Pole Properties of a Resonance: When to subtract partial-decay widths to obtain the pole width.

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References:

Z.-Q. Wang, X.-W. Kang, J.A. Oller, L. Zhang, Phys. Rev. D105, 074016 (2022)

V. Burkert[a], V. Crede[b], E. Klempt[c], K. V. Nikonov[c], J. A. Oller[f], J. R. Peláez[g], J. Ruiz de Elvira[1], A. Sarantsev[c], L. Tiator[d], U. Thoma[c], R. Workman[e] Phys. Lett. B 844, 138070 (2023)

Excited QCD 2024 Workshop
Benasque Science Center

Problem: I. Peculiar Riemann sheet II. $\left.\begin{array}{l} g_i \\ \Gamma_{\pi\pi} \end{array}\right\}$ are taken as physical couplings and width of the resonance

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Evidence that the $a_0(980)$ and $f_0(980)$ are not elementary particles

V. Baru [a,b], J. Haidenbauer [b], C. Hanhart [b], Yu. Kalashnikova [a], A. Kudryavtsev [a]

$$F_{K\bar K} = -\frac{1}{2k}\frac{\Gamma_K}{E - E_f + i\frac{\Gamma_K}{2} + i\frac{\Gamma_P}{2} + O(m^2E/\beta^2)}, \tag{17}$$

$\Gamma_K = g_{K\bar K}\sqrt{m_K E}$

$\Gamma_P = \Gamma_{\pi\pi}$ or $\Gamma_{\pi\eta}$

$f_0(980)$ PDG : $\Gamma_{f_0} \sim 40\text{-}70$ MeV

$K\bar K$ momentum (MeV)

Ref.	M_R	$\Gamma_{\pi\pi}$	$\bar g_{K\bar K}$	E_f	r_e	a	k_1
[22]	969.8	196 50	2.51	−151.5	−0.63	$1.15 - i0.74$	$-58 \oplus i107$
[23]	975	149 51	1.51	−84.3	−1.05	$0.99 - i0.88$	$-65 \oplus i97$
[21]	973	253 56	2.84	−154	−0.56	$1.09 - i0.89$	$-69 \oplus i100$
[24]	996	128.8 11	1.31	+4.6	−1.22	$-0.14 - i1.99$	$-84 \oplus i17$

$$\frac{k_1^2}{m_K} \to \Gamma_{pole} = -2\,\mathrm{Im}\,E_{pole}$$

+ sign

Γ_{pole}(MeV) = 50, 51, 56, 12 MeV << $\Gamma_{\pi\pi}$

Pole in a noncontiguous Riemann sheet: Unexpected Relation between Γ_{pole} and Γ_i

The pole position of the $f_0(980)$ from the dispersive analysis

Garcia-Martin, Kaminski, Peláez, Ruiz de Elvira, Phys. Rev. Lett. 107, 072001 (2011)

- Forward dispersion relations
- Roy and GKPY equations ($\sqrt{s} \leq 1100$ MeV)

- $g^2 = \lim\limits_{s \to s_R} (s - s_R)\, T^{LI}_{\pi\pi\to\pi\pi}(s)$ $K\bar{K}$ threshold is taken at 992 MeV

$\sqrt{s_p} = M_{pole} - i\,\Gamma_{pole} = (996 \pm 7) - i\,(25 \pm^{10}_{6})$ MeV, $|g| = 2.3 \pm 0.2$ GeV.

1. $\Gamma_{\pi\pi} = \dfrac{|g|^2\, q_\pi}{8\pi M_{f_0}^2} = 100^{+20}_{-17}$ MeV $\gg$ $\Gamma_{pole} = 50^{+20}_{-12}$ MeV

How can a partial decay width be larger than the total width from the pole??

Change of Riemann sheet:

Let us consider the Lippmann-Schwinger equation

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$$

$$\mathcal{T}(E) = \mathcal{V} + \mathcal{V}\,\frac{1}{\mathcal{H}_0 - E}\,\mathcal{T}(E) \longrightarrow \mathcal{T}(E)^\dagger = \mathcal{T}(E^*)$$

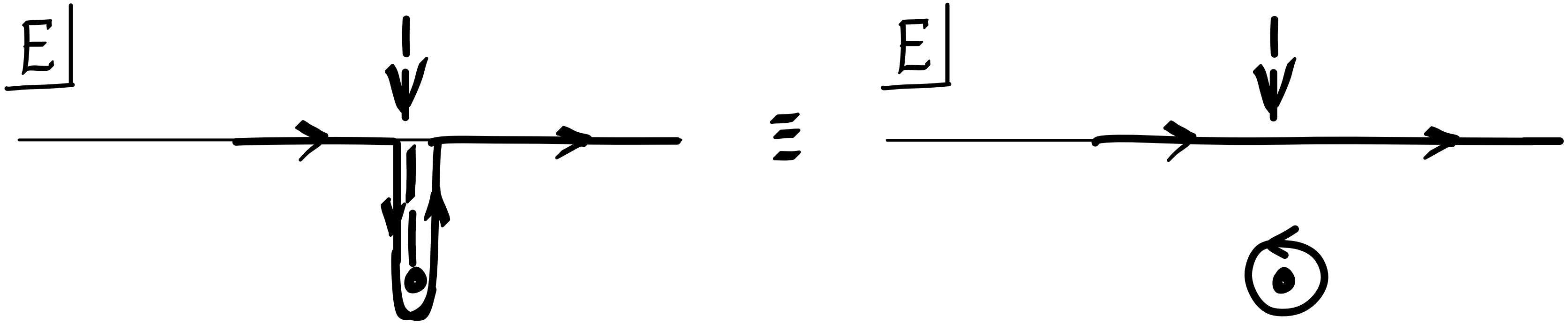
When crossing the real E-axis above the threshold the unitarity cut is generated

$$\mathrm{Im}\,\mathcal{T}(E+i\varepsilon) = \frac{p}{8\pi\sqrt{s}}\,\mathcal{T}(E+i\varepsilon)\,\mathcal{T}(E+i\varepsilon)^\dagger \qquad E > \text{threshold}$$

$$= \frac{1}{2i}\Big(\mathcal{T}(E+i\varepsilon) - \mathcal{T}(E-i\varepsilon)\Big) = \frac{p}{8\pi\sqrt{s}}\,\mathcal{T}(E+i\varepsilon)\,\mathcal{T}(E-i\varepsilon)$$

Discontinuity

Crossing smoothly the real axis above threshold → 2nd RS.



