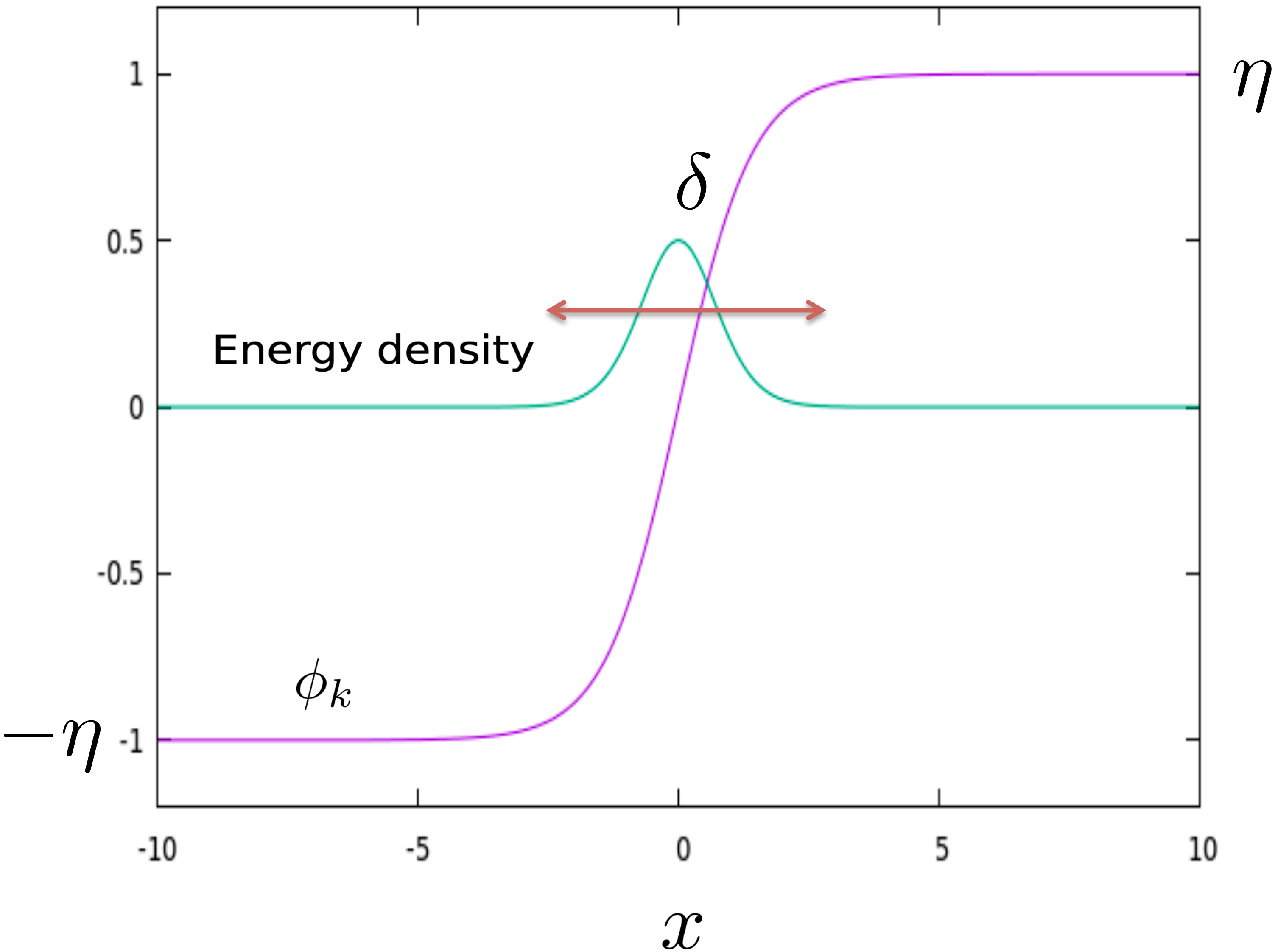
$$\phi_k(x) = \eta \tanh\left(\eta\sqrt{\frac{\lambda}{2}}(x - x_0)\right)$$

Properties of the solution:

- It interpolates between the vacua
- Its thickness is of the order of

$$\delta \sim \frac{1}{m}$$

Its energy density is concentrated on this thickness $\delta \sim \frac{1}{m}$



Kink solution

The total energy of the configuration is given by:

$$M_k = \frac{2\sqrt{2}}{3}\left(\frac{m^3}{\lambda}\right)$$

At small coupling these objects are much heavier than the elementary particles:

$$M_k >> m_{\delta\phi}$$

These are non-perturbative objects.

Ex: Boost the solution to a moving field configuration and compute the energy and momentum of the solution. What happens to the shape of the solution?

Kink stability

Let's now consider small perturbations around our domain wall solution

$$\phi(x,t) = \phi_k(x) + \psi(x,t)$$

The equation of motion for the fluctuations is:

$$\ddot{\psi} - \psi'' + \lambda\left[3\phi_k^2(x) - \eta^2\right]\psi = 0$$

Assuming: $\psi(x,t) \propto e^{-iwt} f(x)$, we arrive at :

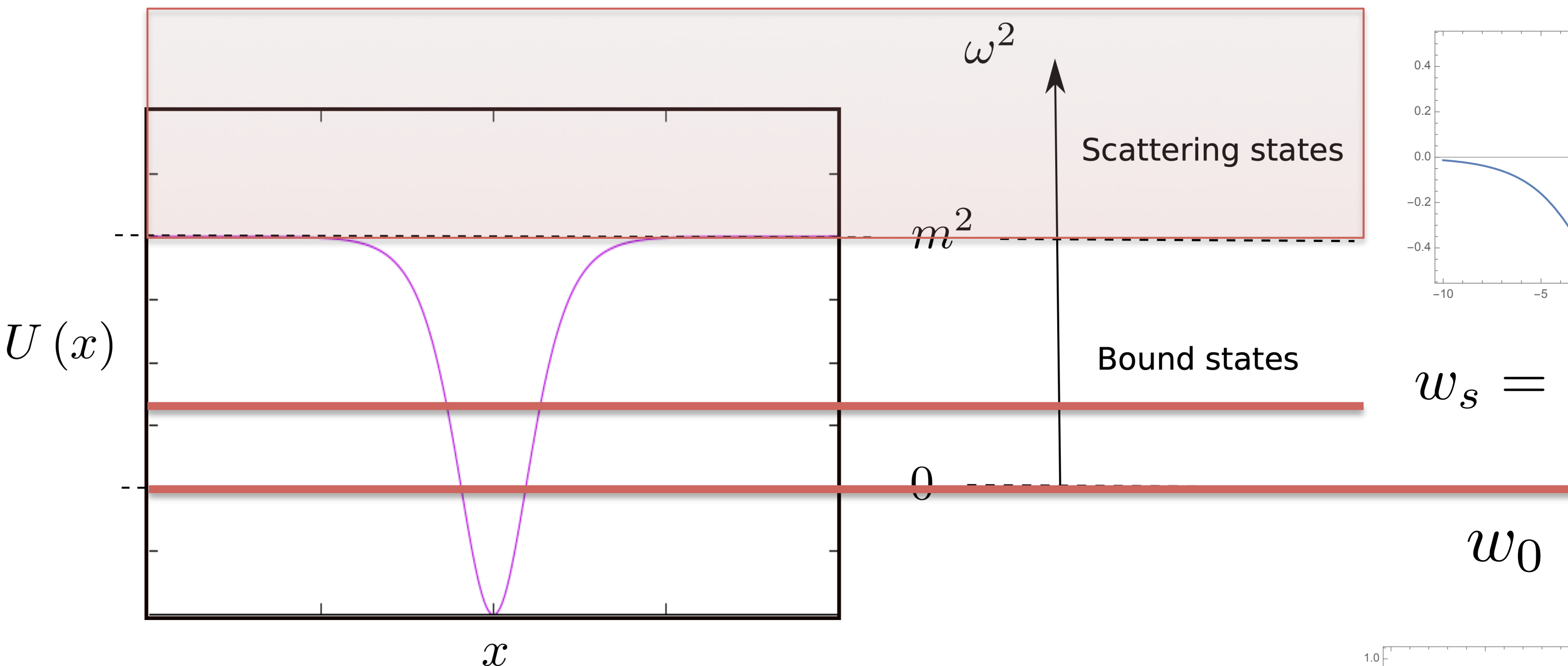
$$-f''(x) + U(x)f(x) = \omega^2 f(x)$$

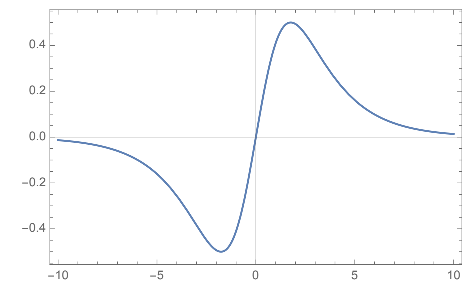
Spectrum of perturbations

(Rajaraman '82).

The associated Schrodinger equation is

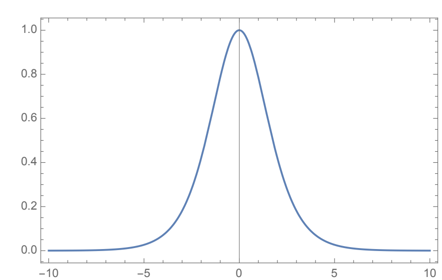
$$-f''(x) + U(x)f(x) = w^2 f(x)$$





$$w_s = \frac{\sqrt{3}}{2}m$$

$$w_0 = 0$$



The zero mode describes small rigid displacements

Topological aspects of the kink solution

Let us consider the conserved current defined as:

$$J^\mu = -\frac{1}{N}\epsilon^{\mu\nu}\partial_\nu W(\phi) \Longrightarrow \partial_\mu J^\mu = 0$$

So there is charge:

$$Q = \int dx J^0 = -\frac{1}{N}\left[W(\phi(\infty)) - W(\phi(-\infty))\right]$$

Defining, N appropiately and,

$$W(\phi) = \sqrt{\frac{\lambda}{2}}\left(\frac{1}{3}\phi^3 - \eta^2\phi\right) \Longrightarrow$$

$$Q(\phi_k) = +1$$

Topological aspects of the kink solution

The charge of the solution depends only on the boundary conditions of the solution. That is why this charge is a topological charge.

The solution with the same vacua on both ends would have topological charge equal to zero.

Let's now show that in fact the kink is the lowest energy configuration with this topological charge.

Bogomol'nyi Bound

With our previous definitions we can see that in our case we have:

(Bogomol'nyi '76)

$$V(\phi) = \frac{1}{2}\left(\frac{\partial W}{\partial \phi}\right)^2$$

This allows us to write the total energy of the static configuration in the form:

$$E = \int dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right] = \int dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{\partial W}{\partial \phi}\right)^2\right]$$

$$E(Q=+1) = -\Delta W + \int dx \frac{1}{2}\left[\frac{d\phi}{dx} + \frac{\partial W}{\partial \phi}\right]^2$$

Bogomol'nyi Bound

This means that given the boundary conditions the energy is bounded from below

$$E \geq -\Delta W = M_k$$

And this bound is saturated when

$$\frac{d\phi}{dx} = -\frac{\partial W}{\partial \phi} \quad \Longrightarrow \quad \frac{d\phi}{dx} = -\sqrt{2V(\phi)}$$

Which is exactly the equation we used to find the kink solution.

So, indeed this solution is the minimal energy configuration with the given boundary conditions !

Going to higher dimensions

Derrick's Theorem: (Derrick '64)

Let's consider a generic Langrangian density in (d+1) spacetime:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$$

The energy of the soliton static configuration can be expressed as

$$E = E_k + E_p = \int d^dx \frac{1}{2}(\nabla\phi)^2 + \int d^dx V(\phi)$$

Let's now consider a scaling transformation of the form,

$$x^i \rightarrow \lambda x^i$$

Going to higher dimensions

With this transformation the energy scales as,

$$E[\lambda] = \lambda^{2-d} E_k + \lambda^{-d} E_p$$

For the soliton solution to be stable we need:

$$\left(\frac{dE[\lambda]}{d\lambda} \right)_{|\lambda=1} = 0 \quad \Longrightarrow \quad (2-d)E_k - (d)E_p = 0$$

RESULTS:

$$d = 1 \quad \Longrightarrow \quad E_k = E_p$$

$$d = 2 \quad \Longrightarrow \quad E_p = 0$$

Going to higher dimensions

The most important result:

$$d > 2 \Longrightarrow E_p = E_k = 0$$

Therefore:

"there are no stable, static, finite energy soliton solutions in a scalar field theory with canonical kinetic terms in more than 2 spacetime dimensions."

This looks like a very strong limitation, however, the theorem has also important assumptions.

Going to higher dimensions

A way out can be to violate one of the assumptions:

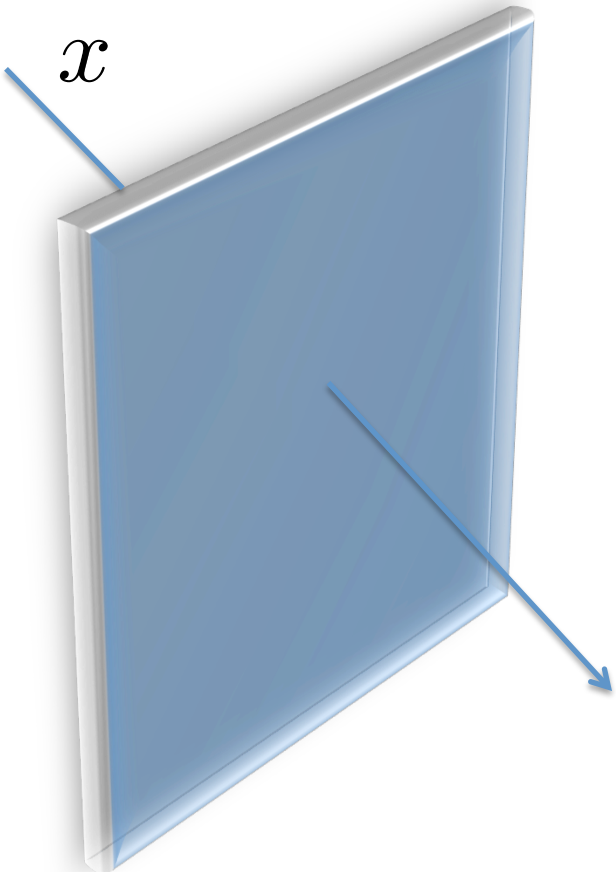
- Finite energy
  - Extended objects along some spatial direction (domain walls)
  - Having a physically motivated cut-off of the energy (vortex solitons)

- Only a scalar field theory
  - Adding other kind of fields avoids the conclusions (Gauge fields for example)

- Time independence
  - With time dependent configurations one can have stable, localized configurations (Q-balls)

- Canonical kinetic terms
  - Adding higher derivative terms modify Derrick's theorem. (Skyrme model)

Domain Wall solution



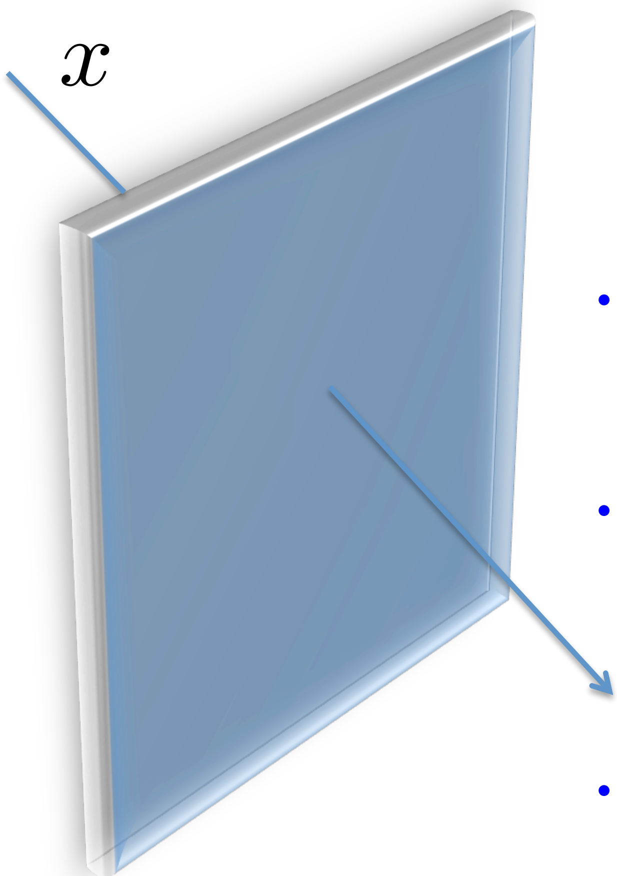
- Let's now consider the same kind of kink solution we studied earlier but in (3+1)d.

- The solution is the same:

$$\phi_k(x) = \eta \tanh\left(\eta\sqrt{\frac{\lambda}{2}}(x - x_0)\right)$$

- In (3+1)d the region of high energy density forms a 2d surface parallel to the (y,z) plane, a membrane or wall (domain wall).

Domain Wall solution



- The equation of state of these objects is quite interesting. **(Ex.)**

$$T^{\mu}_{\nu} = f(x)\ \mathrm{diag}\,(1, 0, 1, 1)$$

- The object is invariant along (y,z). The only motion is the transverse one.

- The tension in the (y,z) directions is equal to the surface density.

$$\sigma = \int T^0_0(x)dx$$

- Similar situation in other defects.

This gives rise to peculiar dynamical and gravitational properties to these objects.

Global Vortex solution (2+1)d

- Let's follow the same strategy as before and make the fields approach different vacua at infinity.

- In (2+1)d, spatial infinity is a circle so this suggest to look for a field theory with a vacuum manifold given by a circle. The simplest case is:

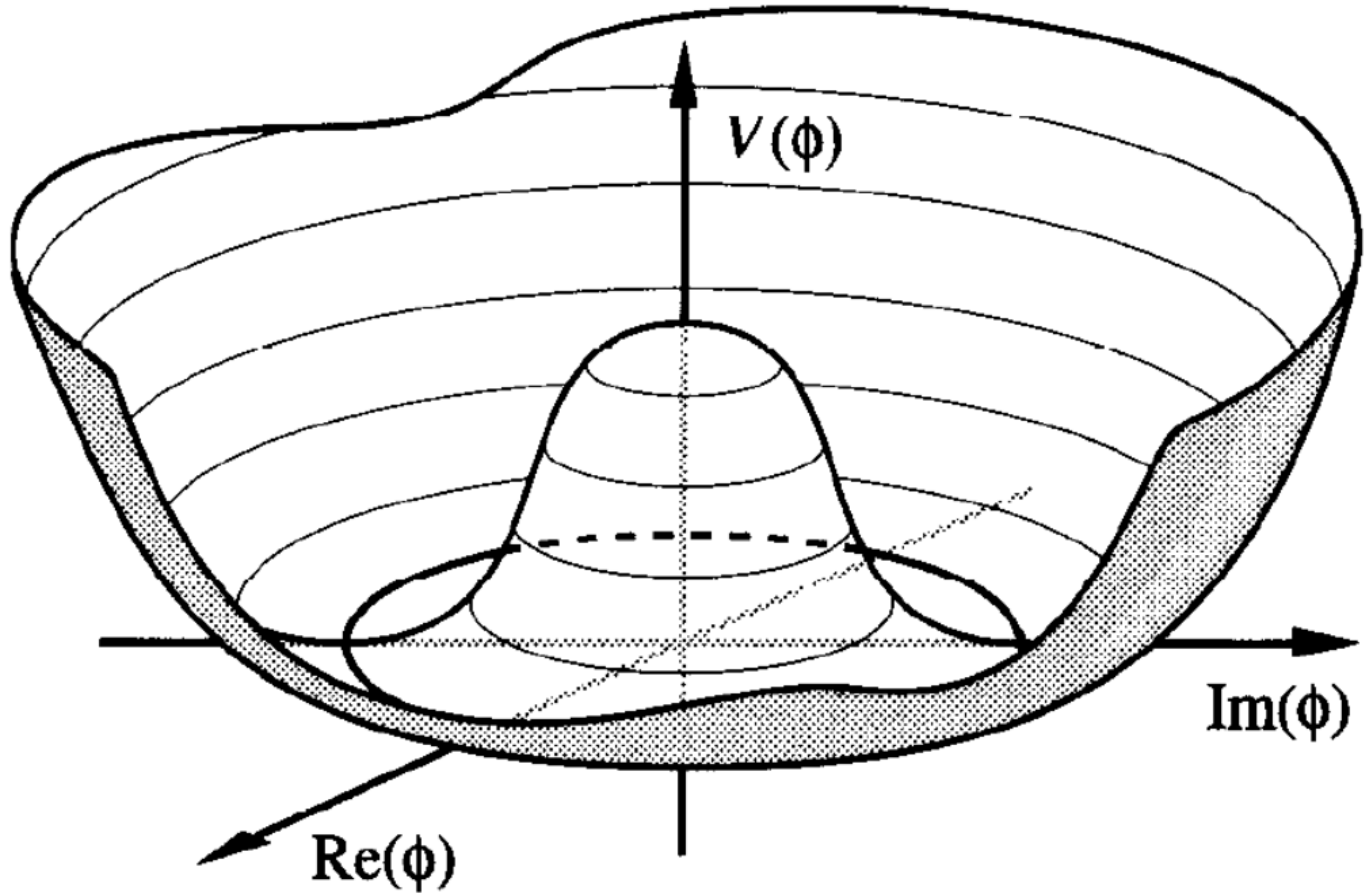
$$S = \int d^3x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi^* - \frac{\lambda}{4} \left(|\phi|^2 - \eta^2 \right)^2 \right]$$