1st Riemann sheet

2nd Riemann sheet

$$\frac{1}{H_0 - E + i\epsilon} \to \frac{1}{H_0 - E - i\epsilon} + 2\pi i\,\delta(W - E)$$

$$W = \sqrt{m_1^2 + \vec{p}^2} + \sqrt{m_2^2 + \vec{p}^2}$$

T-matrix in the 2nd RS: $T^{II}(k,k';E)$

$$T^{II}(k,k';E) = V(k,k') + \int_{m_1+m_2}^{\infty} \frac{d\hat{p}}{(2\pi)^3}\frac{|\vec{p}|\,dW'}{4W} V(k,p)\,\frac{1}{W' - \sqrt{s}}\,T^{II}(p,k';E)$$

$$+ \frac{i|\vec{p}_{on}|}{4\pi\sqrt{s}} \int \frac{d\hat{p}}{4\pi}\, V(k, |\vec{p}_{on}|\hat{p})\, T^{II}(|\vec{p}_{on}|\hat{p}, k'; E) \longleftarrow \text{Added term}$$

$$|\vec{p}_{cm}| = {}^{\mathrm{II}}\sqrt{\frac{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}{4s}}$$

This is the right expression for momentum with only unitarity cut

$${}^{\mathrm{I}}\sqrt{z}\,,\quad \arg z \in [0, 2\pi)\,.$$

$${}^{\mathrm{II}}\sqrt{z}\,,\quad \arg z \in [2\pi, 4\pi)$$

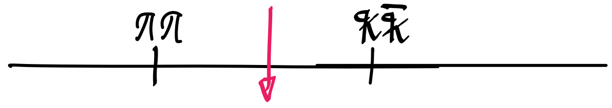
$${}^{\mathrm{I}}\sqrt{z} = -\,{}^{\mathrm{II}}\sqrt{z}$$

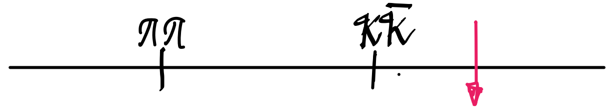
This is why a sign convention is used to designate the Riemann sheets

$(\pi\pi, K\bar{K})$

1st or physical RS : $(+\,+)$

Contiguous Riemann sheets $\left\{\begin{array}{l}\text{2nd RS : } (-\,+) \\ \text{3rd RS : } (-\,-)\end{array}\right.$





4th RS : $(+\,-)$

Flatté parameterization

Z.-Q. Wang, X.-W. Kang, J.A. Oller, Lu Zhang, Phys. Rev. D 105, 074016 (2022)

Let us illustrate the process with a Flatté parameterization   Flatté, Phys. Lett. B63, 224 (1976)

$$\text{Near } K\bar{K} \text{ threshold} \begin{cases} \pi\pi - K\bar{K} \text{ scattering} \text{ — } f_0(980) \\ \pi\eta - K\bar{K} \text{ scattering} \text{ — } a_0(980) \end{cases}$$

One could proceed analogously e.g. for energy-dependent Breit-Wigner's

$$E = \sqrt{s} - 2m_K$$

$$t_{ij}(E) = \frac{g_i g_j}{D(E)} = \frac{g_i g_j}{E - E_f + i\frac{\tilde{\Gamma}_1}{2} + \frac{i}{2} g_2 \sqrt{m_K E}}$$

$D(E)$ = Dressed propagator of the resonance.

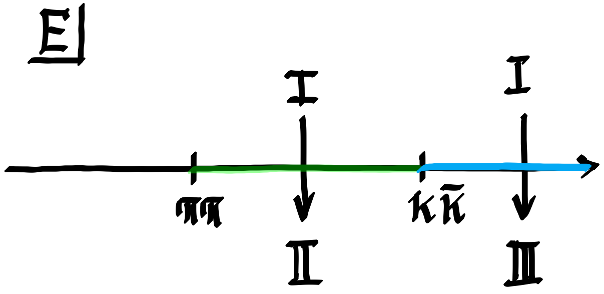
E_f : bare resonance mass

g_i : bare couplings squared

$$\tilde{\Gamma}_1 = \frac{\sqrt{M^2/4 - m_1^2}}{8\pi M^2} g_1$$

bare width to channel 1 ($\pi\pi$)

Kaon momentum [Nonrelativistic kinematics] : $p_2 = \sqrt{m_K E}$



1st RS: Im E > 0 (+ +)

$$t_{ij} = \frac{g_i g_j}{E - E_f + \frac{i}{2}\tilde{\Gamma}_1 + \frac{i}{2} g_2 \, {}^{I}\!\sqrt{m_K E}}$$

2nd RS: Im E < 0 (− +)

$$t_{ij} = \frac{g_i g_j}{E - E_f + \frac{i}{2}\tilde{\Gamma}_1 + \frac{i}{2} g_2 \, {}^{I}\!\sqrt{m_K E}}$$

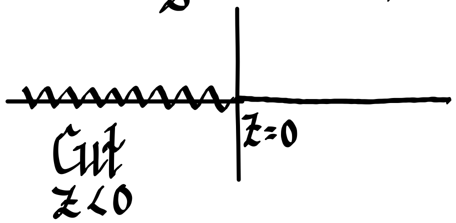
3rd RS: Im E < 0 (− −)

$$t_{ij} = \frac{g_i g_j}{E - E_f + \frac{i}{2}\tilde{\Gamma}_1 - \frac{i}{2} g_2 \, {}^{I}\!\sqrt{m_K E}} = \frac{g_i g_j}{E - E_f + \frac{i}{2}\tilde{\Gamma}_1 + \frac{i}{2} g_2 \underbrace{{}^{II}\!\sqrt{m_K E}}_{p_2^{II}}}$$

$\tilde{\Gamma}_1$ is fixed : Energy-independent width

Fortran, Mathematica convention:

$\sqrt{z}$, $\arg z \in (-\pi, \pi)$



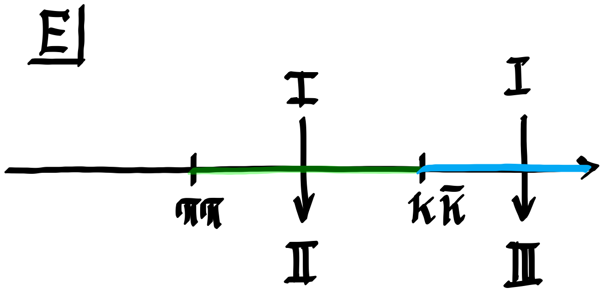
Im E > 0 ${}^{I}\!\sqrt{E}$

Im E < 0 ${}^{II}\!\sqrt{E}$

Smooth transition above the KK̄ threshold

$\mathrm{Re}\sqrt{E \pm i\varepsilon} > 0$, $E > 0$

Kaon momentum [Nonrelativistic kinematics] : $p_2 = \sqrt{m_K E}$



1st RS: $\operatorname{Im} E > 0$ (+ +)

$$t_{ij} = \frac{g_i g_j}{E - E_f + \frac{i}{2}\tilde{\Gamma}_1 + \frac{i}{2} g_2 \, {}^{I}\!\sqrt{m_K E}}$$