- Where the is a complex scalar field

$$\phi(x) = \varphi_1(x) + i\varphi_2(x) = \rho(x) e^{i\theta(x)}$$

Global Vortex solution (2+1)d

- The potential is given by:



Global Vortex solution (2+1)d

- This theory has a global U(1) symmetry.
- The scalar field potential in minimized by $|\phi| = \eta$
- Looking at the perturbations around this vacuum we see that there are 2 elementary types of excitations:
  - Massive excitation (Radial excitation)

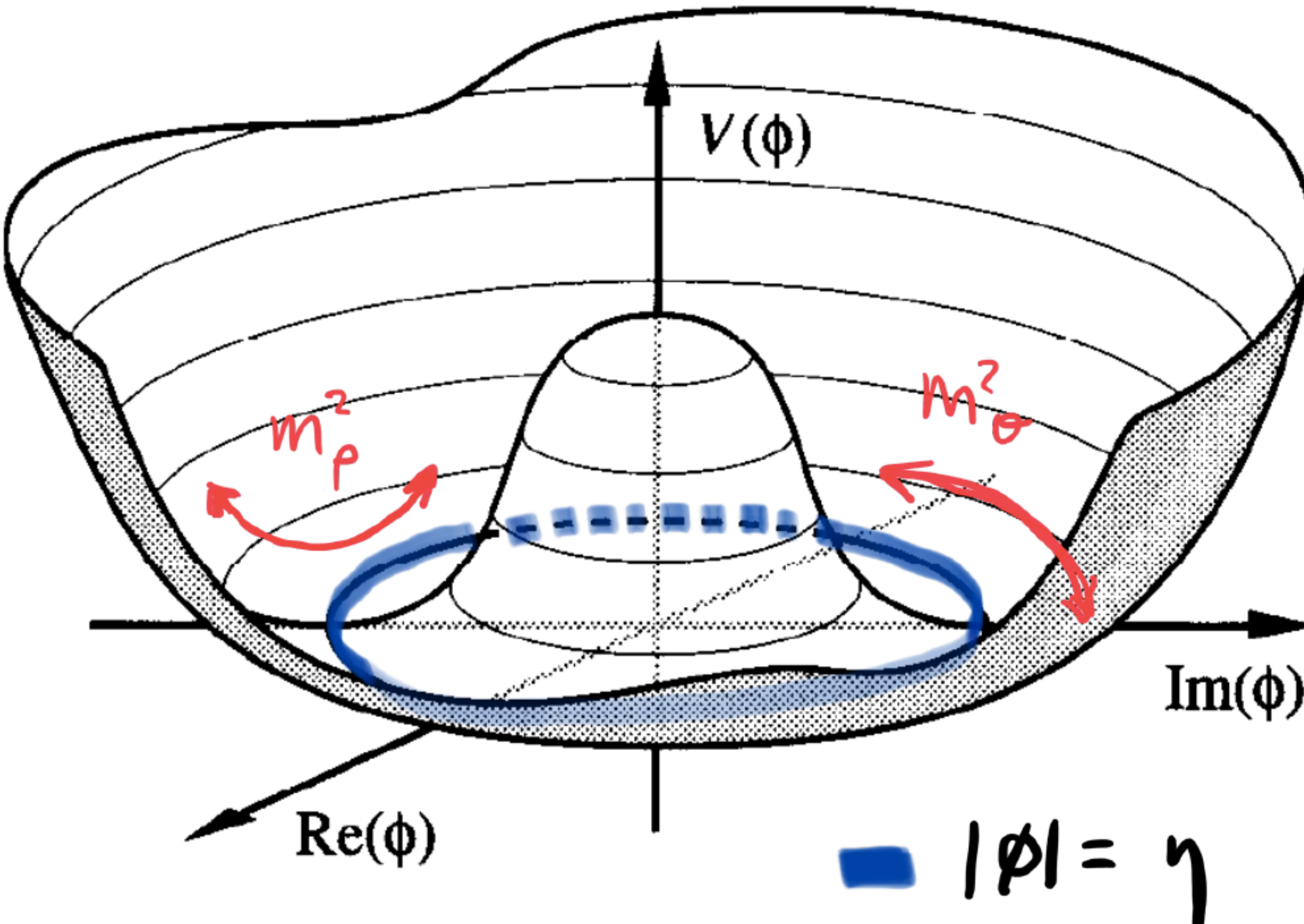
$$m_\rho = \sqrt{2}m$$

  - Massless excitation (Angular excitation)

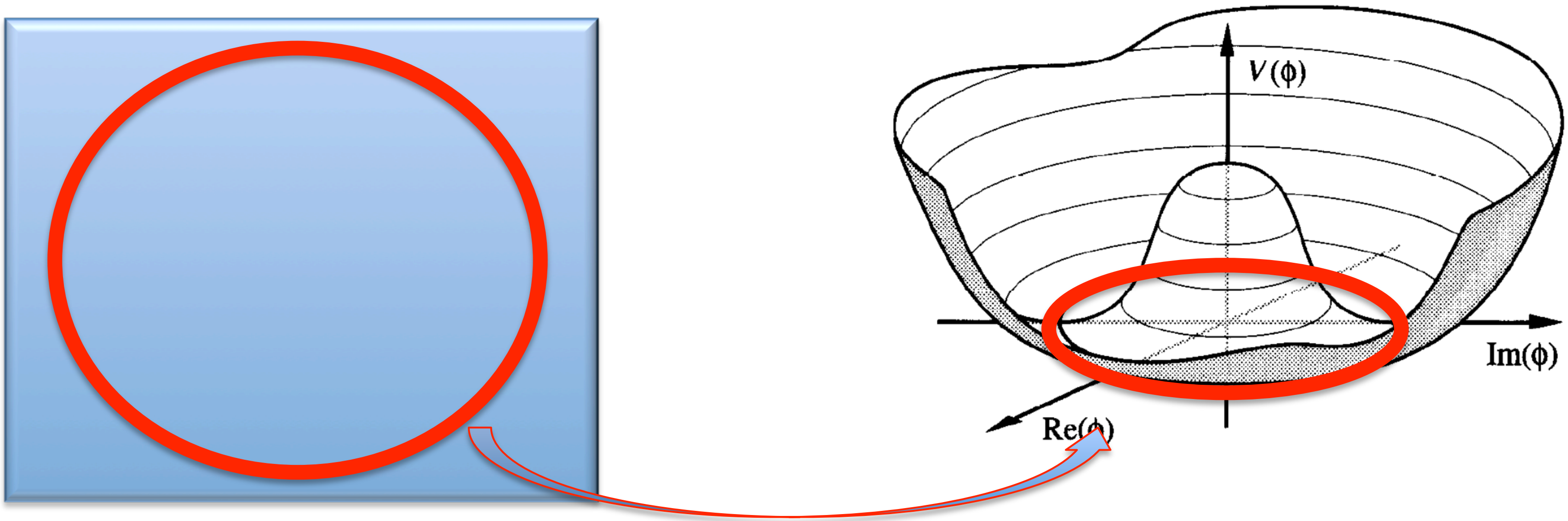
$$m_\theta = 0$$

Are there soliton excitations in this theory?

Global Vortex solution (2+1)d



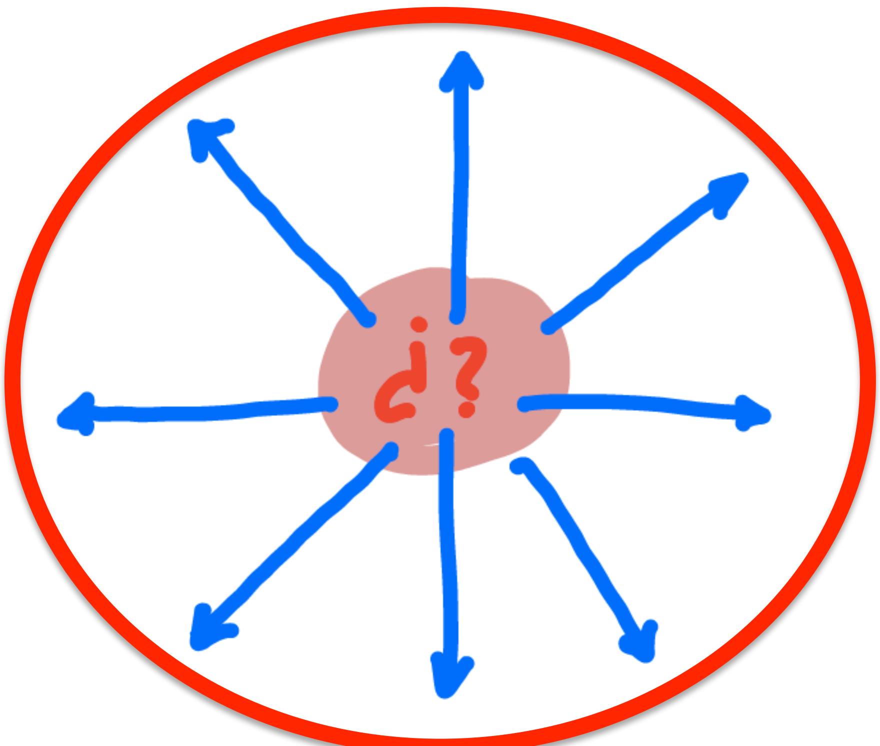
Global Vortex solution (2+1)d



- Following the strategy in the kink solution we want to map the asymptotic circle at infinity in real space in 2d to the vacuum manifold.

Global Vortex solution (2+1)d

- Following the strategy in the kink solution we want to map the asymptotic circle at infinity in real space in 2d to the vacuum manifold.



- The red circle represents the circle at infinity in physical space.
- The blue arrows indicate the configuration of the complex scalar field in field space

Global Vortex solution (2+1)d

- One can see that the solutions are characterized by a topological charge:

$$n_w = \frac{1}{2\pi} \oint_c dl \cdot \nabla \left(\arg\phi \right)$$

- This is called the winding number and is a integer.

- So given this asymptotic boundary conditions with a non-zero winding number, what is the lowest energy configuration of the field.

Something interesting must happen at the center of this configuration.

Global Vortex solution (2+1)d

- In order to find the form of the solution everywhere we look for solutions of the equations of motion, which in the static case are:

$$\nabla^2\phi - \lambda\phi\left(|\phi|^2 - \eta^2\right) = 0$$

- Using the ansatz in polar coordinates:

$$\phi_v(x) = \eta f(\rho) e^{i\theta} \quad \Longrightarrow \quad n_w = 1$$

- We obtain:

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r} - \lambda\left(\eta^2 - f^2\right) f = 0$$

Global Vortex solution (2+1)d

- We want to solve this equation:

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r} - \lambda\left(\eta^2 - f^2\right) f = 0$$

- With the following boundary conditions:

$f(0) \approx 0$ → Top of the potential

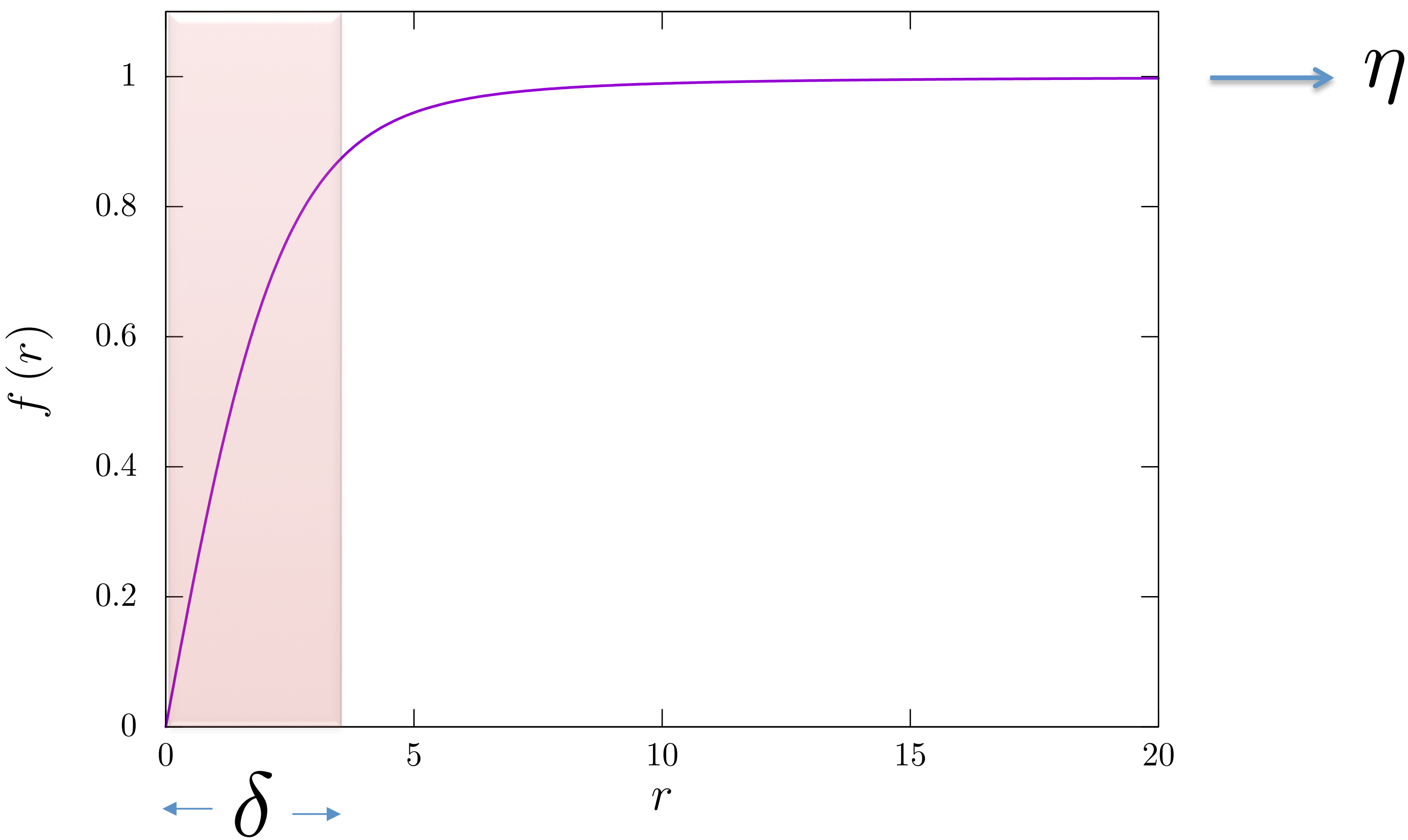
$f(r \to \infty) \approx \eta$ → Vacuum manifold

- In fact one can show that:

$f(r) \approx c \cdot r + \ldots$ → Single shooting parameter (c)

$f(r) \approx \eta - \mathcal{O}(1/r^2) + \ldots$

Global Vortex solution (2+1)d



Global Vortex solution (2+1)d

- Let's compute the energy of this configuration:

$$E_v = \int d^2x \left[\frac{1}{2}(\nabla\rho)^2 + \frac{1}{2}\rho^2(\nabla\theta)^2 + V(\rho)\right]$$

- The gradient term of the phase is problematic (Recall Derrick's Theorem)

$$E_\theta = \int d^2x \frac{1}{2}\rho^2(\nabla\theta)^2 = \int d^2x \, \frac{1}{2}\left(\frac{f^2}{r^2}\right)$$

$$\lim_{r\to\infty} E_\theta \approx \pi\eta^2 \int dr \, \frac{1}{r}$$

Global Vortex solution (2+1)d

- We can regularize this by introducing a long distance cutoff (R) .
- Physically this could be due to the size of the box or the distance to the nearest vortex.
- Taking this into account the energy of the vortex becomes:

$$E_0 \approx E_{\rm core} + 2\pi\eta^2 \log\left(\frac{R}{\delta}\right)$$

- This logarithm dependence resembles the energy of charged particles in 2+1 dimensions.

Global Vortex solution (2+1)d

- In fact this analogy can be made more precise by a duality transformation between the massless scalar field and a (2+1)d Maxwell field:

$$\partial_\alpha \theta = \epsilon_{\alpha\beta\gamma} F^{\beta\gamma}$$

- This duality demonstrates that vortices behave as charged objects in (2+1)d.

- This leads to an important point, these objects have long range interactions. This has important consequences in cosmological models

- This suggests that n=2 vortices are unstable (This is indeed demonstrated by looking at small perturbations around the field theory solution with n=2.)

Local vortices in (2+1)d

- Let's now consider the local U(1) symmetric Abelian-Higgs Lagrangian

$$S_{AH} = \int d^4x \left[|D_\mu \phi|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{4} \left(|\phi|^2 - \eta^2 \right)^2 \right]$$

- The vacuum manifold is still the same:

$$|\phi| = \eta$$

- However the propagating degrees of freedom are now both massive.

$$m_\phi = \sqrt{2\lambda}\eta \qquad m_A = e\eta$$

- The Higgs mechanism.

This has very important consequences for the solution and the dynamics of the soliton.

Local vortices in (2+1)d

- There is a way now to cure the divergent contribution to the gradient energy since we have

$$D\phi \sim e^{i\theta} i\rho \left(\nabla\theta - e\mathbf{A}\right)$$

- So turning on a non-zero vector potential asymptotically we can kill this contribution. This can be accomplished by the ansatz that asymtotically is of the form,

$$\left.\begin{aligned} \phi(\infty) &= \eta e^{in\theta} \\ A_\theta(\infty) &= \frac{n}{er} \end{aligned}\right\} \qquad D\phi \to 0 \qquad F_{\mu\nu} \to 0 \qquad r \to \infty$$

Possibility of a Finite energy configuration

Local vortices in (2+1)d

- In order to find the complete solution, let's look for an ansatz of the form,

$$\phi(r) = \eta f(r) e^{in\theta}$$

$$A_\theta(r) = n\frac{\alpha(r)}{er}$$

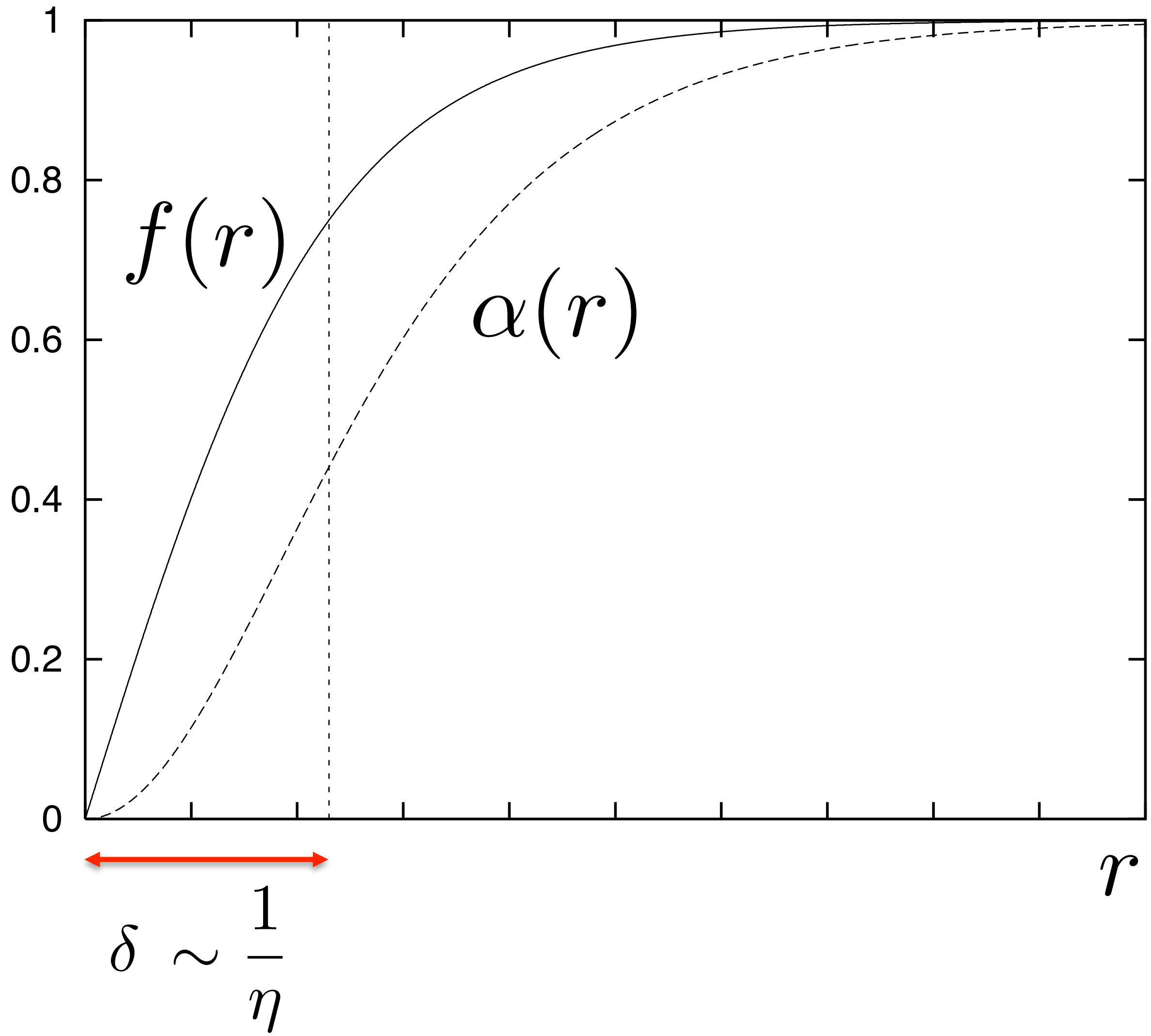
- With the boundary conditions:

$$f(r \to 0) \approx 0 \qquad f(r \to \infty) \approx 1$$

$$\alpha(r \to 0) \approx 0 \qquad \alpha(r \to \infty) \approx 1$$

Local vortices in (2+1)d

(Nielsen & Olesen'73)



$$\phi(r) = \eta f(r) e^{in\theta}$$

$$A_\theta(r) = n\frac{\alpha(r)}{er}$$

Local vortices in (2+1)d

- Properties of the solution:
  - They are characterized by a quantum winding number:

$$n_w = \frac{1}{2\pi} \oint_c dl \cdot \nabla \left(\arg\phi\right)$$

  - Due to the asymptotic form of the solution, this is also connected to the magnetic flux.

$$n_w = \frac{1}{2\pi} \oint_c dl \cdot \nabla(\arg\phi) = \frac{e}{2\pi} \oint_c dl \cdot \mathbf{A} = \frac{e}{2\pi} \int d^2 x B(x)$$

$$\Phi_{\rm mag} = \frac{2\pi n}{e}$$

Relativistic version of the Abrikosov vortices in superconductors

Local vortices in (2+1)d

- Properties of the solution (cont.)
  - Exponential approach to the vacuum

$$\alpha(r) \approx 1 - \sqrt{r}e^{-m_A r} + \ldots \qquad f(r) \approx 1 - \frac{1}{\sqrt{r}}e^{-m_i r} + \ldots$$

  - Finite energy solutions

$$E_v = \pi\eta^2 g(\beta) \qquad \beta = \left(\frac{m_\phi}{m_A}\right)^2 \qquad g(1) = 1$$

  - Interaction between vortices

$$\beta < 1 \longrightarrow \text{Attractive}$$

$$\beta > 1 \longrightarrow \text{Repulsive}$$

Local vortices in (2+1)d

- Critical case (supersymmetric one), no force between vortices

$$\beta = 1 \quad \Longrightarrow \quad m_\phi = m_A$$

- Similarly to what happens in the kink solution we can write the static energy of the vortex in the critical case in terms of an inequality

$$E_v \geq \pi \eta^2 n$$

- This is saturated if the vortex solution satisfies a first order equations of motion.