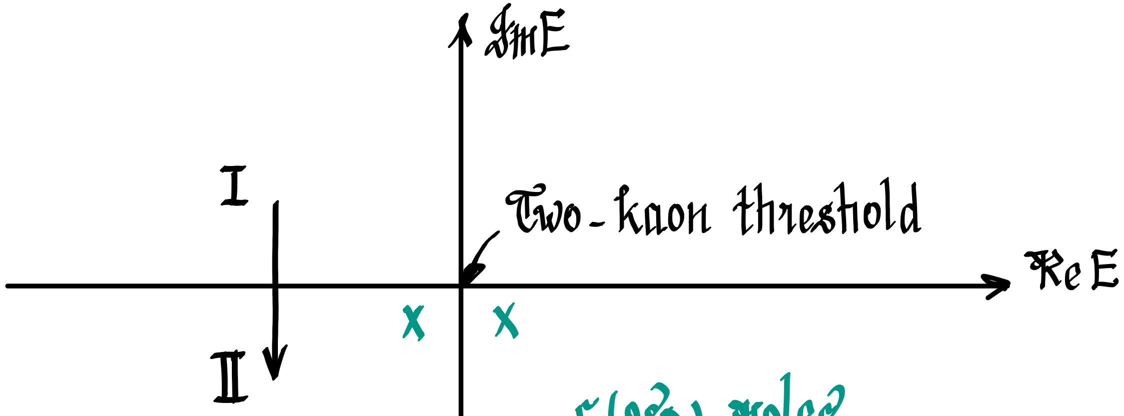
2nd RS: $\operatorname{Im} E < 0$ (− +)

$$t_{ij} = \frac{g_i g_j}{E - E_f + \frac{i}{2}\tilde{\Gamma}_1 + \frac{i}{2} g_2 \, {}^{I}\!\sqrt{m_K E}} = \frac{g_i g_j}{E - E_f + \frac{i}{2}\tilde{\Gamma}_1 - \frac{i}{2} g_2 \, {}^{II}p_2}$$

Fortran, Mathematica convention

3rd RS: $\operatorname{Im} E < 0$ (− −)

$$t_{ij} = \frac{g_i g_j}{E - E_f + \frac{i}{2}\tilde{\Gamma}_1 - \frac{i}{2} g_2 \, {}^{I}\!\sqrt{m_K E}} = \frac{g_i g_j}{E - E_f + \frac{i}{2}\tilde{\Gamma}_1 + \frac{i}{2} g_2 \, {}^{II}p_2}$$



Analytic continuation

$f_0(980)$ poles → Why does it matter?

in the $\underbrace{\text{2nd RS}}$ → $\Gamma^{II}_{pole} = \Gamma_1 - \Gamma_2$

3rd RS → $\Gamma^{III}_{pole} = \Gamma_1 + \Gamma_2$

Simple Breit-Wigner Model for two-channel scattering

$m_1 \equiv m_1^a = m_2^a \ll m_2 \equiv m_1^b = m_2^b$ Mimicking $\pi\pi - K\bar{K}$

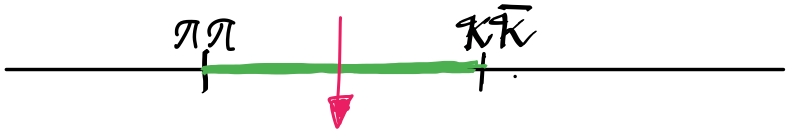
$$t_{ij}^{I}(s) = \frac{g_i\, g_j}{M_B^2 - s - i\left(g_1^2\rho_1(s) + g_2^2\rho_2(s)\right)}$$

$$\rho_i(s) = \frac{\sqrt{1 - 4m_i^2/s}}{16\pi}$$ phase space

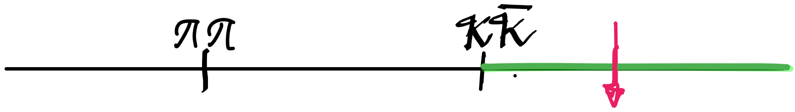
M_B is the Breit-Wigner or bare resonance mass

+, - : Fortran, Mathematica convention

$$t_{ij}^{II}(s) = \frac{g_i\, g_j}{M_B^2 - s - i\left(\underset{+}{-}g_1^2\rho_1(s) \underset{-}{+} g_2^2\rho_2(s)\right)}$$



$$t_{ij}^{III}(s) = \frac{g_i\, g_j}{M_B^2 - s - i\left(\underset{+}{-}g_1^2\rho_1(s) \underset{+}{-} g_2^2\rho_2(s)\right)}$$



The $f_0(980)$ pole position by

García-Martín, Kaminski, Peláez, Ruiz de Elvira, Phys. Rev. Lett. 107, 072001 (2011)

at $\sqrt{s_{pole}} = (996 \pm 7) - i\left(25^{+10}_{-6}\right)$ MeV lies in the 2nd Riemann sheet

Corolaries:

① $\Gamma_{pole} = \Gamma_{\pi\pi} - \Gamma_{K\bar{K}} > 0 \rightarrow \Gamma_{\pi\pi} > \Gamma_{K\bar{K}}$ Upper bound for $\Gamma_{K\bar{K}}$.

② $\Gamma_{\pi\pi} = \Gamma_{pole} + \Gamma_{K\bar{K}} \rightarrow \Gamma_{\pi\pi} > \Gamma_{pole}$ Lower bound for $\Gamma_{\pi\pi}$.

Numerics:

$$\Gamma_{\pi\pi} = \frac{|g|^2}{8\pi M^2} = 100^{+20}_{-17} \text{ MeV}$$ because $|g| = 2.3 \pm 0.2$ GeV.

$$\Gamma_{K\bar{K}} = \Gamma_{\pi\pi} - \Gamma_{pole} = 50^{+26}_{-21} \text{ MeV}$$ by the knowledge of $|g|$ and s_{pole}.

$$r_{K\bar{K}/\pi\pi} \equiv \frac{\Gamma_{K\bar{K}}}{\Gamma_{\pi\pi}} = 0.49 \pm 0.11$$

Garcia-Martin, Kaminski, Peláez, Ruiz de Elvira, Phys. Rev. Lett. 107, 072001 (2011)

$\sqrt{s_{pole}} = (996 \pm 7) - i\left(25^{+10}_{-6}\right)$ MeV lies in the 2nd Riemann sheet