Cosmic Strings

- Extending the vortex solution to a (3+1)d spacetime, we have a straight string; a cosmic string
- The tension of the string will have the same magnitude as the energy density.
- The dynamics are therefore relativistic.
- Furthermore, the Newtonian potential of an infinitely straight string is zero

$$p_3 = -\rho$$
$$p_1 = p_2 = 0$$
$$\Longrightarrow \quad \nabla^2 \Phi = 4\pi G \left(\rho + p_1 + p_2 + p_3 \right)$$

Other solitons

- It is easy to generalize the global vortex model to a global theory whose vacuum manifold is a 2-sphere.
- This would lead to a global monopole (also divergent in energy).
- This could be cured by a local theory. This would give rise to magnetic monopole solutions.
- However, the existence of a phase transitions that produces stable magnetic monopoles is problematic.
- They do not annihilate fast enough and become the dominant form of energy in the universe.
- These models are ruled out. Inflation is a natural way out of this problem.

Other localized solutions

- There are other field theory configurations that make use of their non-linear properties to create long lasting localized energy configurations.

- Most well know one Oscillons.

    - In 1+1 can be understood as bound states of kink-antikink configurations.

    - They do not have a topological stability (no-charge)

- Others

Summary part I

- Many extensions of the Standard Model of Particle Physics allow for the possibility of stable soliton configurations.
- There are different types of objects depending on the field theory and in particular on its vacuum manifold:
    - Domain walls, strings, monopoles…
- Many of these models are associated with new physics scale typically at high energies.
- This energy scale characterizes the object, mass, size, etc..
- Their long lifetime can leave some imprint in our universe.

Thanks

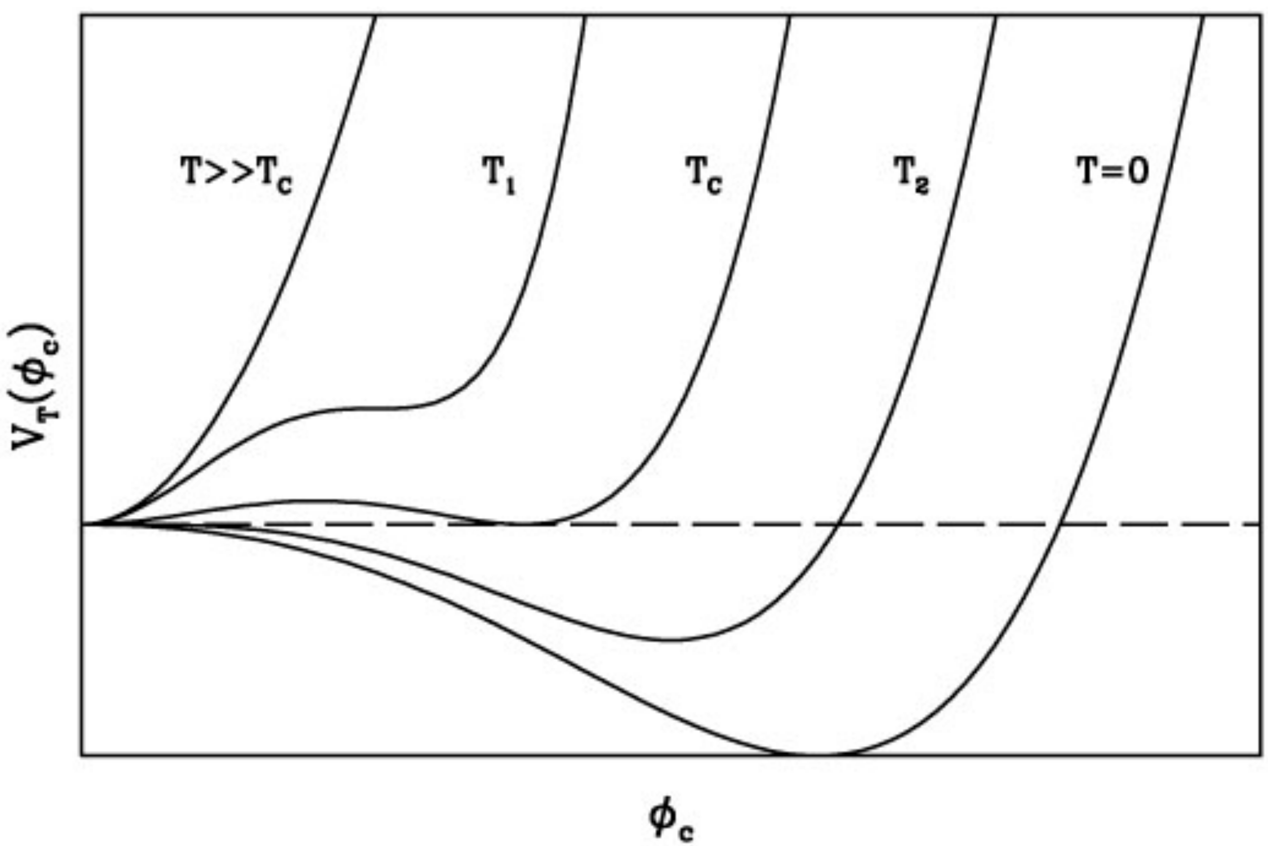
Lecture II

Outline of 2nd lecture

- Second Lecture:
    - Cosmological Formation of Defects
    - Viable models
    - Observational signatures
        - Domain wall networks
        - Cosmic string networks

Cosmological Phase Transitions

- As the universe expands the temperature of the universe decreases.
- The effective potential for different fields changes with temperature.



Cosmological Phase Transitions

- Depending of the pattern of symmetry breaking different defects can be formed.

Table 3.1. Topological classification of defects with the homotopy groups $\pi_n(\mathcal{M})$.

Topological defect	Dimension	Classification
Domain walls	2	$\pi_0(\mathcal{M})$
Strings	1	$\pi_1(\mathcal{M})$
Monopoles	0	$\pi_2(\mathcal{M})$
Textures	—	$\pi_3(\mathcal{M})$

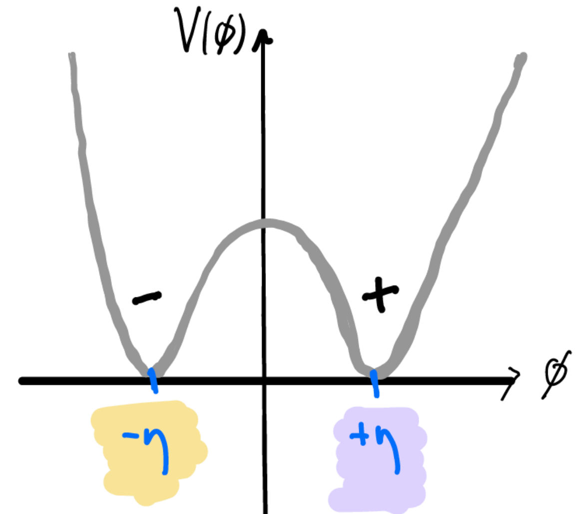
- We will concentrate for now on the case such that vacuum manifold is not simply connected. In this case we will form domain walls.

The Kibble mechanism (Kibble '76).

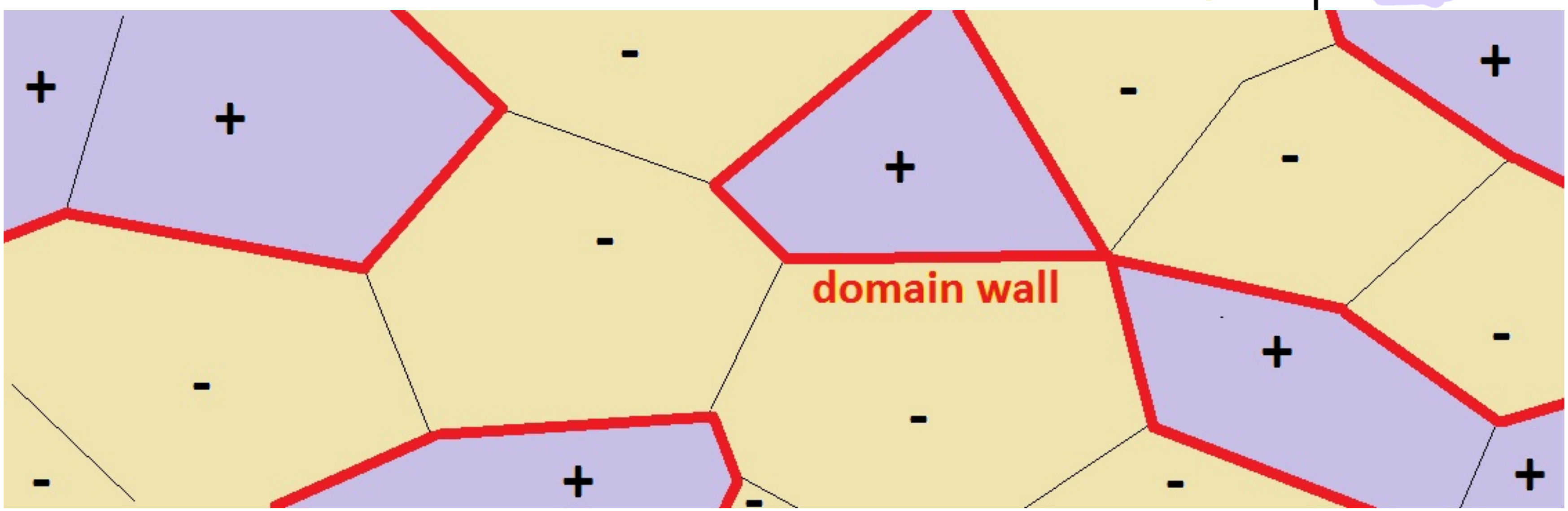
- In an expanding universe, there are parts of the universe separated by a distance larger than the distance travelled by light since the beginning of the universe.
- This length is the horizon distance. $d_h \sim H^{-1}$
- The fields can not be correlated over distances bigger than the horizon.
- That means that in our example there will be regions that settle to one vacuum while others choose the other vacuum.
- Domain walls will form between these vacua.

The Kibble mechanism (Kibble ‘76).

Cartoon picture of the Kibble mechanism



Horizon distance



Cosmological models with defects

- In order to have a viable cosmological model, the energy in these defects can not dominate the energy density of the universe.
- This is an important constraint and rules out models with monopoles.
- For domain walls we will see that this alone restricts the type of models allowed but does not kill this possibility completely.
- We will explain that strings are indeed allowed due to their special dynamics.

Domain wall dynamics

- At low energies, we can take as the effective action for the massless degrees of freedom the Nambu-Goto action. (Thin wall approximation)

$$S_{NG} = -\sigma \int \sqrt{-\gamma}\, d^3\xi$$

where σ is the energy per unit area of the domain wall and $\gamma_{\alpha\beta}$ denotes the induced metric

$$\gamma_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$$

and $X^\mu(\xi^\alpha)$ parametrizes the position of the worldvolume of the domain wall.

Domain wall dynamics

Simple example: (Ex: Work out the details)

- Let's consider a spherical domain wall which we parametrize in terms of its radius $R(t)$
- Plugging this ansatz in the NG action we get

$$S = -4\pi\sigma \int dt R^2 \sqrt{1 - \dot{R}^2}$$

whose equation of motion imply,

$$\ddot{R} = -2\left(\frac{1-\dot{R}^2}{R}\right) \Longrightarrow \frac{\text{Force}}{\text{area}} \sim \frac{\sigma}{R}$$

Curved walls collapse under their own tension

Domain wall networks

- Simulations show that networks in a cosmological background enter a period of *scaling*.
- This means that all the distances in the problem scale with the horizon.
- We can picture this scaling as the configuration where there is a wall per horizon volume.
- That means that the energy density in domain wall is

$$\rho_{DW} \sim \frac{\sigma H^{-2}}{H^{-3}}$$

- Therefore the fraction of energy in the universe in walls will be:

$$\Omega_{DW} = \frac{\rho_{DW}}{\rho_c} \sim \frac{\sigma H G}{H^2} \sim \frac{\sigma G}{H}$$

Domain wall networks

- Taking, $H \sim t^{-1}$
- We see that domain walls in a scaling solution would start dominating the energy density in the universe at the time roughly of the order

$$t_* = \frac{1}{G\sigma}$$

- Taking this time to be at least the current age of the universe imposes the constraint on the domain wall tension

$$\sigma < 1 \ \mathrm{G}eV$$

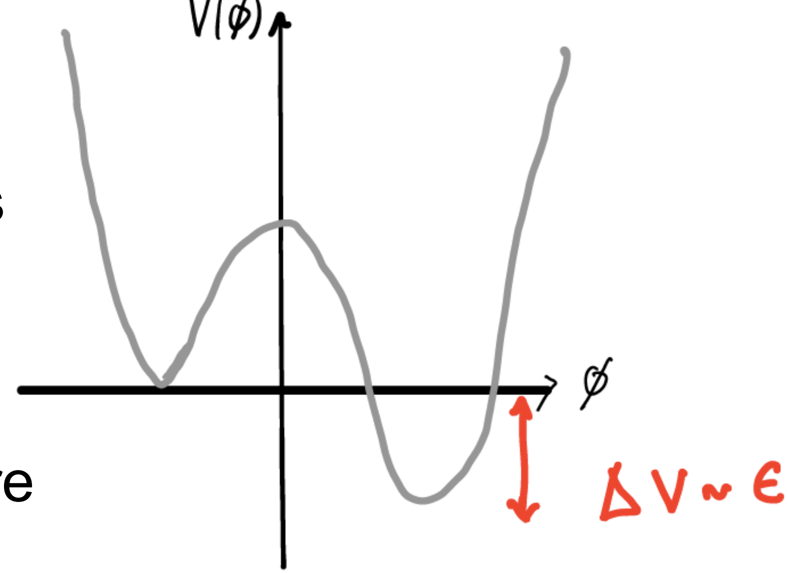
This is quite a very important constraint on models with domain walls !

Biased Domain wall networks

- This constraints can be avoided if the domain walls dissapear at some point.

- This can happen if, for example, there is a small bias towards one of the vacua.

$$\Delta V \sim \epsilon$$



- This produces a source of small pressure towards one of the vacua. This force becomes important when

$$\epsilon \sim \frac{\sigma}{R}$$

- This should happen before wall domination

$$R < \frac{1}{G\sigma} \Longrightarrow \epsilon > G\sigma^2$$

Gravitational waves from domain walls

- Taking this rough scenario we can estimate the spectrum of the stochastic background of gravitational waves produced by these walls.

$$\Omega_{GW} = \frac{1}{\rho_c}\left(\frac{d\rho_{GW}}{d\ln f}\right)$$

- First we would like to estimate the height of the spectrum:

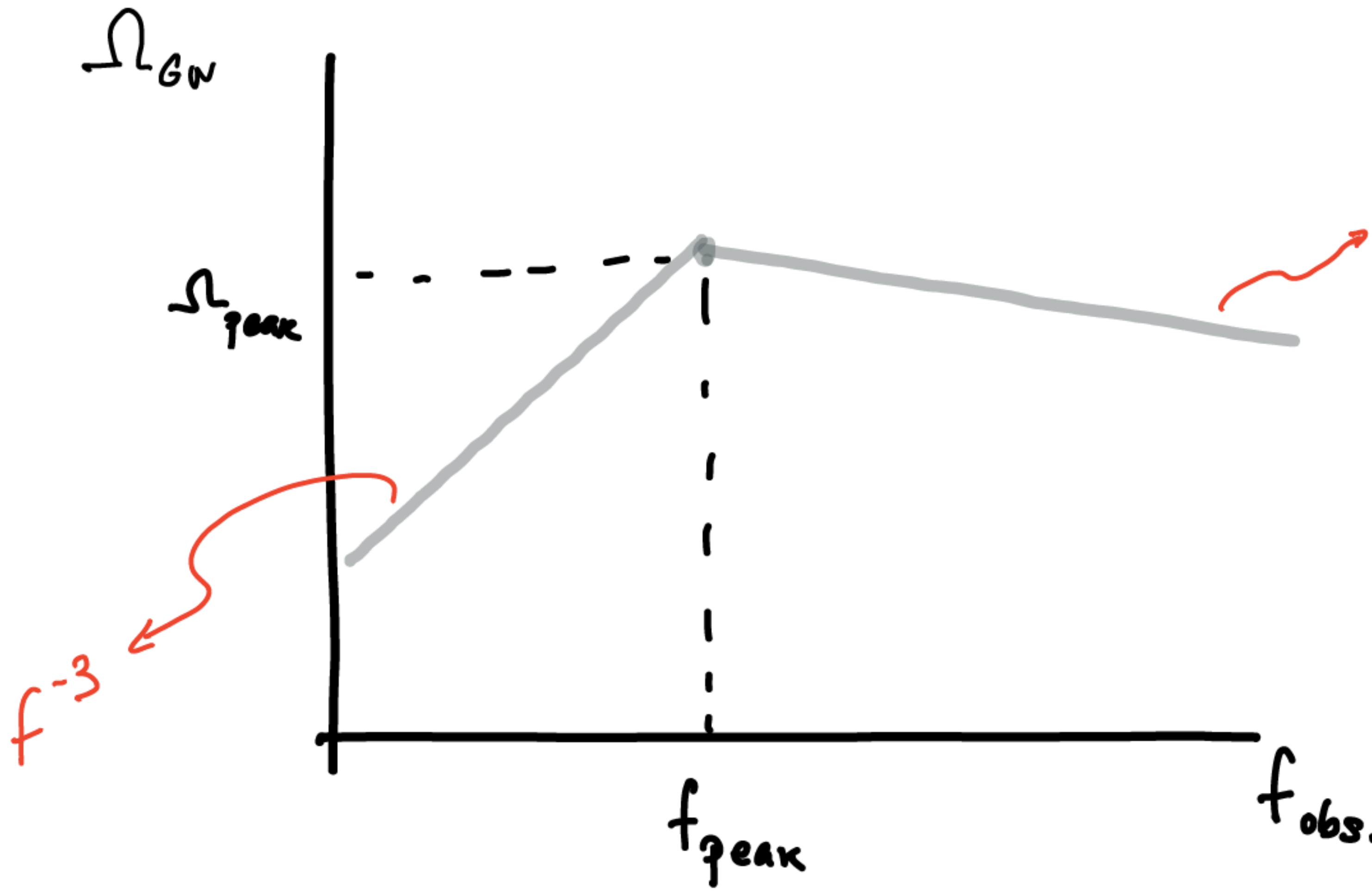
$$\left(\Omega_{GW}h^2\right)_{\rm peak} \sim 3\times 10^{-18}\left(\frac{\sigma}{1{\rm TeV}^3}\right)^2\left(\frac{T_{an}}{10{\rm MeV}}\right)^4$$

- And the position of the peak:

$$f_{\rm peak} \sim 10^{-9}{\rm Hz}\left(\frac{T_{an}}{10{\rm MeV}}\right)$$

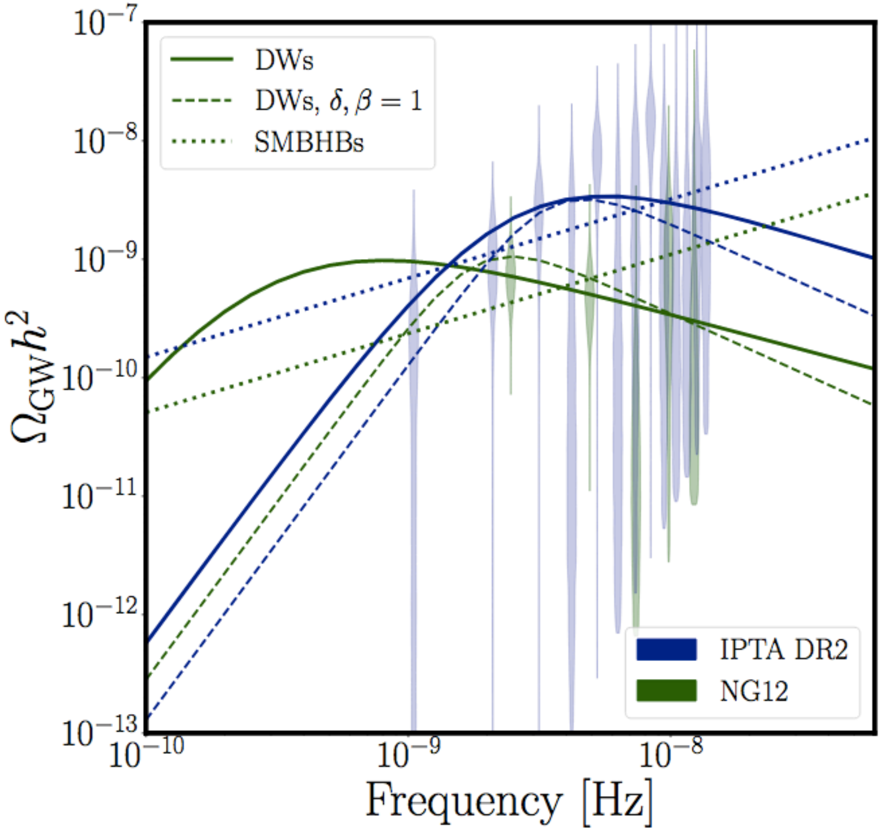
Gravitational waves from domain walls

- Changing the parameters of the model, (σ, T_{ann}) , we can move this spectrum in different bands of observation and amplitude.



Gravitational waves from domain walls in the PTA band

For example:

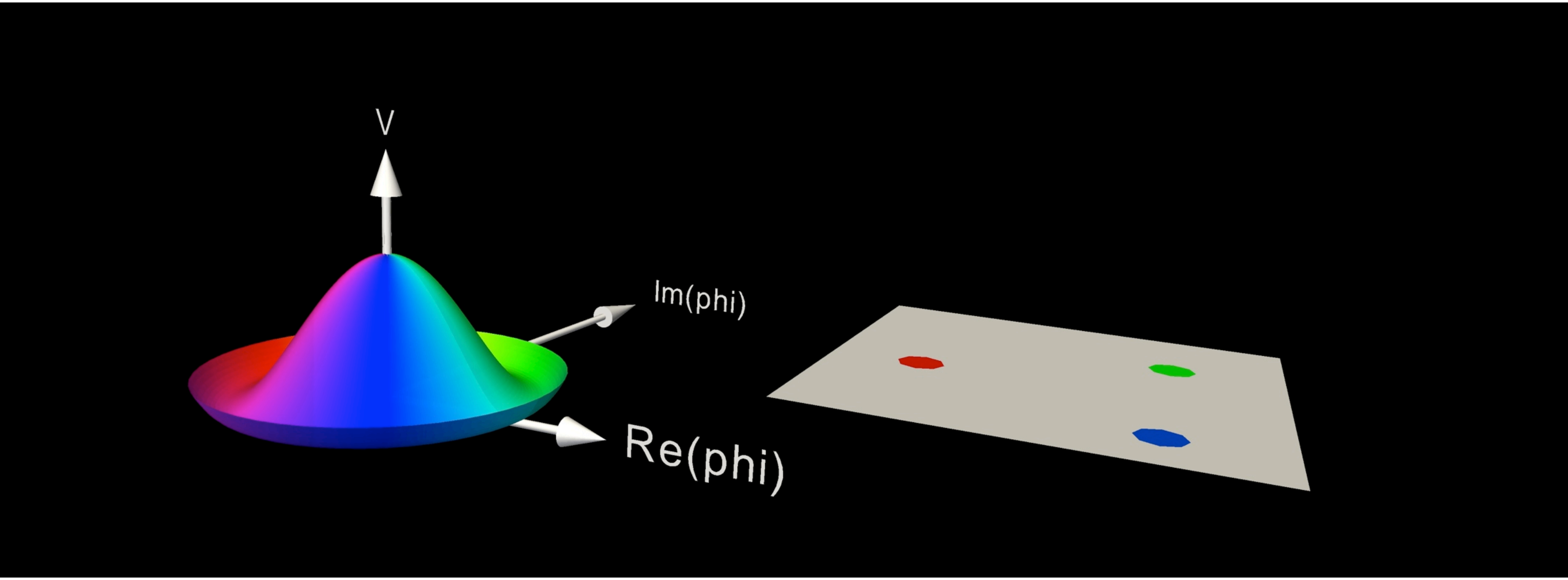


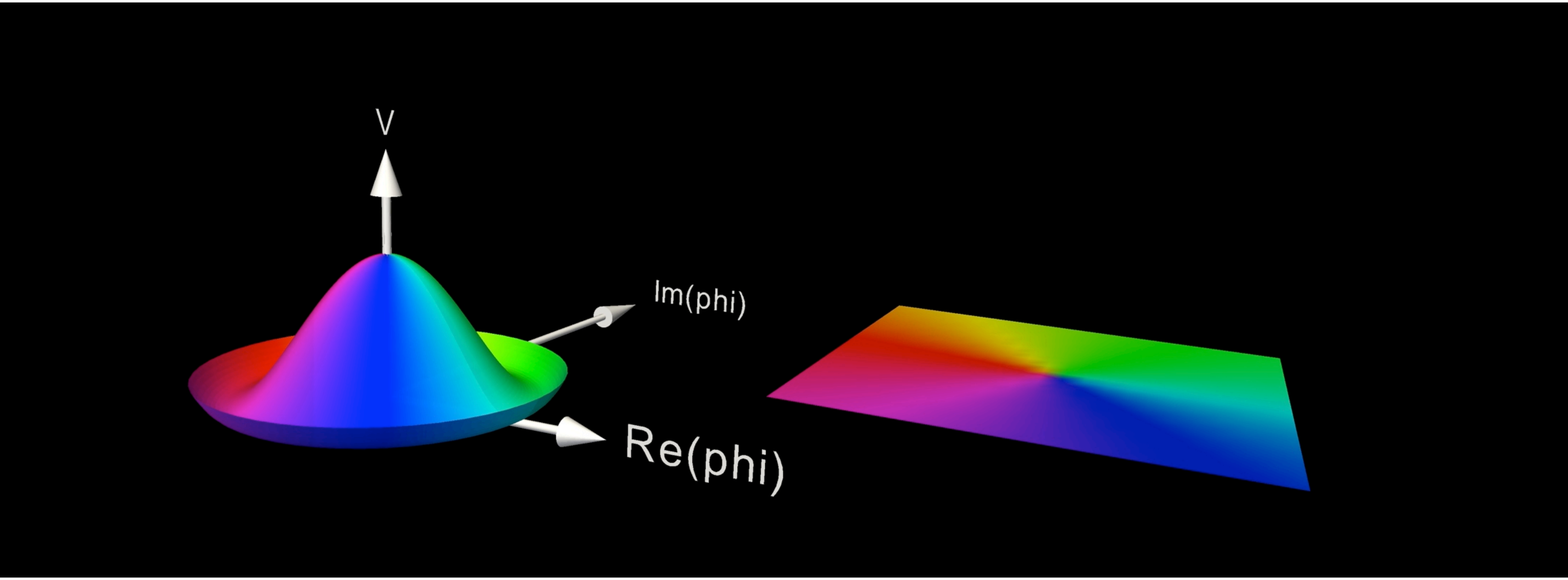
(R. Ferreira et al. '22).

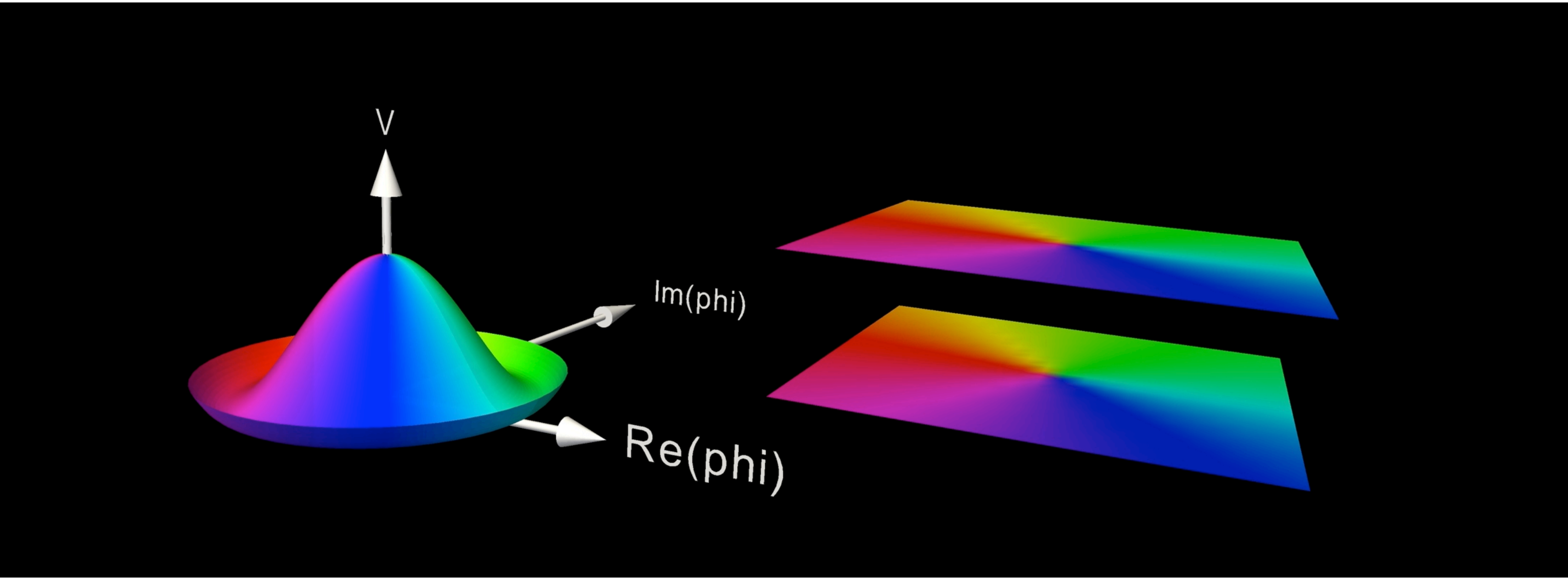
See also the discussion of these types of models in the NANOGrav Collaboration paper:

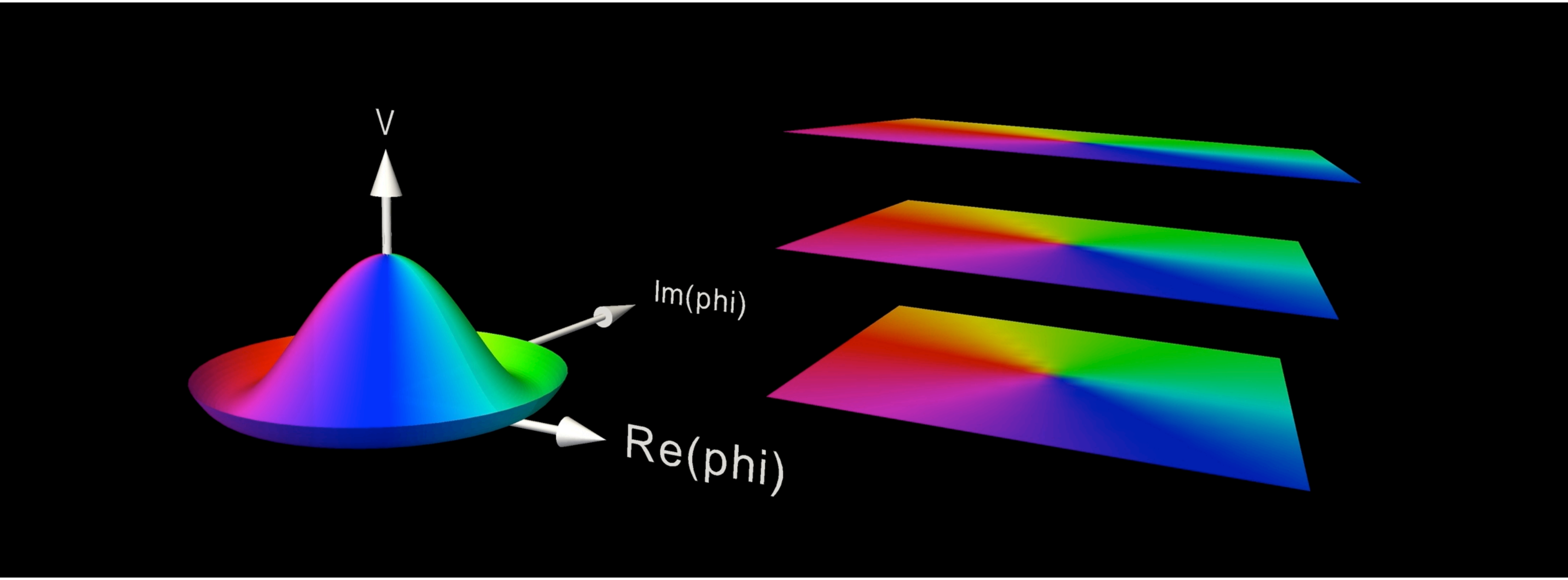
"The NANOGrav 15-year Data Set: Search for Signals from New Physics", A. Afzal et al. (2023)

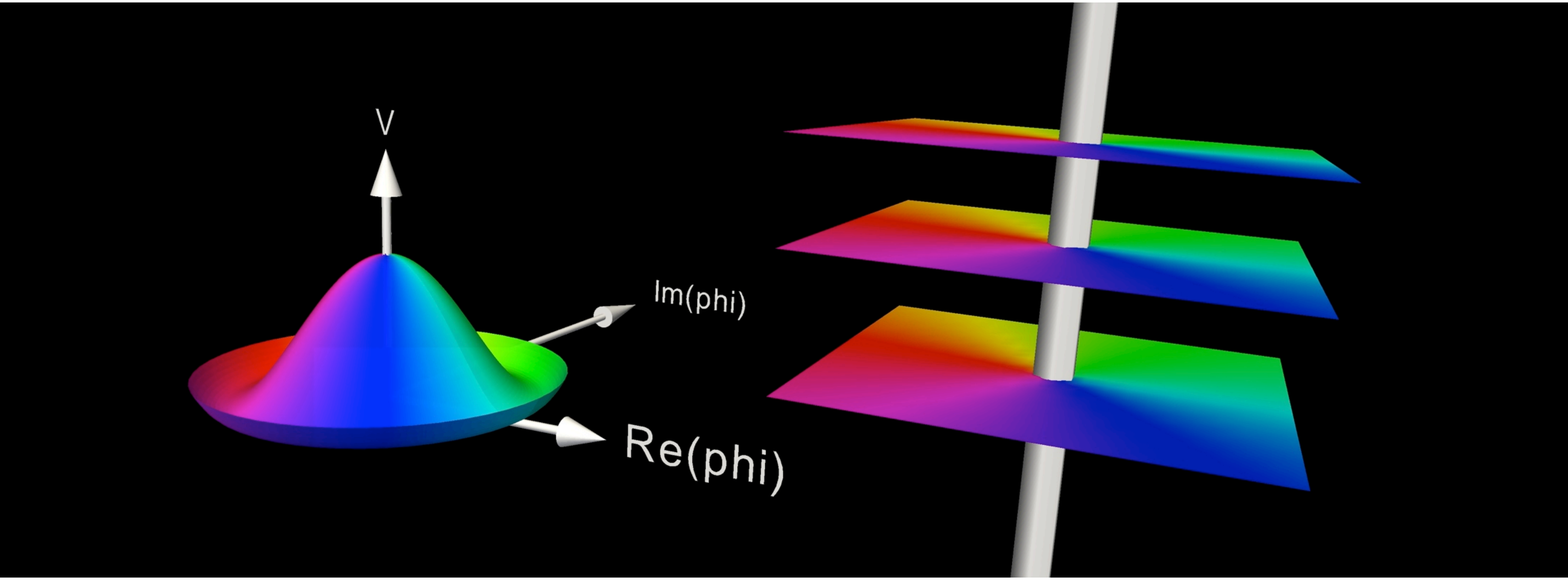
Cosmic Strings


V
Im(phi)
Re(phi)


V
Im(phi)
Re(phi)


V
Im(phi)
Re(phi)


V
Im(phi)
Re(phi)


V
Im(phi)
Re(phi)

What is a cosmic string?

- Physical properties of the strings:
    - They are topological stable objects, they have no ends.
    - They are Lorentz invariant.

      Tension = Energy density per unit length

    - They are not coupled to any massless mode, except gravity.

(This is the simplest version of strings that we will consider here)

The String Scale

- Thickness, energy density and tension of the string are controlled by the symmetry breaking scale. η
- For a Grand Unified Theory scale: $\eta \approx 10^{16}\mathrm{GeV}$
- Thickness: $\delta = 10^{-30}\mathrm{cm}$
- Linear mass density: $\mu = 10^{22}\mathrm{gr/cm}$
- Tension : $T = 10^{37}N$
- Gravitational effects depend on:

$$G\mu = \left(\frac{\eta}{M_{Pl}}\right)^2 \sim 10^{-6}$$

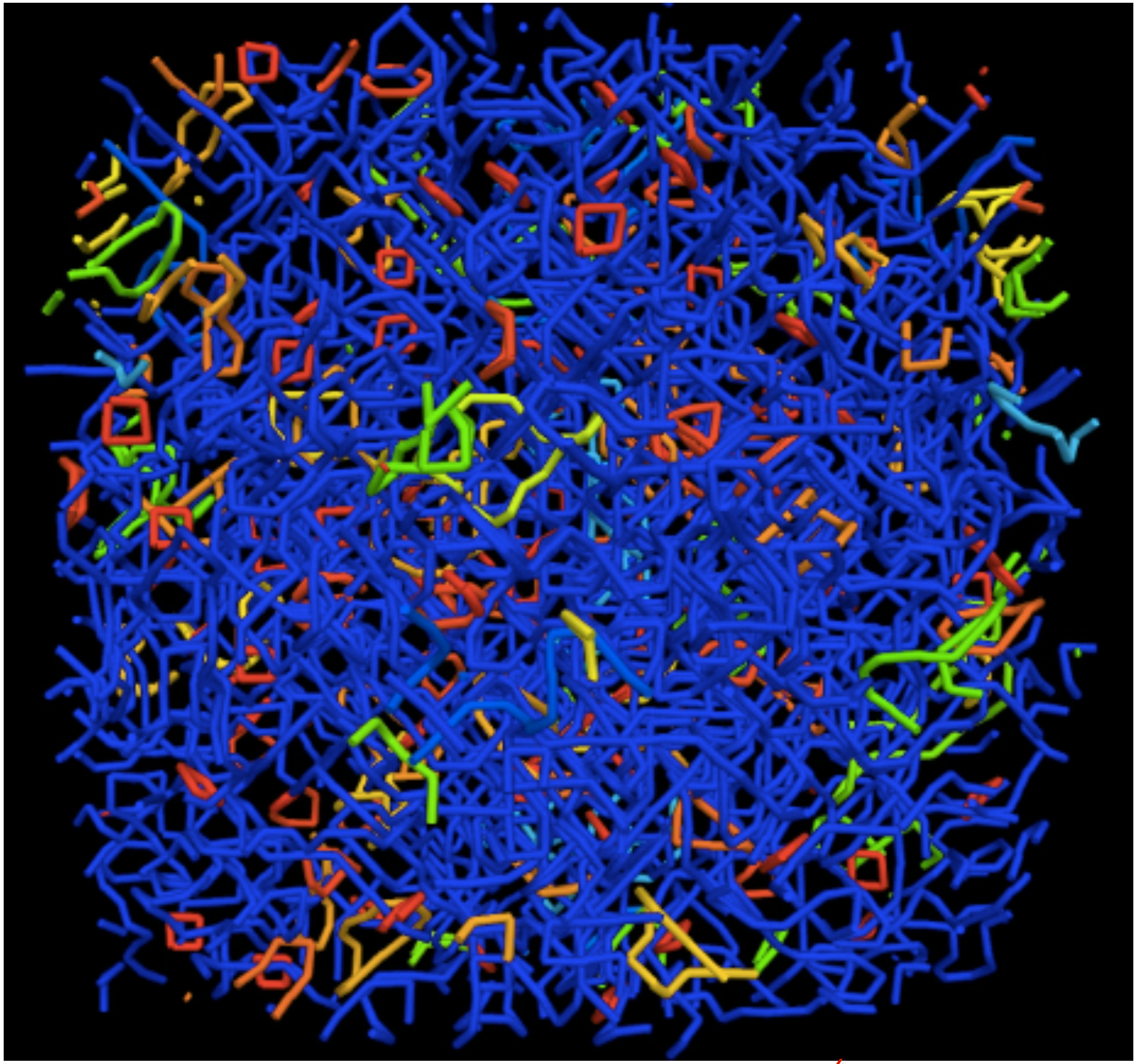
Cosmological Formation

(Kibble '76).

- Strings get formed at a cosmological phase transition.
- In order for strings to be cosmologically relevant this should happen at the end of inflation.
- They could be formed in models of hybrid inflation or during reheating.
- In String Theory they are produced at the end of brane inflation. (They have some characteristic properties that make them special)

Initial Conditions

(Vachaspati & Vilenkin '84).



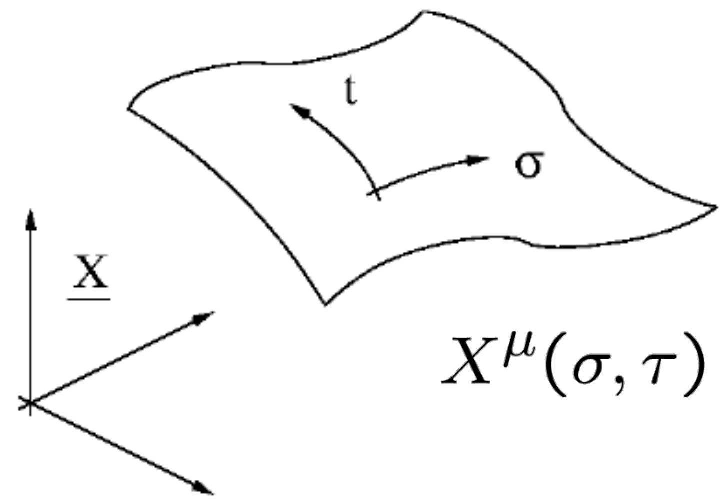
(B-P., Olum and Shlaer '12).