Definitions:

$$\Gamma_{total} = \Gamma_{\pi\pi} + \Gamma_{K\bar{K}} = 151^{+44}_{-37} \text{ MeV}$$

$$\left.\begin{aligned} BR_{\pi\pi} &\equiv \frac{\Gamma_{\pi\pi}}{\Gamma_{total}} = 0.67 \pm 0.07 \\ BR_{K\bar{K}} &\equiv \frac{\Gamma_{K\bar{K}}}{\Gamma_{total}} = 0.33 \pm 0.07 \end{aligned}\right\}$$

Pole position for $f_0(980)$ from Z.-H. Guo, J.A.O, J. Ruiz de Elvira, Phys. Rev. D 86, 054006 (2012)

Unitarization of one-loop
U(3) Chiral Pert. Theory.
QCD spectral sum rules,
semilocal duality, large N_c QCD.

$$\sqrt{s_p} = (978^{+7}_{-11}) - i(29^{+9}_{-11})\ \text{MeV}$$

$$|g_{\pi\pi}| = 2.0^{+0.2}_{-0.3}\ \text{GeV}$$

$$|g_{K\bar K}| = 4.7 + 0\,8\ \text{GeV}$$

Direct application:

$$\Gamma_{\pi\pi} = \frac{p_\pi |g_{\pi\pi}|^2}{8\pi M_K^2} = 63 \pm 17\ \text{MeV}$$

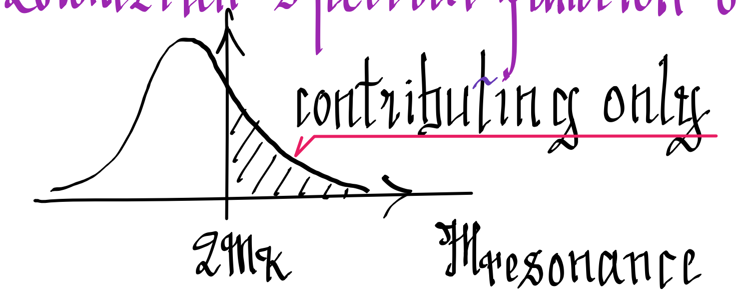
$$\Gamma_{K\bar K} = \Gamma_{\pi\pi} - \Gamma_{\text{pole}} = 5 \pm 27\ \text{MeV}$$

[0 - 32 MeV]

$$r_{K\bar K/\pi\pi} = 0.08 \pm 0.43$$

Physical insight test:

Lorentzian spectral function of a resonance



$$\Gamma_{K\bar K} = \frac{|g_{K\bar K}|^2}{8\pi} \int_{2M_K}^{M+n\frac{\Gamma}{2}} dW\, \frac{p_2(W)}{W^2}\, \frac{\Gamma_{\text{pole}}/2\pi}{(M-W)^2 + \Gamma_{\text{pole}}^2/4}$$

$n = 1 \quad \Gamma_{K\bar K} = 5.5 \pm 3.9$ MeV

$n = 2 \quad \Gamma_{K\bar K} = 15.5 \pm 7.1$ MeV

For more examples see

Z.-Q. Wang, X.W. Kang, J.A. Oller, Lu Zhang, Phys. Rev. D 105, 074016 (2022)

$\Gamma_{pole} = \Gamma_1 - \Gamma_2$ A new result generally applicable
to a resonance lying nearby to a higher-mass threshold

It was first deduced in
Z.-Q. Wang, X.W. Kang, J.A. Oller, Lu Zhang, Phys. Rev. D 105, 074016 (2022)
there applied also to the $a_0(980)$

It is worth looking for other resonances where to apply it

Dressing or renormalization of bare parameters

Z.-Q. Wang, X.W. Kang, J.A. Oller, Lu Zhang, Phys. Rev. D 105, 074016 (2022)

Let us illustrate the process with a Flatté parameterization Flatté, Phys. Lett. B 63, 224 (1976)

One could proceed analogously e.g. for energy-dependent Breit-Wigner's

Near $K\bar{K}$ threshold resonance

$$E = \sqrt{s} - 2m_K$$

$$t_{ij}(E) = \frac{g_i g_j}{D(E)} = \frac{g_i g_j}{E - E_f + i\frac{\tilde{\Gamma}_1}{2} + \frac{i}{2} g_2 \sqrt{m_K E}}$$

$D(E)$ = Dressed propagator of the resonance.

E_f : bare resonance mass

g_i : bare squared couplings

$$\tilde{\Gamma}_1 = \frac{\sqrt{M^2/4 - m_1^2}}{8\pi M^2} g_1$$ bare width to channel 1 ($\pi\pi$)

Problem: $\left.\begin{matrix} g_i \\ \tilde{\Gamma}_1 \end{matrix}\right\}$ are taken as physical couplings and width of the resonance

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Bare parameters

Evidence that the $a_0(980)$ and $f_0(980)$ are not elementary particles

V. Baru [a,b], J. Haidenbauer [b], C. Hanhart [b], Yu. Kalashnikova [a], A. Kudryavtsev [a]

$$F_{K\bar{K}} = -\frac{1}{2k}\frac{\Gamma_K}{E - E_f + i\frac{\Gamma_K}{2} + i\frac{\Gamma_P}{2} + O(m^2E/\beta^2)}, \quad (17)$$

$$\frac{k_1^2}{m_K} = E_{pole} \qquad \Gamma_{pole} = -2\,\mathrm{Im}\,E_{pole}$$