Classical Relativistic String

(Nambu,' 71; Goto '70).

- The Lagrangian of a relativistic string:

$$S_{NG} = -\mu \int d^2\xi \sqrt{-\gamma}$$



- String tension equal to the energy density:

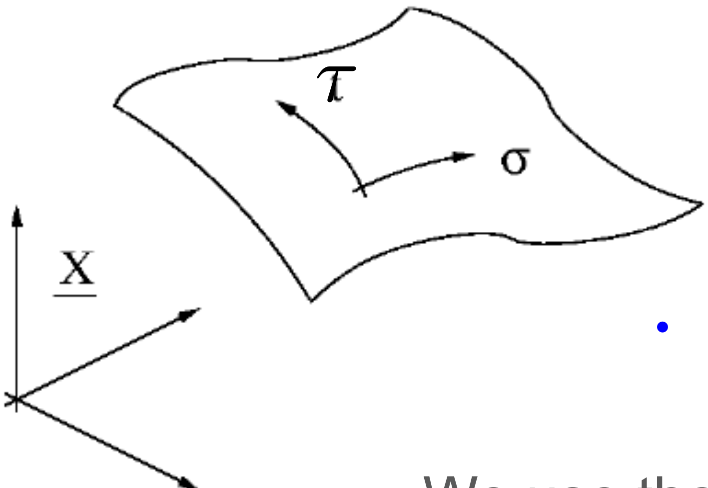
$$\mu$$

- Induced metric on the worldsheet:

$$\gamma_{\alpha\beta} = g_{\mu\nu} \, \partial_\alpha X^\mu \partial_\beta X^\nu$$

Cosmic String Dynamics

(Nambu, '71; Goto '70).



- A relativistic string dynamics has an action of the form,

$$S_{NG} = -\mu \int \sqrt{-\gamma}\, d^2\xi$$

- The string is described by: $\mathbf{X}(\sigma, \tau)$

- We use the gauge conditions: $\mathbf{X}'^2 + \dot{\mathbf{X}}^2 = \mathbf{1}$

$$\mathbf{X}' \cdot \dot{\mathbf{X}} = \mathbf{0}$$

- The e.o.m. become:

$$\mathbf{X}'' = \ddot{\mathbf{X}}$$

$$\mathbf{X}(\sigma, \tau) = \frac{1}{2}\left[\mathbf{a}(\sigma - \tau) + \mathbf{b}(\sigma + \tau)\right]$$

$$|\mathbf{a}'| = |\mathbf{b}'| = \mathbf{1}$$

Classical Non-relativistic String

- There are two important parameters in this problem:
  - The energy per unit length of the string μ
  - Tension of the string T
- The equations of motion for the transverse motion of the string is given by:

$$\frac{\partial^2 Y}{\partial x^2} - \left(\frac{\mu}{T}\right)\frac{\partial^2 Y}{\partial t^2} = 0$$

$$\frac{\partial^2 Y}{\partial x^2} - \left(\frac{1}{v^2}\right)\frac{\partial^2 Y}{\partial t^2} = 0$$

The wave equation

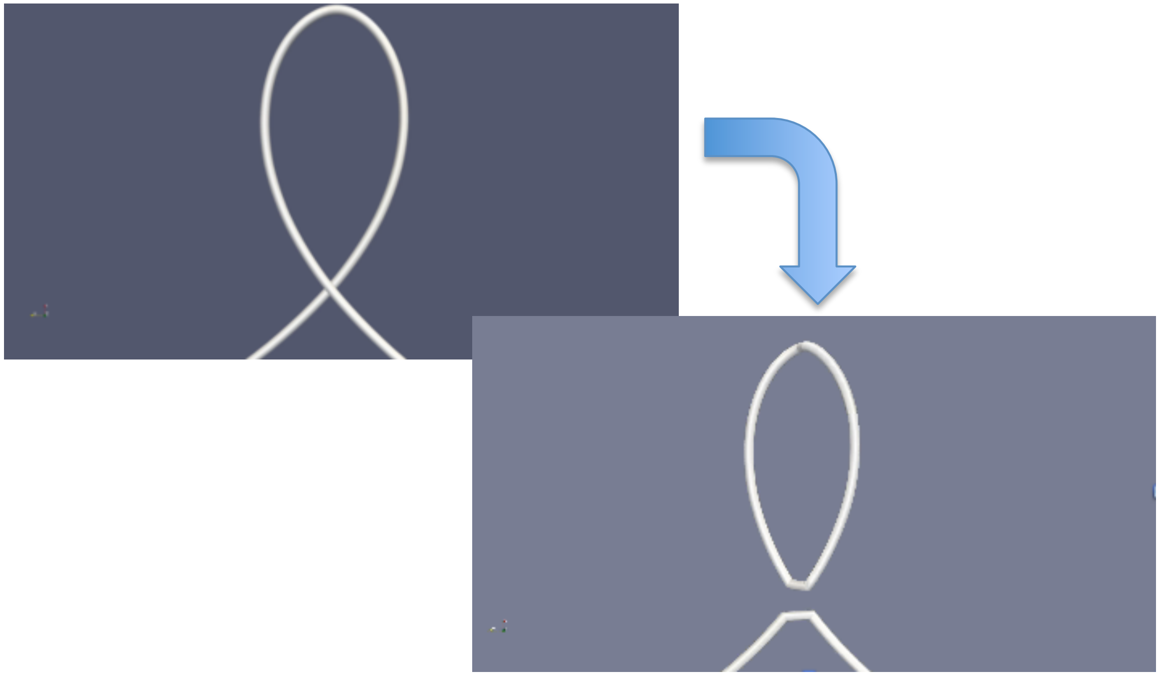
Cosmic String Dynamics

$$\mathbf{X}(\sigma,\tau) = \frac{1}{2}\left[\mathbf{a}(\sigma-\tau) + \mathbf{b}(\sigma+\tau)\right]$$

The function a and b represent travelling waves moving on the string in opposite directions.

What about interactions?

- Strings interact by exchanging partners, creating loops.



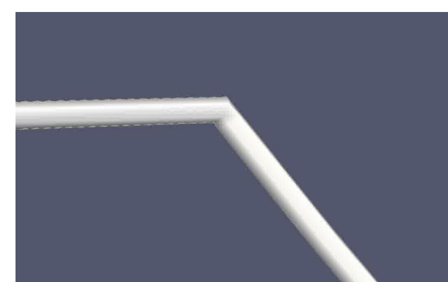
- For field theory strings

$$P_{\rm inter} = 1$$

for Superstrings

$$P_{\rm inter} < 1$$

- This mechanism produces kinks on strings.
- This builds up small scale structure on strings.



Classical Relativistic String

- Simplest possible example:

$$X^\mu = [t, R(t)\cos(\sigma), R(t)\sin(\sigma), 0]$$

- With this ansatz, we have:

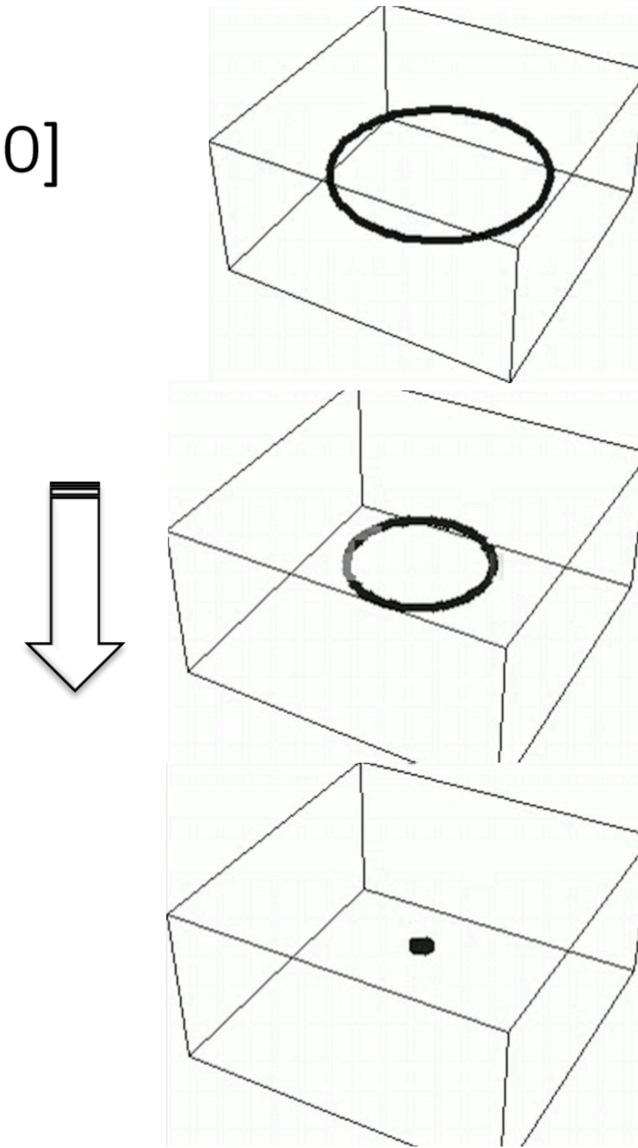
$$S = -2\pi\mu \int dt R \sqrt{1-\dot{R}^2}$$

- Equation of motion:

$$\dot{R}^2 + \left(\frac{R}{R_0}\right)^2 = 1$$

- Solution:

$$R(t) = R_0 \cos\left(\frac{t}{R_0}\right)$$



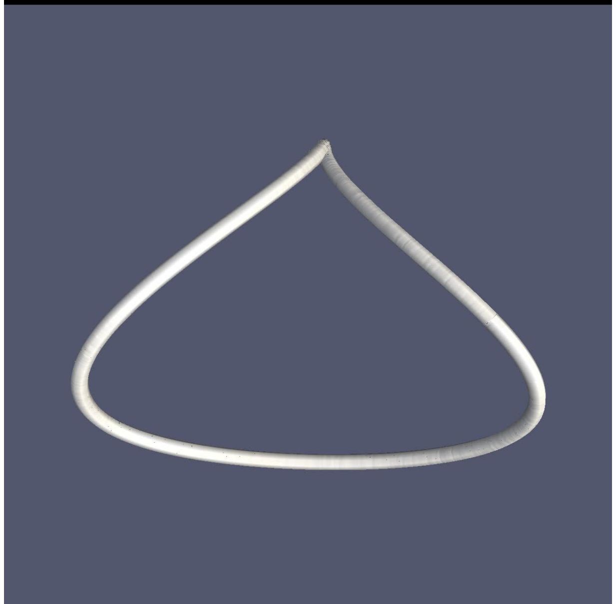
Cosmic String Dynamics (Loops)

$$\mathbf{X}(\sigma,\tau) = \frac{1}{2}\left[\mathbf{a}(\sigma-\tau) + \mathbf{b}(\sigma+\tau)\right]$$

- The solutions for closed loops are periodic.
- The loops oscillate under their tension.
- The strings move typically relativistically.
- During its evolution a loop may have points where the string reaches the speed of light: A cusp

$$|\dot{\mathbf{X}}| = \frac{\mathbf{1}}{\mathbf{2}}|\mathbf{b}' - \mathbf{a}'| \qquad\qquad \mathbf{b}' = -\mathbf{a}'$$

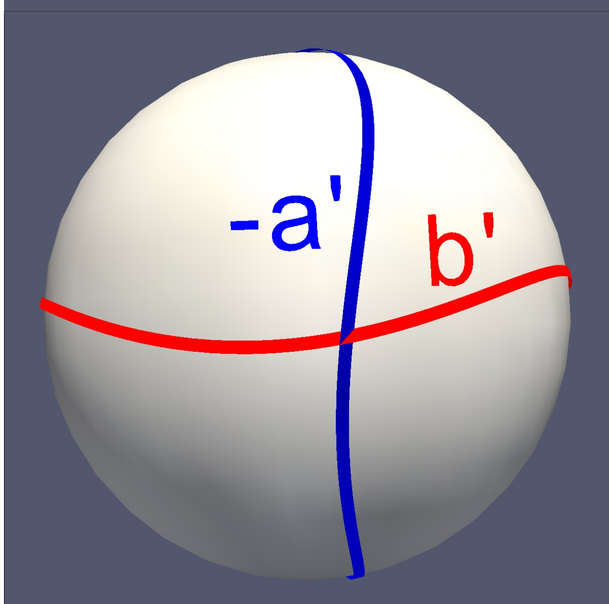
Cosmic String Cusps

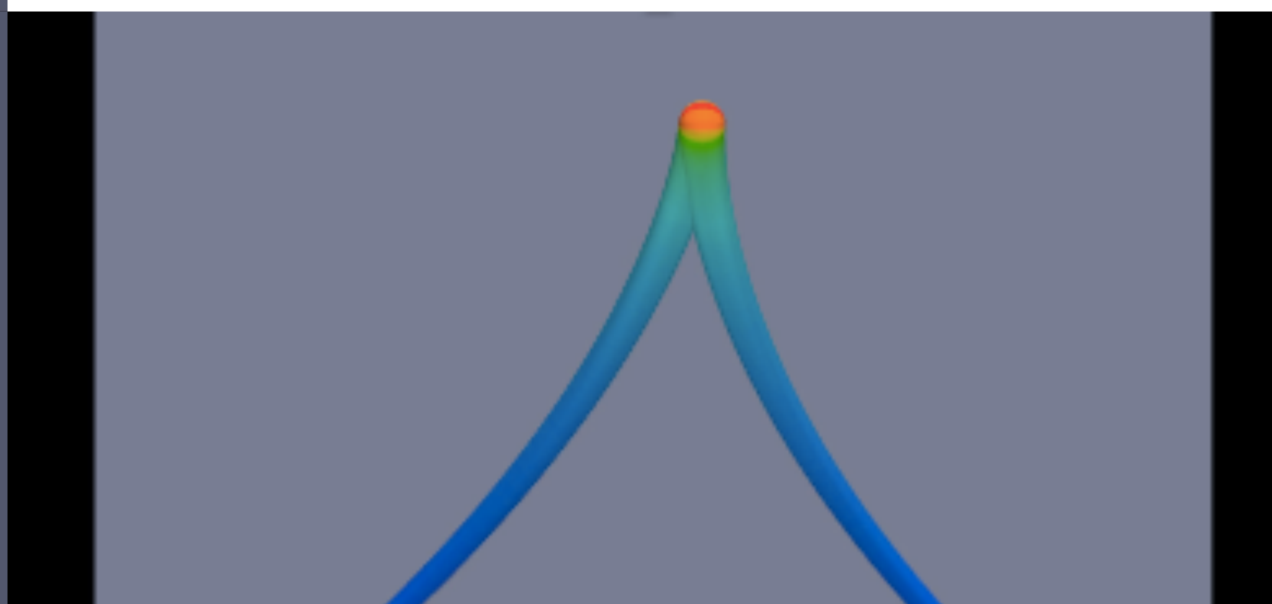


(Turok ‘84).

- Loops will typically have a cusp in each oscillation.
- The string doubles back on itself.

$$\mathbf{X'^2} + \dot{\mathbf{X}}^2 = 1$$





The importance of Loops

- Without any mechanism for energy loss strings would dominate the energy density of the universe.

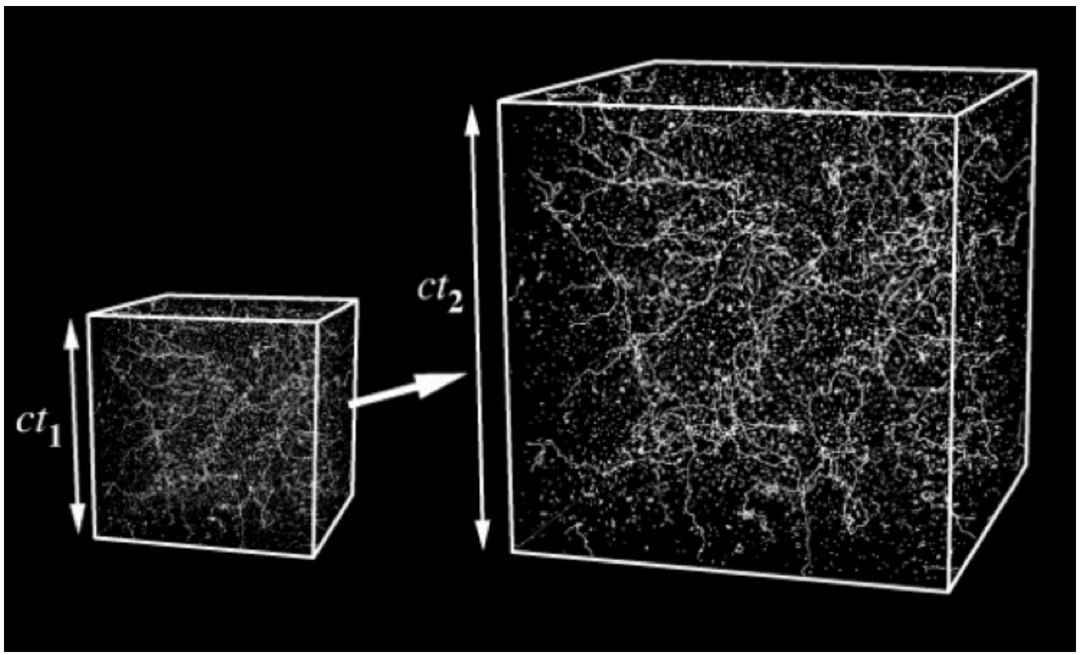
- Loops oscillate under their tension and lose energy by gravitational radiation.

- This mechanism allows strings to be subdominant part of the energy budget.

- No “monopole problem” for strings.

Cosmic String Networks

- As the string network evolves it reaches a scaling solution where the energy density of strings is a constant fraction of the energy density in the universe.

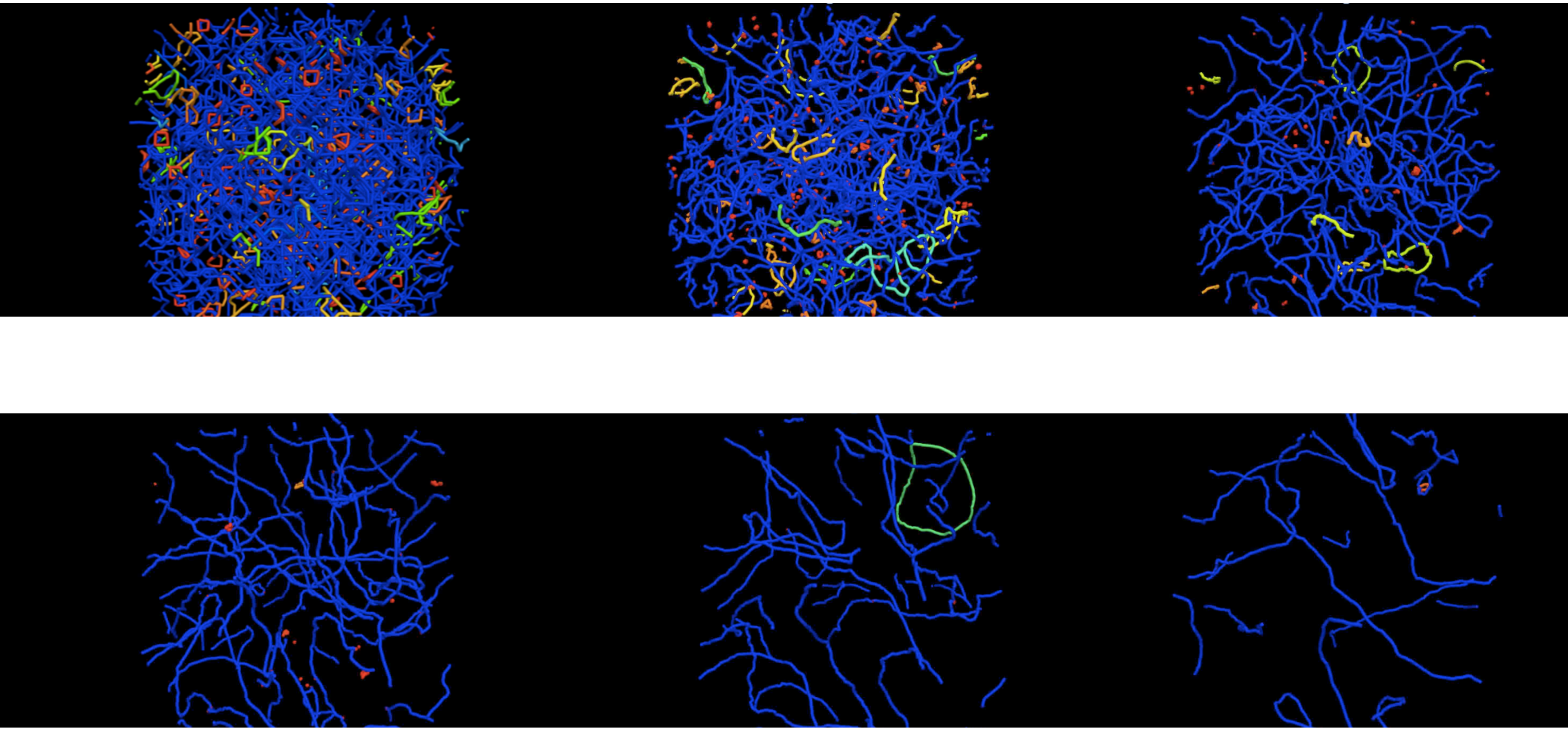


$$\frac{\rho_\infty}{\rho} = \text{constant}$$

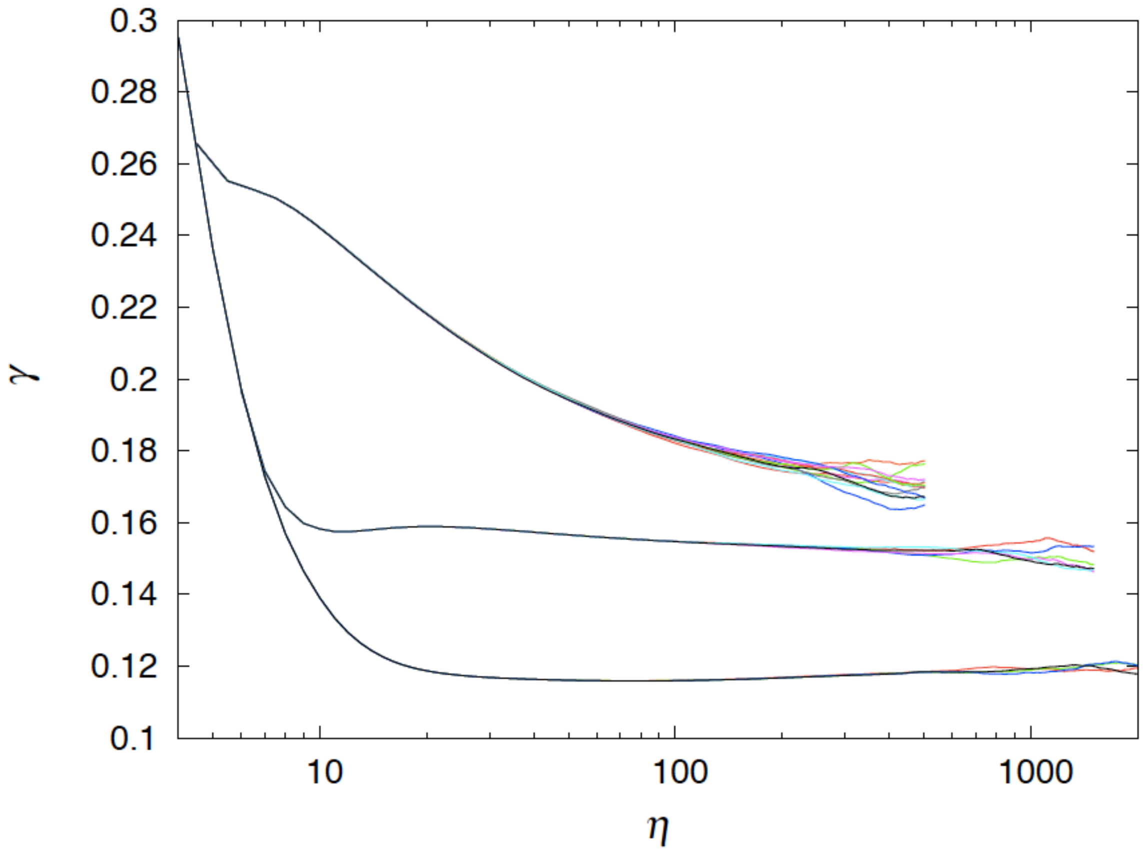
- All statistical properties scale with the horizon distance.