$$\Gamma_K = g_{K\bar{K}}\sqrt{m_K E}$$

$f_0(980)$

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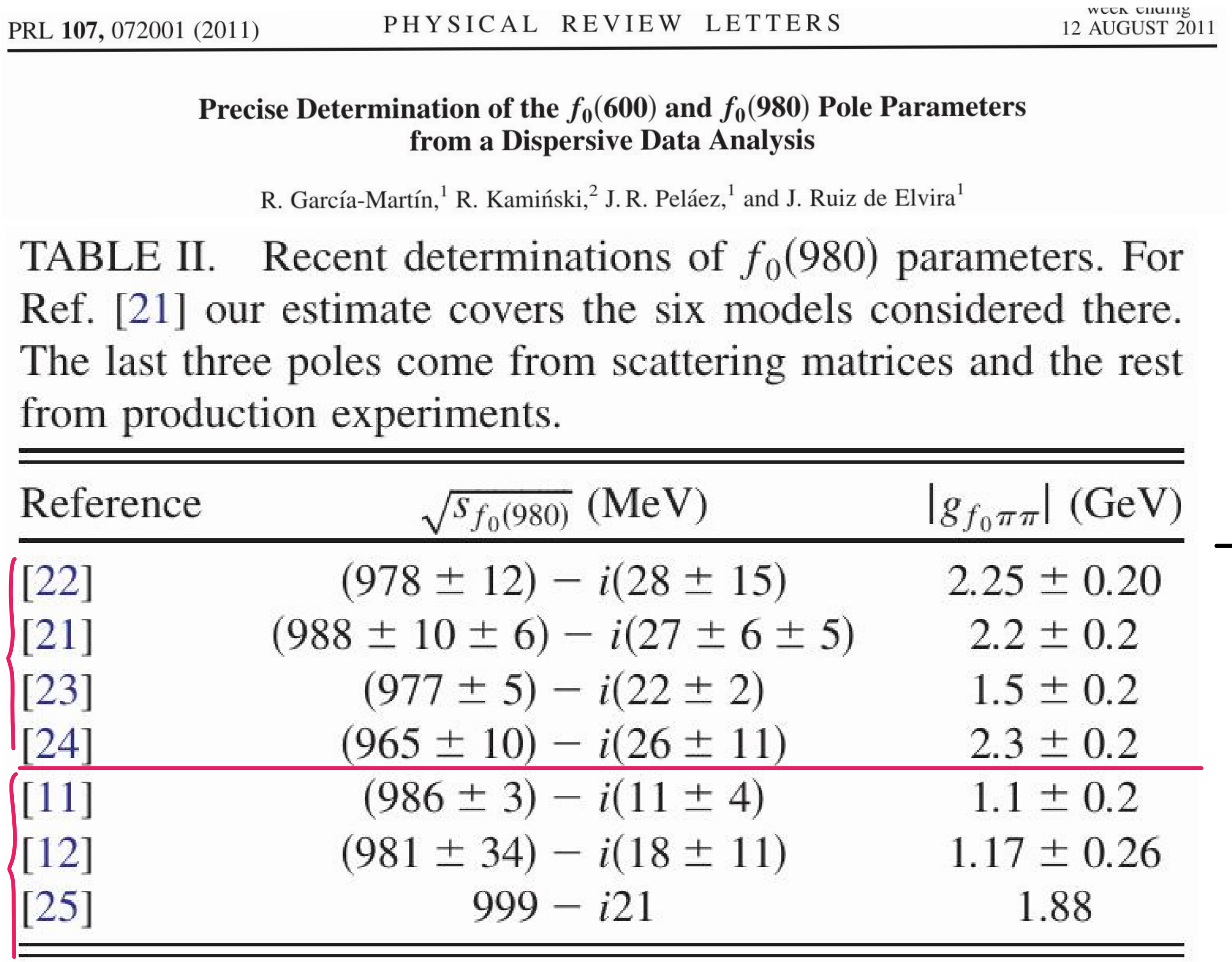
[22]	969.8	196 50	2.51	-151.5	-0.63	$1.15 - i0.74$	$-58 + i107$
[23]	975	149 51	1.51	-84.3	-1.05	$0.99 - i0.88$	$-65 + i97$
[21]	973	253 56	2.84	-154	-0.56	$1.09 - i0.89$	$-69 + i100$
[24]	996	128.8 11	1.31	$+4.6$	-1.22	$-0.14 - i1.99$	$-84 + i17$

$a_0(980)$

Ref.	M_R	$\Gamma_{\pi\eta}$	$\bar{g}_{K\bar{K}}$	E_f	r_e	a	k_1
[18]	1001	70 45	0.224	9.6	-7.1	$-0.16 - i0.59$	$-104 + i55$
[19]	999	146 77	0.516	7.6	-3.1	$-0.07 - i0.69$	$-134 + i71$
[20]	1003	153 46	0.834	11.6	-1.9	$-0.16 - i1.05$	$-129 + i44$
[20]	992	145.3 74	0.56	0.6	-2.8	$-0.01 - i0.76$	$-126 + i73$
[21]	984.8	121.5 80	0.41	-18.0	-3.9	$0.18 - i0.61$	$-102 + i97$



PRL **107,** 072001 (2011) PHYSICAL REVIEW LETTERS week ending 12 AUGUST 2011

Precise Determination of the $f_0(600)$ and $f_0(980)$ Pole Parameters from a Dispersive Data Analysis

R. García-Martín,[1] R. Kamiński,[2] J. R. Peláez,[1] and J. Ruiz de Elvira[1]

TABLE II. Recent determinations of $f_0(980)$ parameters. For Ref. [21] our estimate covers the six models considered there. The last three poles come from scattering matrices and the rest from production experiments.

Reference	$\sqrt{s_{f_0(980)}}$ (MeV)	$\lvert g_{f_0\pi\pi}\rvert$ (GeV)
[22]	$(978 \pm 12) - i(28 \pm 15)$	2.25 ± 0.20
[21]	$(988 \pm 10 \pm 6) - i(27 \pm 6 \pm 5)$	2.2 ± 0.2
[23]	$(977 \pm 5) - i(22 \pm 2)$	1.5 ± 0.2
[24]	$(965 \pm 10) - i(26 \pm 11)$	2.3 ± 0.2
[11]	$(986 \pm 3) - i(11 \pm 4)$	1.1 ± 0.2
[12]	$(981 \pm 34) - i(18 \pm 11)$	1.17 ± 0.26
[25]	$999 - i21$	1.88

Flatté or BW bare couplings

Residue

$$\frac{g^2_{\pi\pi}\, p_\pi}{8\pi M^2} = \Gamma_{\pi\pi}$$

99 (MeV)
94
44
104
23
27
68

Pole position

$$E - E_f + i\frac{\tilde{\Gamma}_1}{2} + \frac{i}{2} g_2 \sqrt{m_K E} = 0$$

$$E_R \equiv M_R - \frac{i}{2}\Gamma_R = E_f - \frac{m_K}{8} g_2^2 - \frac{i}{2}\tilde{\Gamma}_1 + \sigma \sqrt{\frac{g_2^2 m_K}{4}} \sqrt{\frac{m_K g_2^2}{16} - E_f + \frac{i}{2}\tilde{\Gamma}_1}$$

$\sigma = \pm 1$

$+ :$ RS II

$- :$ RS III

Because

$$\sqrt{E_R} = \frac{2i}{g_2\sqrt{m_K}}(M_R - E_f) + \frac{\Gamma_R - \tilde{\Gamma}_1}{g_2\sqrt{m_\pi}}$$

$M_R - E_f > 0 \longrightarrow \operatorname{Im}\sqrt{E_R} > 0$ RS II $(\sigma = +1)$

$M_R - E_f < 0 \longrightarrow \operatorname{Im}\sqrt{E_R} < 0$ RS III $(\sigma = -1)$

Renormalized Couplings

Residue: $$t_{ij} = \frac{g_i g_j}{E - E_f + \frac{i}{2}\tilde{\Gamma}_1 + \frac{i}{2} g_2^{I}\sqrt{m_K E}} \xrightarrow[E \to E_R]{} \frac{\beta g_i g_j}{(E - E_R)}$$

$$\beta = \lim_{E \to E_R} \frac{E - E_R}{E - E_f + i\frac{\tilde{\Gamma}_1}{2} + \frac{i}{2} g_2 \sqrt{m_K E}} = Z^{-1}$$

wave function renormalization
in QFT terminology

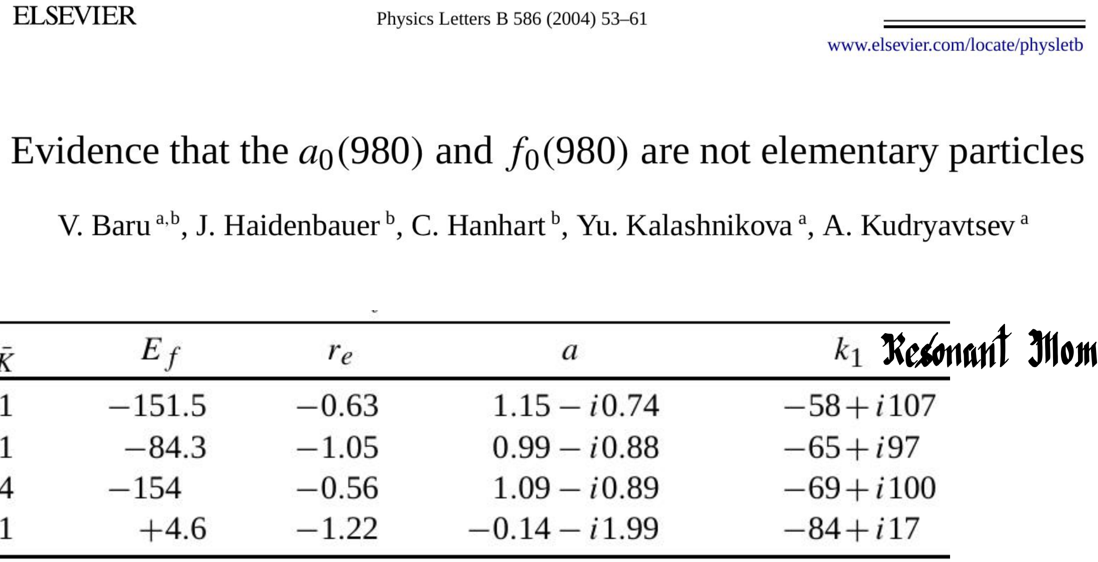
Closed expression:

$$\beta = \frac{1}{\left| 1 + \frac{i}{4} g_2 \sqrt{\frac{m_K}{E_R}} \right|} = \frac{4 |E_R|^{1/2}}{\left\{ m_K g_2^2 + 16 |E_R| + 4\sigma g_2 \sqrt{2 m_K (|E_R| - M_R)} \right\}^{1/2}}$$

$$\sigma = \begin{cases} + : & \text{RS II} \\ - : & \text{RS III} \end{cases}$$

$$\boxed{\begin{aligned} g_i^{\text{renormalized}} &= \beta g_i \\ \Gamma_1 &= \beta \tilde{\Gamma}_1 \end{aligned}}$$

Examples: Table on the $f_0(980)$ from



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V. Baru [a,b], J. Haidenbauer [b], C. Hanhart [b], Yu. Kalashnikova [a], A. Kudryavtsev [a]

Ref.	M_R	$\Gamma_{\pi\pi}$	$\bar{g}_{K\bar{K}}$	E_f	r_e	a	k_1
[22]	969.8	196	2.51	−151.5	−0.63	$1.15 - i0.74$	$-58 + i107$
[23]	975	149	1.51	−84.3	−1.05	$0.99 - i0.88$	$-65 + i97$
[21]	973	253	2.84	−154	−0.56	$1.09 - i0.89$	$-69 + i100$
[24]	996	128.8	1.31	+4.6	−1.22	$-0.14 - i1.99$	$-84 + i17$

Resonant Momentum

$f_0(980)$ PDG: $\Gamma_{f_0} \sim 40\text{-}70$ MeV

Ref.	$\tilde{\Gamma}_{\pi\pi}$	k_1	Γ_p	β	$\Gamma_{\pi\pi} = \beta\tilde{\Gamma}_{\pi\pi}$	$\Gamma_{K\bar{K}} = \Gamma_{\pi\pi} - \Gamma_p$
[22]	196	$-58 + i107$	50.1	0.29	56.5	6.4
[23]	149	$-65 + i97$	50.8	0.40	59.7	8.9
[21]	253	$-69 + i100$	55.6	0.27	67.2	11.6
[24]	128.8	$-84 + i17$	11.6	0.43	55.7	44.1

It is very important to take β into account

Compositeness analysis:

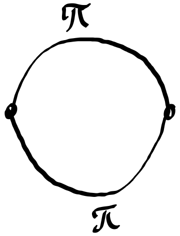
Weight of the meson-meson components

$$X = \underbrace{\beta\, g_1 \left|\frac{\partial G_1}{\partial s}\right|_{s_R}}_{X_1 \equiv X_{\pi\pi}} + \underbrace{\beta\, g_2 \left|\frac{\partial G_2}{\partial s}\right|_{s_R}}_{X_2 \equiv X_{K\bar{K}}} \leq 1$$

γ_i = renormalized couplings

$$\gamma_1^2 = \beta\, g_1$$

$$\gamma_2^2 = \beta\, g_2\, 8\pi M_K^2$$



$$G_1 = 1 \;\bigcirc\; 1 = i\int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[\left(k+\frac{P}{2}\right)^2 - m_1^2\right]\left[\left(k-\frac{P}{2}\right)^2 - m_1^2\right]} = \frac{-1}{16\pi^2} \ln\frac{\sigma(s)-1}{\sigma(s)+1} + \text{ct.} \quad \begin{cases} s = P^2 \\ \sigma(s) = \sqrt{1 - 4m_1^2/s} \end{cases}$$

$$\left.\frac{\partial G_2}{\partial s}\right|_{s_R} = \frac{-i}{128\pi\, M_K^{3/2}\sqrt{E_R}} + O(E_R^0)$$