Nambu-Goto Cosmic String Networks Simulations

(B-P., Olum and Shlaer '12).



The Scaling of Cosmic String Networks



(B-P., Olum and Shlaer '12).

$$\gamma = \frac{1}{d_h}\sqrt{\frac{\mu}{\rho_\infty}}$$

All the properties of the strings scale with the horizon size.

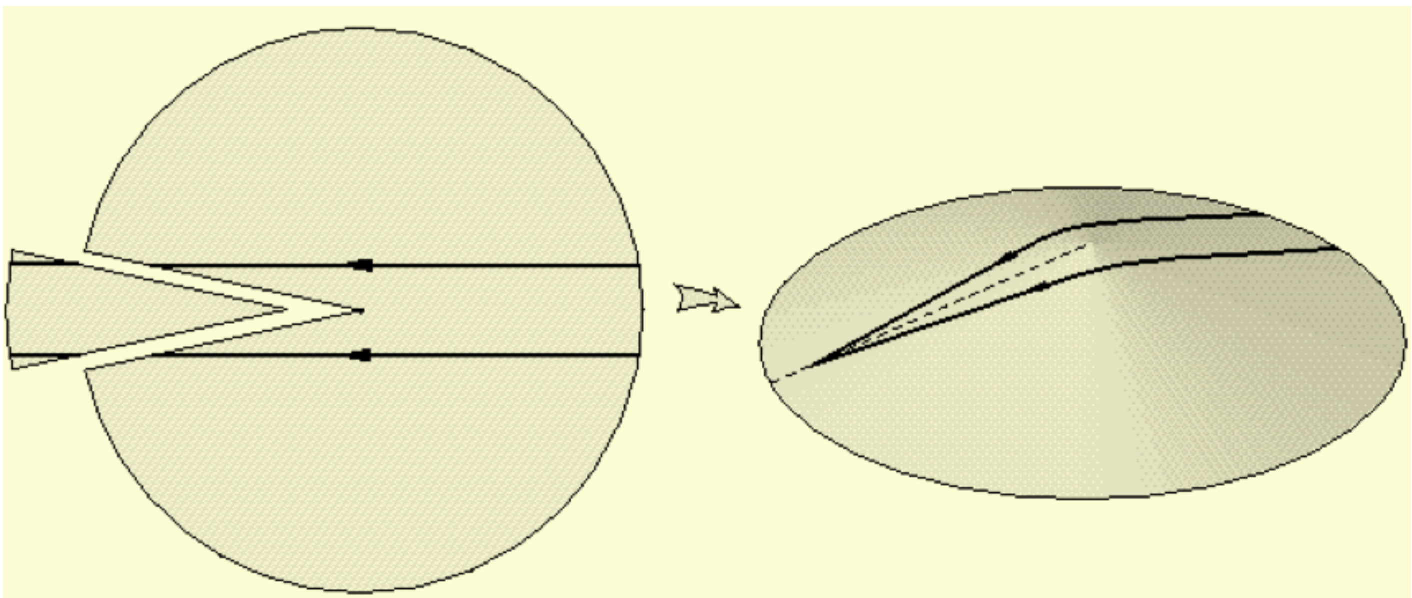
Observational Signatures

- Many different ideas:
  - Lensing.
  - Effects on the CMB.
  - Cosmic Rays.
  - Gravitational waves.
  - Several other ideas…

Spacetime around a string

(Vilenkin '81).

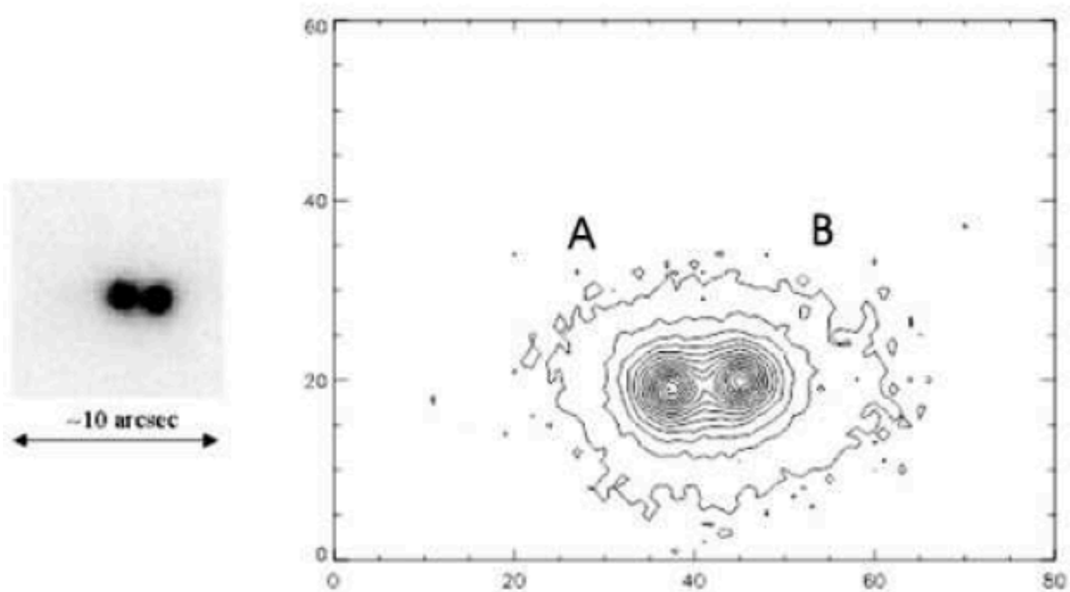
- The spacetime generated by a cosmic string is conical



$$\Delta = 8\pi G\mu$$

Lensing

- This spacetime will create exact copies of the same galaxy at a small angular distance.

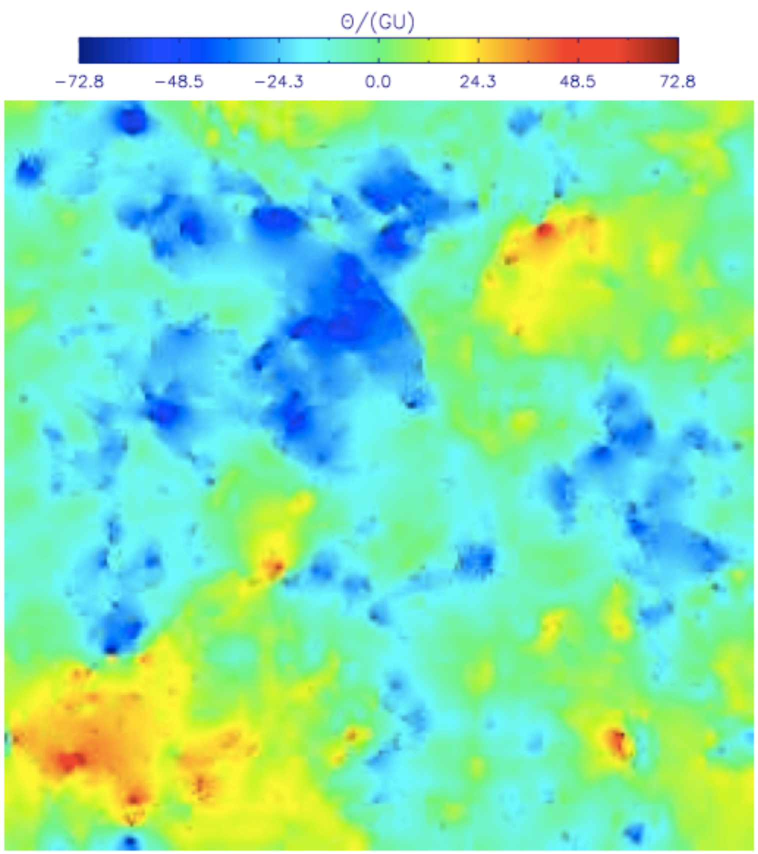


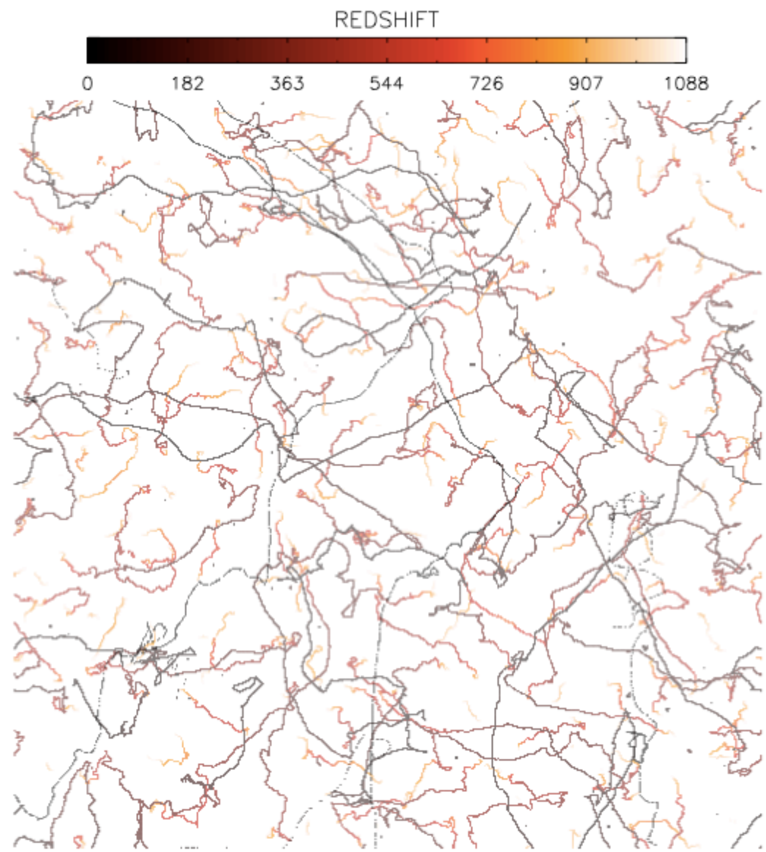
$$\delta\varphi \approx 8\pi G\mu \approx 5'' \left(\frac{G\mu}{10^{-6}}\right)$$

String signatures on the CMB

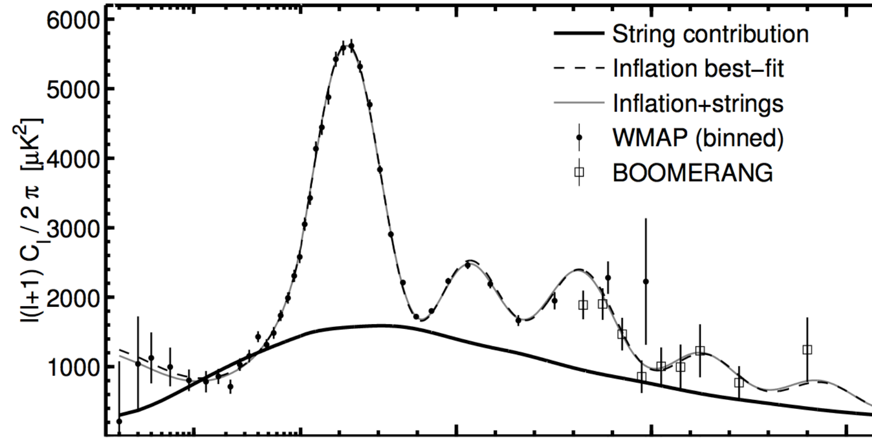
- We can see the effect on the temperature fluctuations by generating maps that include strings.

(Fraisse et al. ‘08).





String signatures on the CMB



- Another approach:
  - Calculate of the power spectrum using the network of strings as a source.

(Bevis et al. 06).

- Conclusions agree:

$$G\mu < 6 \times 10^{-7}$$

  - Only a small percent of the total power in temperature perturbations could be due to strings.

Gravitational Radiation by Loops

- The power of gravitational waves will affect the size of the loops:

$$\dot{M} \sim G(\dddot{Q})^2 \sim GM^2L^4w^6 \sim \Gamma G\mu^2$$

- The total power has been calculated with several sets of loops:

$$P \sim \Gamma G\mu^2 \qquad\qquad \Gamma \sim 50 - 100$$

- Loops will therefore shrink in size so the rest mass of the loop will be:

$$m(t) \sim m(t') - \Gamma G\mu^2(t - t')$$

$$L(t) \sim L(t') - \Gamma G\mu(t - t')$$

Gravitational Waves from the Network

- There are 2 different contributions to gravitational waves from a network of strings:

  - Stochastic background generated by all the modes in the loop.

    (Vilenkin '81, Hogan and Rees '84, Caldwell et al. '92, Siemens et al.; Battye et al., Sanidas et al., Binetruy et al.; Ringeval and Suyama; Kuroyanagi et al…).

  - Burst signals from individual cusps.

    (Damour & Vilenkin '01).

Stochastic background of Gravitational Waves

- The whole network of strings contributes to the stochastic background of GW.

$$\Omega_{gw}(\ln f) = \frac{8\pi G}{3H_o^2} f \int_0^{t_0} dt \left(\frac{a(t)}{a(t_0)}\right)^3 \int_0^{m_{max}} dm \; n(t,m) \left(\frac{dP}{df}\right)$$

$n(t,m)$ ⟵ It depends directly on the number of loops.

$\left(\frac{dP}{df}\right)$ ⟵ It also depends on the spectrum of gw emission by the surviving loops.

The number of cosmic string loops

(B-P., Olum and Shlaer '13).

- We have been able to obtain from the simulations the scaling distribution of loops.

- This allows us to calculate the loop distribution of sizes at any moment in the history of the universe:

$$\frac{n_r(t,l)}{a^3(t)} \approx \frac{0.18}{t^{3/2}(l+\Gamma\mu t)^{5/2}}$$

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$n(t,m)$ ⟵ It depends directly on the number of loops.

$\left(\frac{dP}{df}\right)$ ⟵ It also depends on the spectrum of gw emission by the surviving loops.

Gravitational Radiation by Loops

- Loops are periodic sources so they emit at specific frequencies

$$f_n = \frac{2n}{L}$$

- We have to determine the amount of radiation at each frequency and it total power

$$\Gamma = \sum_{n=1}^{\infty} P_n$$

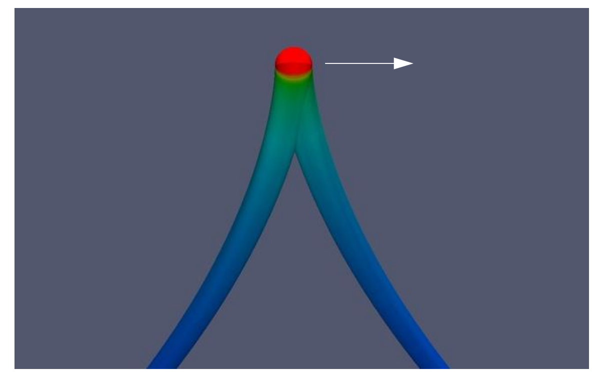
$$L(t) \sim L(t') - \Gamma G\mu(t - t')$$

Gravitational Radiation by Loops

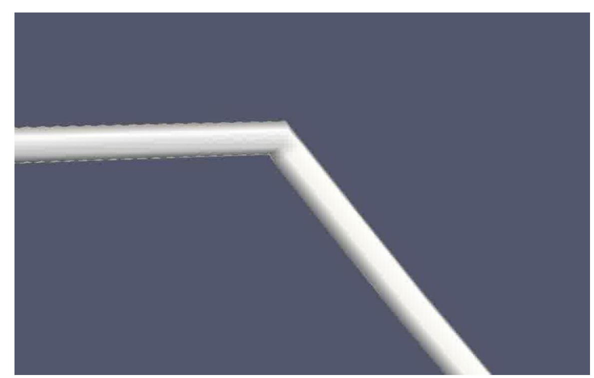
- The spectrum of radiation depends on the shape of the string.
- Different structures on the strings have different spectrum:

(Vachaspati and Vilenkin '85).

$$P_n^{cusps} \sim G\mu^2 n^{-4/3}$$



$$P_n^{kinks} \sim G\mu^2 n^{-5/3}$$



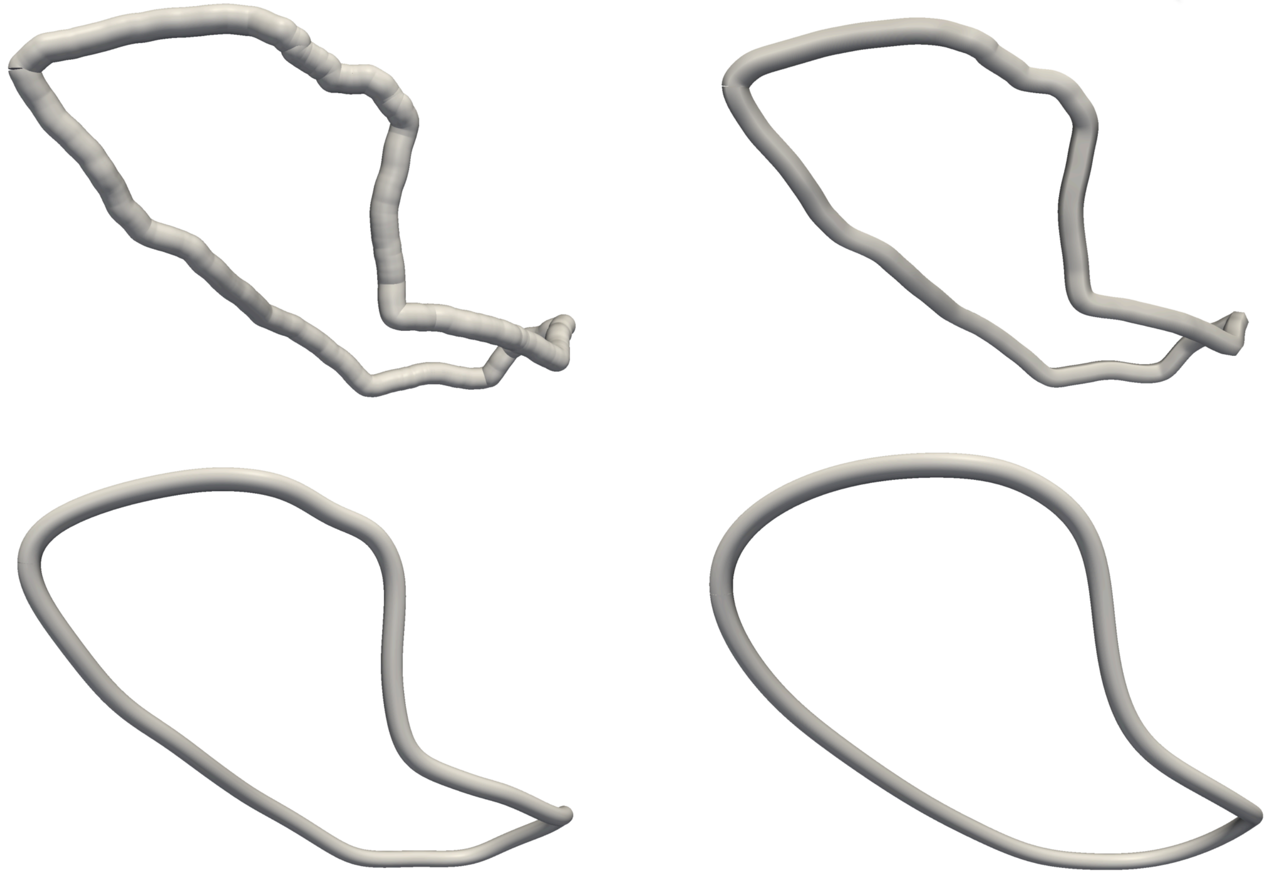
Loops from the Simulation

(B-P., Olum and Shlaer '12).