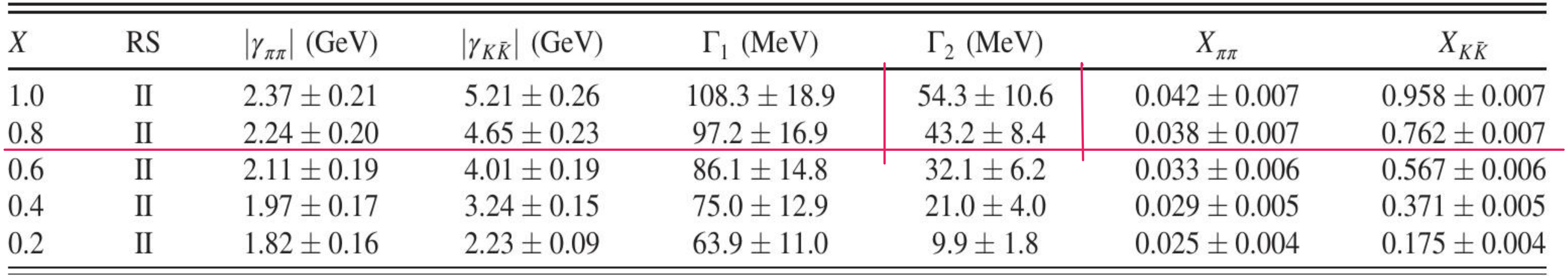
J.A.O, Ann.Phys.396,429 (2018)

In a Flatté parameterization $\{E_f, \tilde{\Gamma}_1, g_2\}$

We take as inputs $\{M_K, \Gamma_{pole}, X\}$

García-Martín, Kaminski, Peláez, Ruiz de Elvira, Phys. Rev. Lett. 107, 072001 (2011)

at $\sqrt{s_{pole}} = (996 \pm 7) - i\left(25^{+10}_{-6}\right)$ MeV lies in the 2nd Riemann sheet



X	RS	$\lvert\gamma_{\pi\pi}\rvert$ (GeV)	$\lvert\gamma_{K\bar{K}}\rvert$ (GeV)	Γ_1 (MeV)	Γ_2 (MeV)	$X_{\pi\pi}$	$X_{K\bar{K}}$
1.0	II	2.37 ± 0.21	5.21 ± 0.26	108.3 ± 18.9	54.3 ± 10.6	0.042 ± 0.007	0.958 ± 0.007
0.8	II	2.24 ± 0.20	4.65 ± 0.23	97.2 ± 16.9	43.2 ± 8.4	0.038 ± 0.007	0.762 ± 0.007
0.6	II	2.11 ± 0.19	4.01 ± 0.19	86.1 ± 14.8	32.1 ± 6.2	0.033 ± 0.006	0.567 ± 0.006
0.4	II	1.97 ± 0.17	3.24 ± 0.15	75.0 ± 12.9	21.0 ± 4.0	0.029 ± 0.005	0.371 ± 0.005
0.2	II	1.82 ± 0.16	2.23 ± 0.09	63.9 ± 11.0	9.9 ± 1.8	0.025 ± 0.004	0.175 ± 0.004

Compatibility. $X \in (0.8, 1.0)$

$X_{K\bar{K}} \gg X_{\pi\pi}$

Numerics:

$$\Gamma_{\pi\pi} = \frac{|g|^2}{8\pi M^2} = 100^{+20}_{-17} \text{ MeV}$$ because $|g| = 2.3 \pm 0.2$ GeV.

$$\Gamma_{K\bar{K}} = \Gamma_{\pi\pi} - \Gamma_{pole} = 50^{+26}_{-21} \text{ MeV}$$ by the knowledge of $|g|$ and s_{pole}.

Conclusions

1.- For a resonance near a higher threshold in the 2nd Riemann sheet

$$\Gamma_{\rm pole} = \Gamma_{\rm lighter} - \Gamma_{\rm higher}$$

New finding that is worth pursing further for near threshold resonances

2.- For resonance parameterizations: Flatté, energy-dependent Breit-Wigner's one has to distinguish between bare couplings and width

$$\Gamma_i = \beta\, \tilde{\Gamma}_i \quad \text{or} \quad \Gamma_i = Z^{-1}\, \tilde{\Gamma}_i$$

$\gamma_i^2 = \beta\, g_i$ γ_i are the residue at the pole position.

β can be very different to 1 !!

Avoiding confussion: Give (also) residues and pole position

3. This is far from just academic

Applied to the $f_0(980)$ Phys. Rev. D105, 074016 (2022) ; Phys. Lett. B 844, 138070 (2023)

$a_0(980)$ Phys. Rev. D105, 074016 (2022)

In non-relativistic QFT the compositeness is an observable

One has to distinguish between being an observable and model independent

As I showed in Ann. Phys. (2018)

$H = H_0 + V$

$\downarrow$

Free

with $H_0|\varphi_\alpha\rangle = E_\alpha|\varphi_\alpha\rangle$

$H_0|\phi_B\rangle = E_B|\phi_B\rangle$

Non-interacting spectrum

$H|\psi_B\rangle = E_B|\psi_B\rangle$ Bound state for n_V-body system.

Compositeness:

$$X = \frac{1}{n_V}\langle\psi_B|N_0|\psi_B\rangle$$

N_0 = Number operator of non-interacting fields.

Symmetry: In NR-QFT particle number is conserved

$\psi \to e^{i\alpha}\psi$, $\psi^\dagger \to e^{-i\alpha}\psi^\dagger$

$$\mathcal{L} = \frac{i}{2}\left[\psi^\dagger \frac{\partial}{\partial t}\psi - \frac{\partial\psi^\dagger}{\partial\psi}\psi\right] - \frac{1}{2m}\vec{\nabla}\psi^\dagger\vec{\nabla}\psi - \psi^\dagger \mathcal{V}(\vec{x})\psi$$

$$N_0(t) = \int d^3x\, \psi^\dagger(\vec{x},t)\,\psi(\vec{x},t) \qquad [H_0, N_0(t)] = 0 \rightarrow \frac{dN_0(t)}{dt} = 0$$

All orders in perturbation theory: J.A.O, Ann.Phys. 396, 429 (2018)

$$\chi = \frac{1}{n_v} \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} dt \int d^3x \, \langle \psi_B | T\Big[e^{-i\int_{-\infty}^{\infty} dt' \underline{V_D(t')}} \underbrace{\sum_{i=1} \psi_i^\dagger(x)\,\psi_i(x)}_{\text{Dirac picture}} \Big] | \psi_B \rangle$$

$$= \frac{1}{n_v} \langle \psi_B | N_H | \psi_B \rangle = \frac{1}{n_v} \int d^3x \, \langle \psi_B | \underbrace{\Psi_i^\dagger(\vec{x},t)\,\Psi_i(\vec{x},t)}_{\text{Heisemberg picture}} | \psi_B \rangle$$

Because of the symmetry

$$[H, N_H(t)] = 0 \rightarrow \frac{dN_H(t)}{dt} = 0$$

$$\chi = \frac{1}{n_V} \int d^3x \, \langle \psi_B | \sum_i \Psi_i^\dagger(\vec{x},t) \, \Psi_i(\vec{x},t) \, | \psi_B \rangle$$

$$= \frac{1}{2\pi\delta(0)} \cdot \frac{1}{n_V} \cdot \underbrace{\int d^4x \, \langle \psi_B | \sum_i \Psi_i^\dagger(\vec{x},t) \, \Psi_i(\vec{x},t) \, | \psi_B \rangle}_{\text{S-matrix element} \longrightarrow \text{Observable}}$$