Loops obtained directly from the simulation have a lot of structure.
However, we need to consider smoothing by gravitational backreaction.

Smoothing the loops (Toy model)

(B-P., Olum '15).

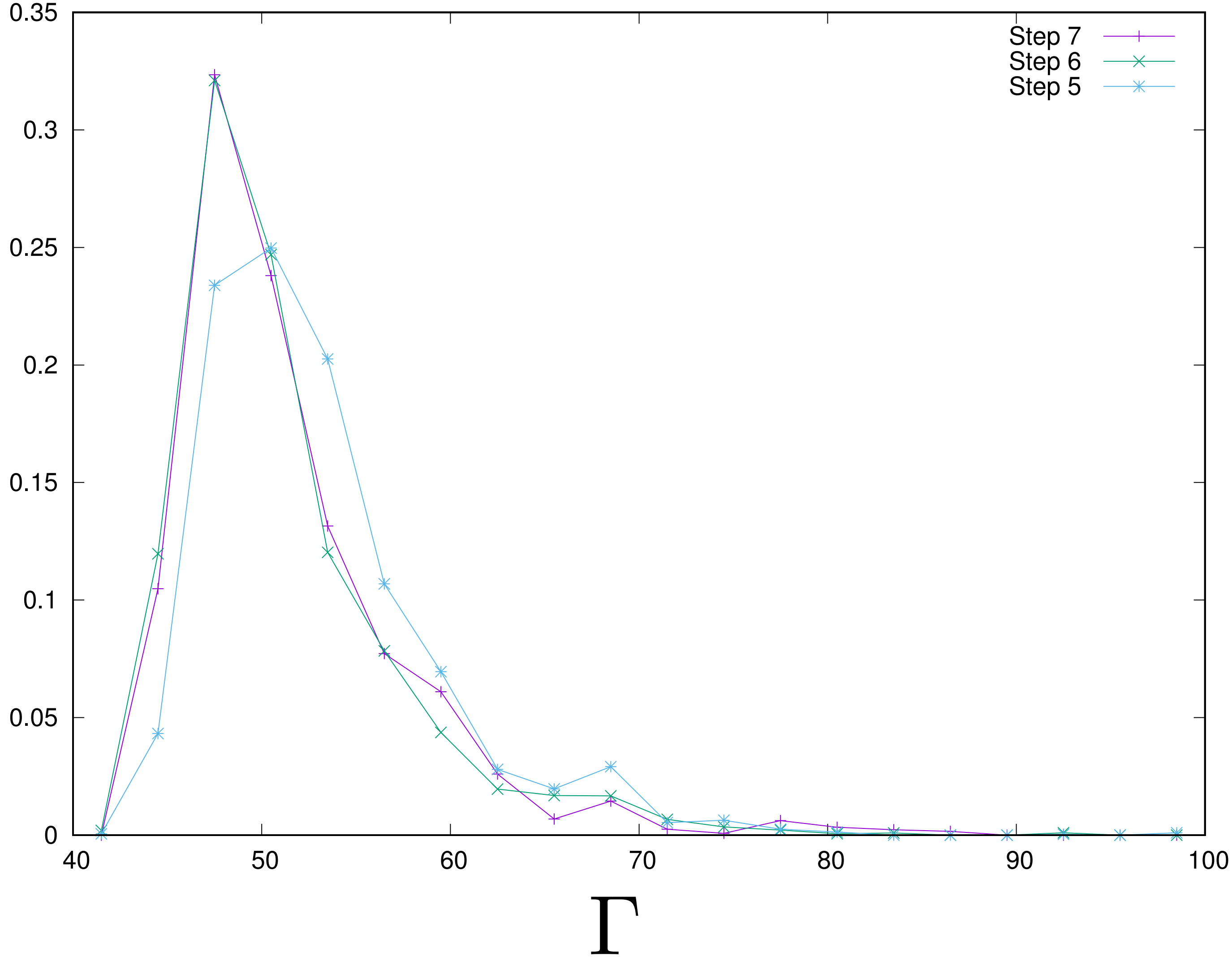


We need to do the actual gravitational smoothing by gravitational backreaction (Work in progress)

Gravitational Radiation by Loops

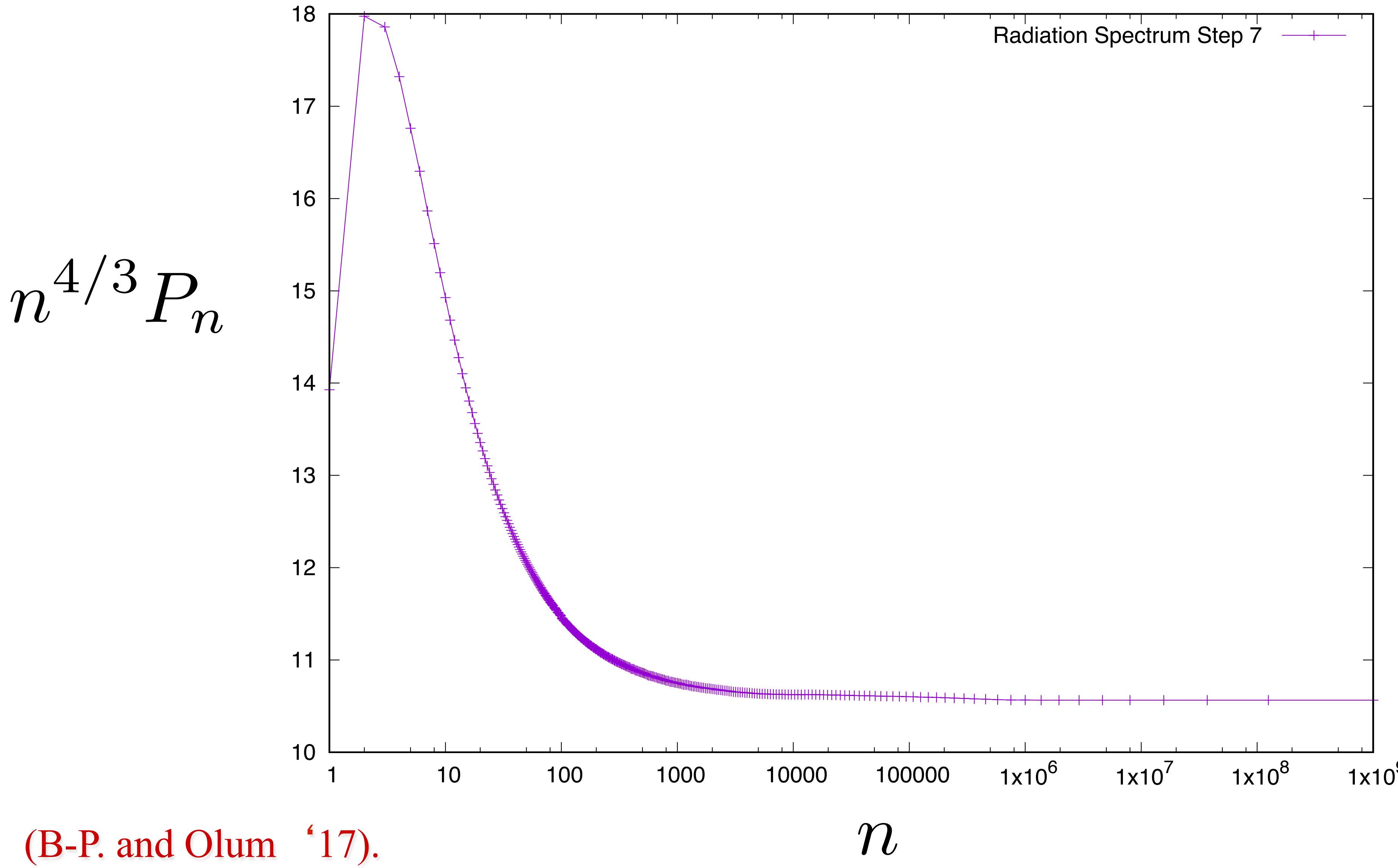
(B-P. and Olum '17).

$$L(t) \sim L(t') - \Gamma G\mu(t - t')$$



Gravitational Radiation by Loops

- Averaging over more than 1000 loops we get a spectrum of the form.



(B-P. and Olum '17).

Stochastic background of Gravitational Waves

- The whole network of strings contributes to the stochastic background of GW.

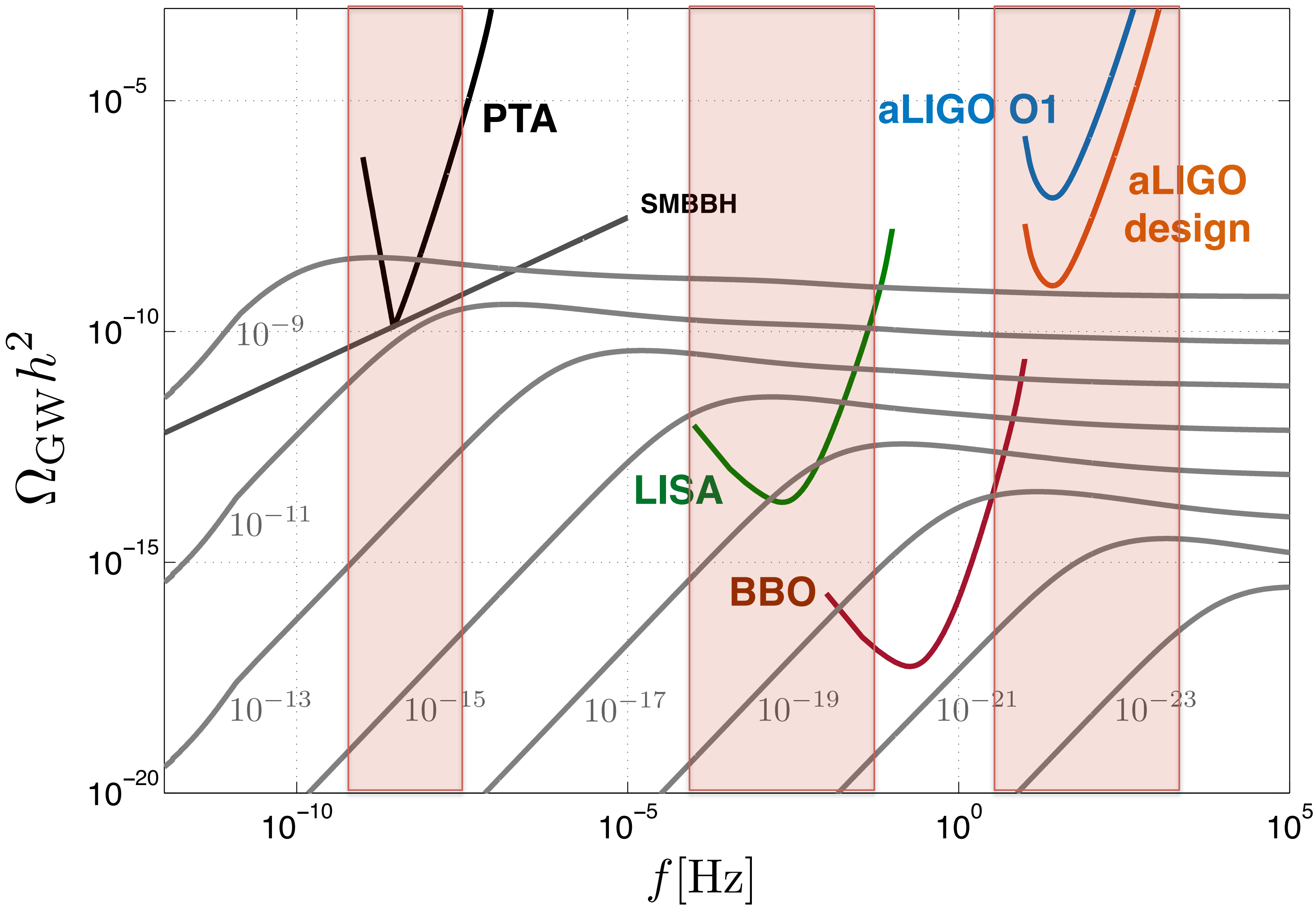
$$\Omega_{gw}(\ln f) = \frac{8\pi G}{3H_o^2} f \int_0^{t_0} dt \left(\frac{a(t)}{a(t_0)}\right)^3 \int_0^{m_{max}} dm\; n(t,m) \left(\frac{dP}{df}\right)$$

✔ $n(t,m)$ ⟵ It depends directly on the number of loops.

✔ $\left(\frac{dP}{df}\right)$ ⟵ It also depends on the spectrum of gw emission by the surviving loops.

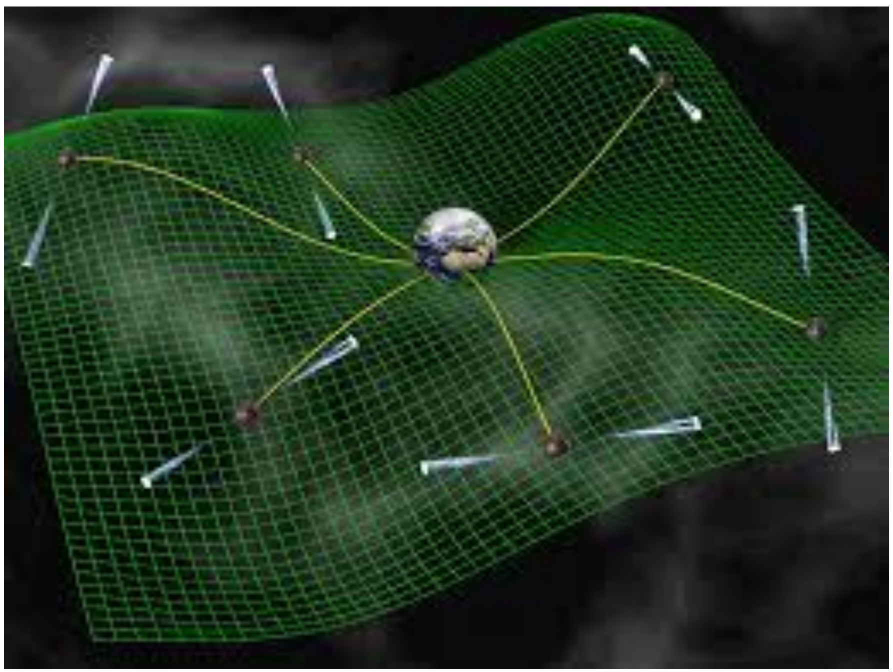
Implications for Future Observations

(B-P. , Olum and Siemens '17).



GW observations from Pulsar Timing Array

There are several PTA observatories that monitor the time of arrival of the pulses that come from many pulsars.



Gravitational waves can create a residual on the time of arrival of these pulses.

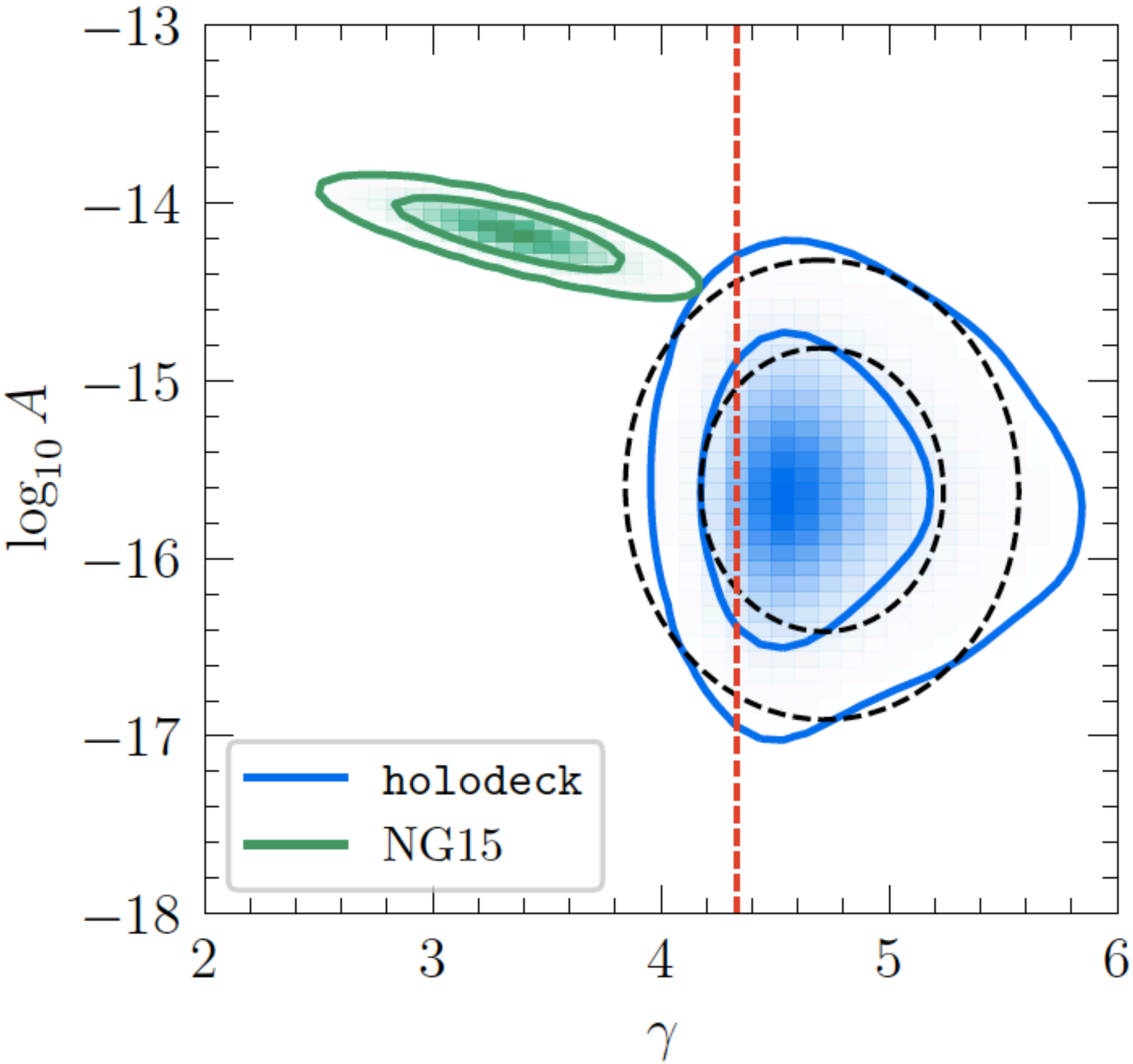
These observatories are sensitive to gravitational waves with a frequency of the order:

$$f_{\mathrm{PTA}} \approx 10^{-9}\mathrm{Hz}$$

NANOGrav 15 year data

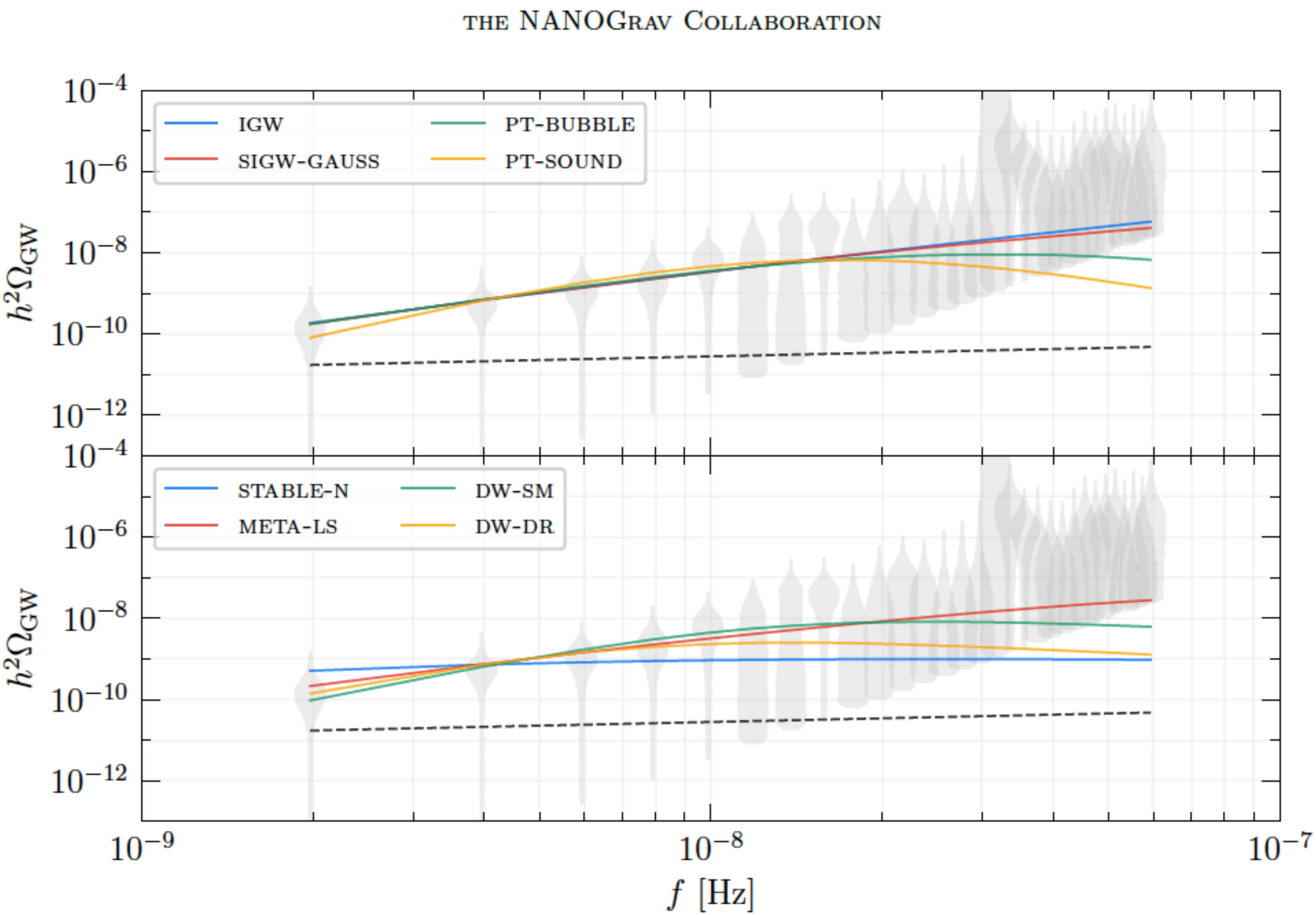
"The NANOGrav 15-year Data Set: Search for Signals from New Physics", A. Afzal et al. (2023)

$$\Omega_{\rm GW} = \frac{2\pi^2}{3H_0^2} A^2 f_{\rm yr}^2 \left(\frac{f}{f_{\rm yr}}\right)^{5-\gamma}$$



Is there room for new physics in this data?

NANOGrav 15 year data



“The NANOGrav 15-year Data Set: Search for Signals from New Physics”, A. Afzal et al. (2023)

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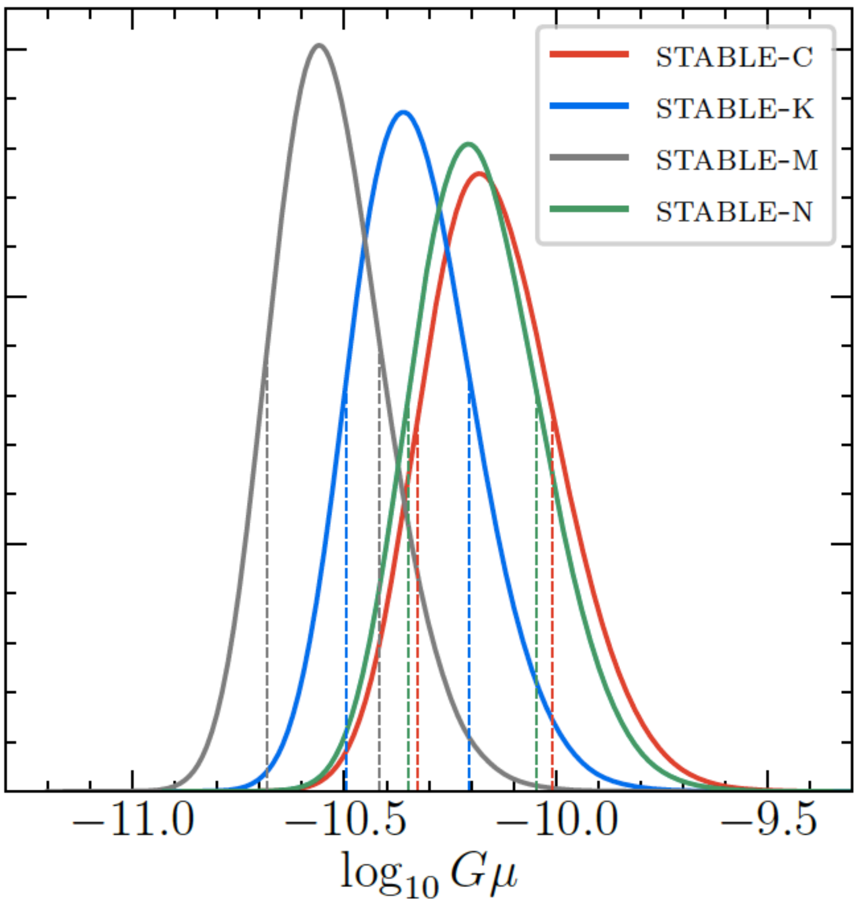


We can use this data to place upper bounds on the tension of the strings:

$$\log{(G\mu)}_{\text{STABLE-C}} < -9.67$$

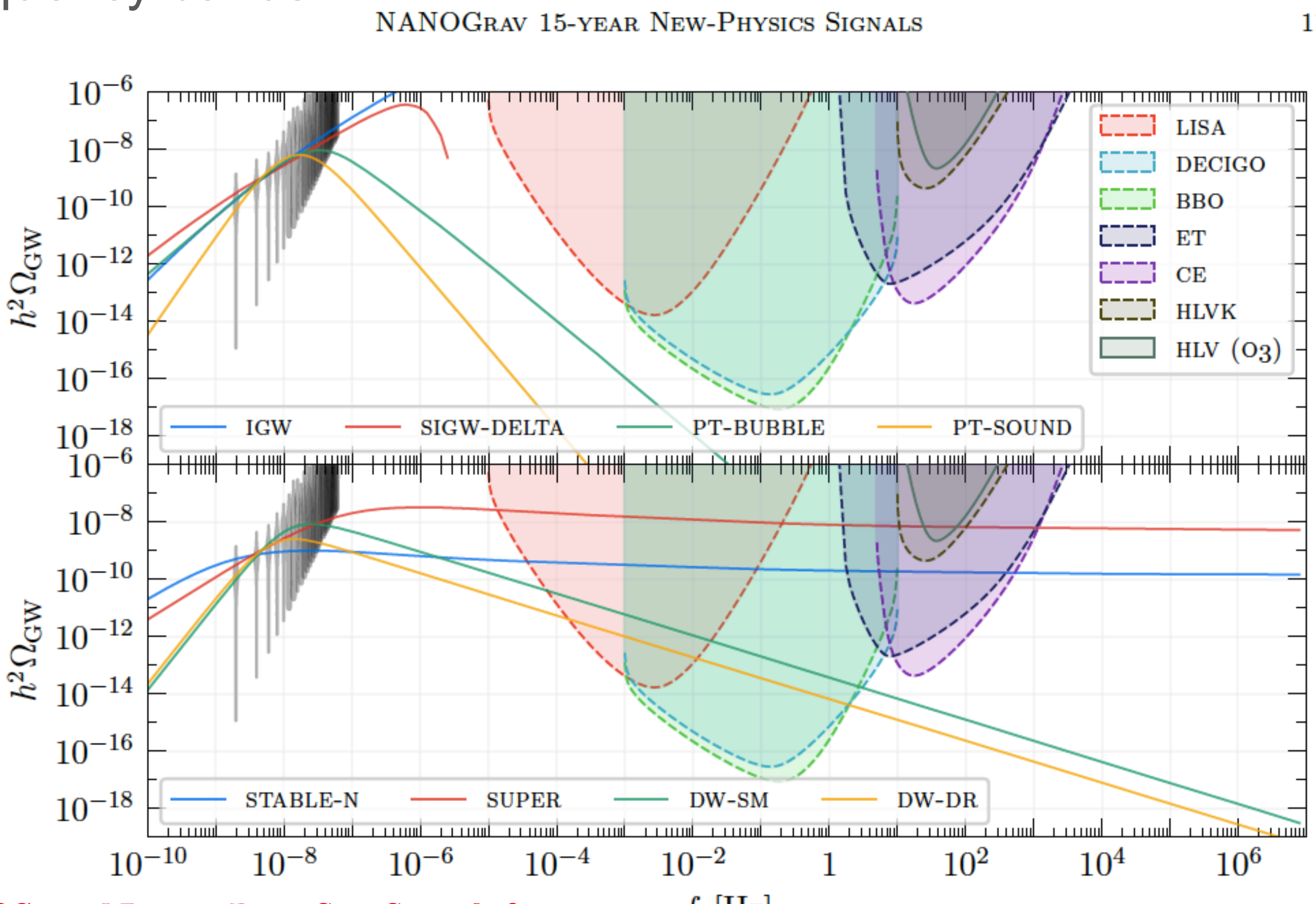
$$\log{(G\mu)}_{\text{STABLE-N}} < -9.71$$

$$\log{(G\mu)}_{\text{STABLE-K}} < -9.87$$

$$\log{(G\mu)}_{\text{STABLE-N}} < -10.10$$

NANOGrav 15 year data

This may have important consequences for possible observations in other frequency bands.

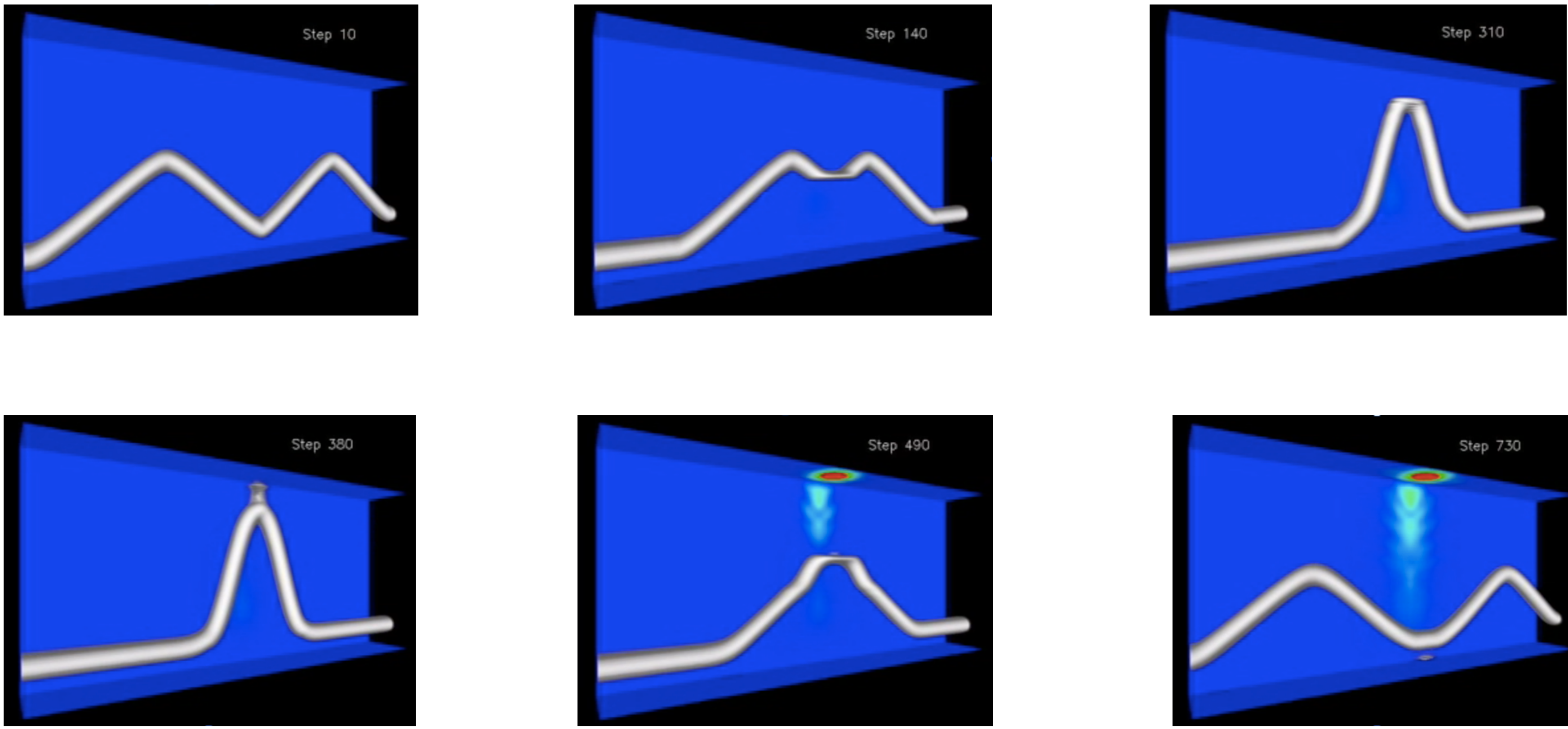


"The NANOGrav 15-year Data Set: Search for Signals from New Physics", A. Afzal et al. (2023)

Cosmic Rays

(B-P. & Olum '99).

Strings can radiate massive particles in regions of high curvature.

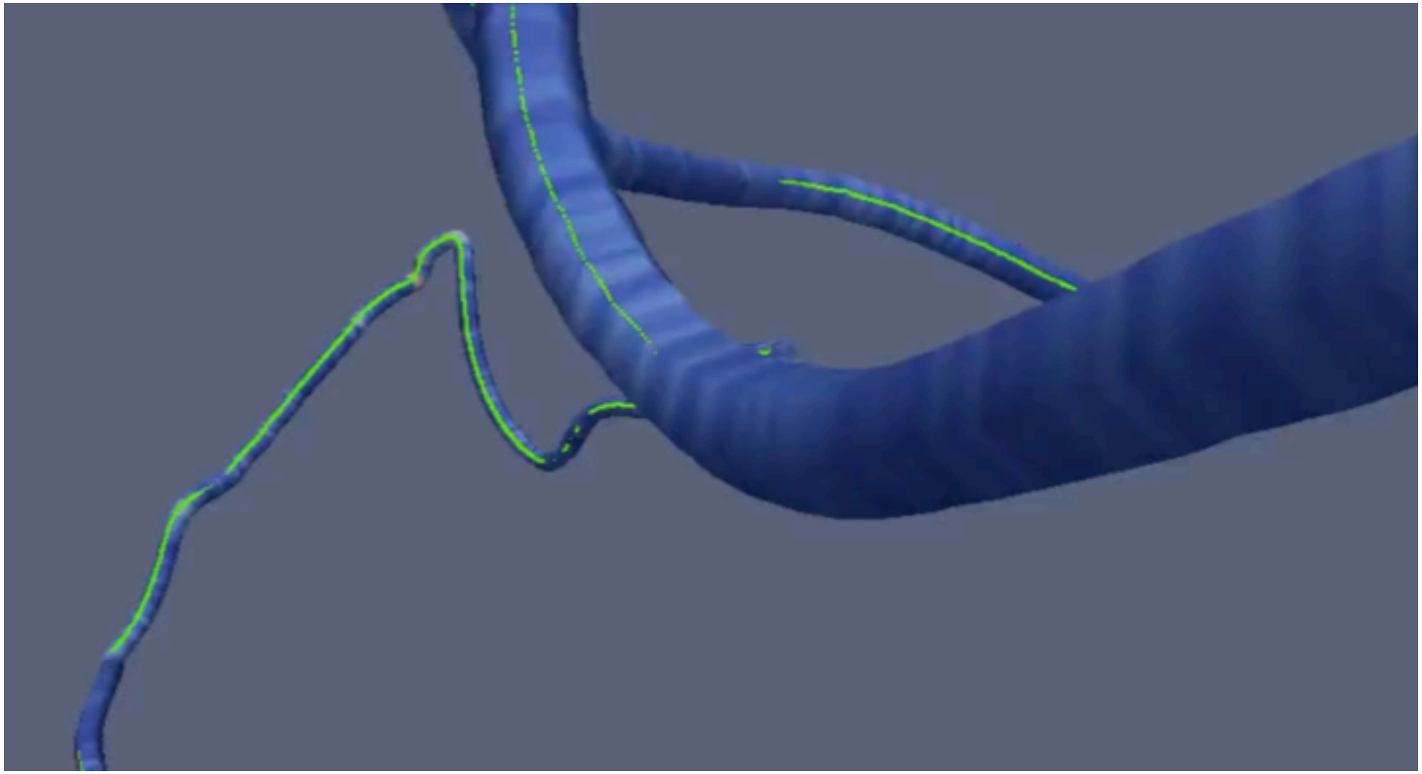


This radiation could be seen in form of cosmic rays. However, we do not expect this to be very significant.

Nambu-Goto dynamics and particle radiation

(BP, Jimenez-Aguilar, Lizarraga, Lopez-Eiguren, Olum, Urio and Urrestilla, ’23).

Strings from cosmological field theory simulations move exactly like Nambu-Goto predicts for most of their evolution.



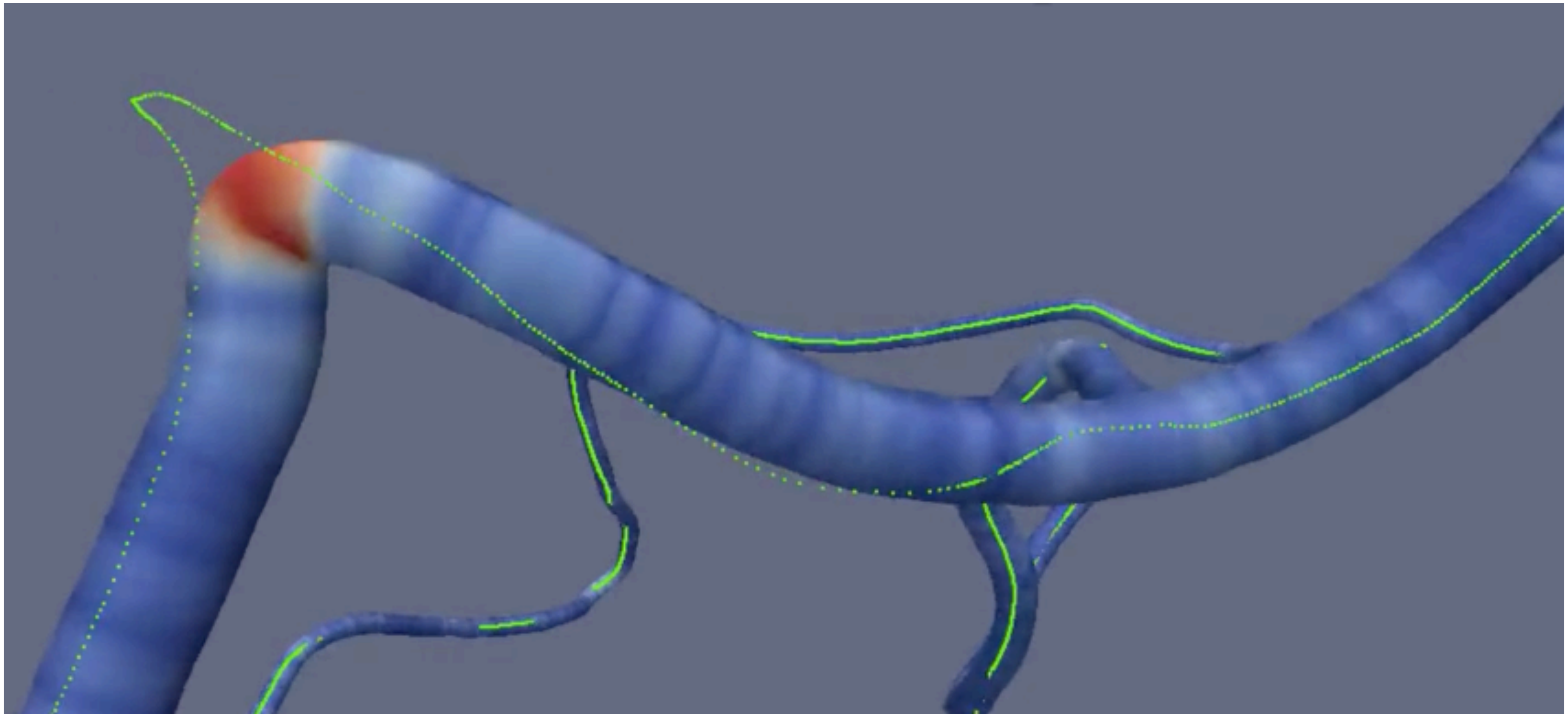
Green line: Nambu-Goto prediction

Blue tube: Field Theory simulation

Nambu-Goto dynamics and particle radiation

(BP, Jimenez-Aguilar, Lizarraga, Lopez-Eiguren, Olum, Urio and Urrestilla, '23).

Except in regions of high curvature where they can radiate particles.



More work in progress

Global Strings (Axionic strings)

- Global strings are coupled to a massless field thefore they have another decay channel apart from gravity.

- This is in fact much more efficient than the gravitational radiation.

- These massless fields should become massive and give rise to a cosmic abundance of axionic dark matter.

- The effective action for these strings should incorporate this coupling:

$$S = -\mu \int \sqrt{-\gamma} d^2\xi + \frac{1}{6} \int d^4x \sqrt{-g} H^2 + 2\pi\eta \int B_{\mu\nu} d\sigma^{\mu\nu}$$

Global Strings (Axionic strings)

- There are several field theory simulations of these axionic strings and the results are not in agreement yet.

Gorghetto et al. ‘18

Buschmann et al. ‘21

Hindmarsh et al.’20 & ‘21

- These networks also have some signature in gravitational waves. However their amplitude is expected to be lower than local strings.

- These strings would eventually dissapear by the formation of domain walls that bound them.

Summary Part II

- Topological defects are predicted in many extensions of the SM.
- They long lifetime could leave some imprint on the different cosmological observable we have access to. (CMB, Stochastic GW Background, Cosmic Rays, etc…)
- We can impose important constraints on some regions of the parameter space from current observations.
- Gravitational wave observatories have a great potential to discover new physics involving topological defects.
- Future observatories like LISA or ET could detect or constrained these scenarios.
- These bounds have an impact on high energy physics of the early universe.

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- “Aspects of Symmetry”, S. Coleman
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- “Topological Solitons” N. Manton nd P. Sutcliffe
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