Reparameterization invariance

$$V(t) \rightarrow V(t) + \int d^3x \sum_{i=1}^{A} \Psi_i^\dagger(\vec{x},t) \, \Psi_i(\vec{x},t) \;\; \epsilon_i(\vec{x},t)$$

External fields $\epsilon_i(\vec{x},t)$

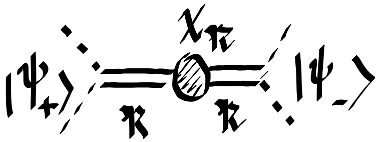
$$\chi = \frac{1}{2\pi\delta(0)} \frac{1}{n_V} \int d^3x \sum_{i=1}^{A} \left[\frac{\delta S_{BB}[\epsilon_1, \ldots, \epsilon_A]}{\delta \epsilon_i(\vec{x},t)} \right]_{\epsilon_i = 0}$$

For resonances take in/out states $|\psi_+\rangle$, $|\psi_-\rangle$

Evaluate

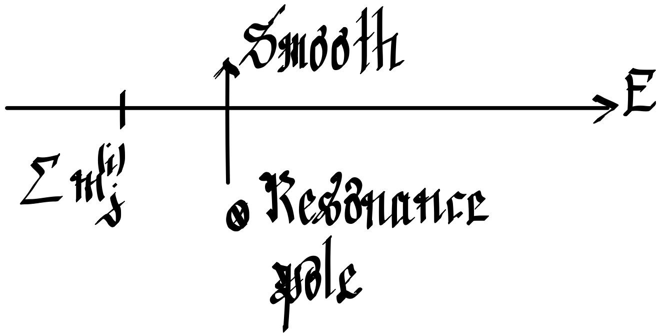
$$\frac{1}{2\pi\delta(0)}\cdot\frac{1}{n_V}\cdot\int d^4x\,\underbrace{\langle\psi_-|\sum_i \Psi_i^{e\dagger}(\vec{x},t)\,\Psi_i(\vec{x},t)\,|\psi_+\rangle}_{S\text{-matrix element}} = \frac{1}{2\pi\delta(0)}\,\frac{1}{n_V}\int d^4x\sum_{i=1}^{A}\left[\frac{\delta S_{out/in}[\epsilon_1,\dots,\epsilon_A]}{\delta\epsilon_i(\vec{x},t)}\right]_{\epsilon_i=0}$$

Extrapolate to the resonance pole position



Get χ_R by stripping away resonance double pole

$|\chi_R^i|$ if $M_R > \sum_j m_j^{(i)}$



It is customary to define $\sqrt{z}$ with the cut to the left, $\arg(z) \in (-\pi,\pi]$

Mathematica, Fortran, ...

the unphysical sheets are then reached for $\mathrm{Im}\, s < 0$

$$t^{I}_{ij}(s) = \frac{g_i\, g_j}{M^2 - s - i\left(g_1^2\, \rho_1(s) + g_2^2\, \rho_2(s)\right)}$$

$$t^{II}_{ij}(s) = \frac{g_i\, g_j}{M^2 - s - i\left(g_1^2\, \rho_1(s) - g_2^2\, \rho_2(s)\right)}$$

+,- : Fortran, Mathematica convention

$$t^{III}_{ij}(s) = \frac{g_i\, g_j}{M^2 - s - i\left(g_1^2\, \rho_1(s) + g_2^2\, \rho_2(s)\right)}$$

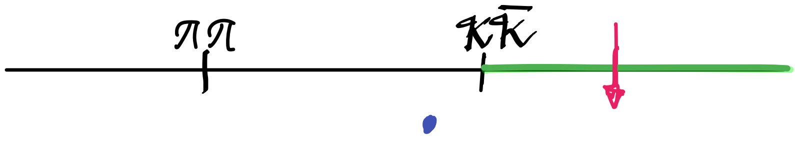
Toy Model: $g_1^2 = 5\ \mathrm{GeV}^2$

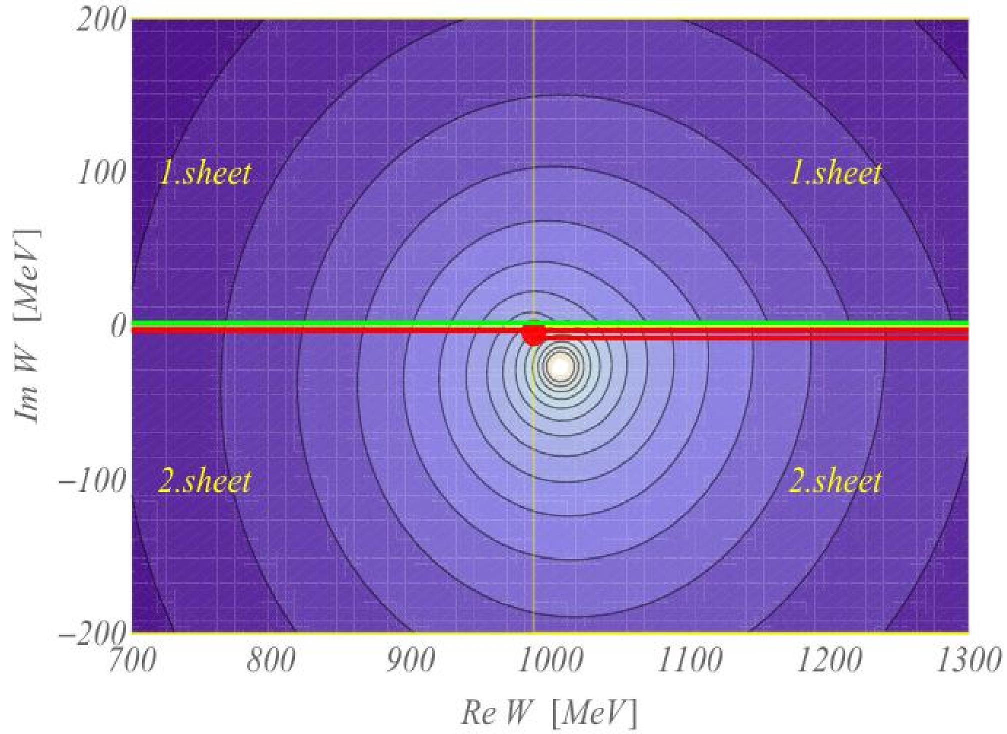
$g_2^2 = 10\ \mathrm{GeV}^2$

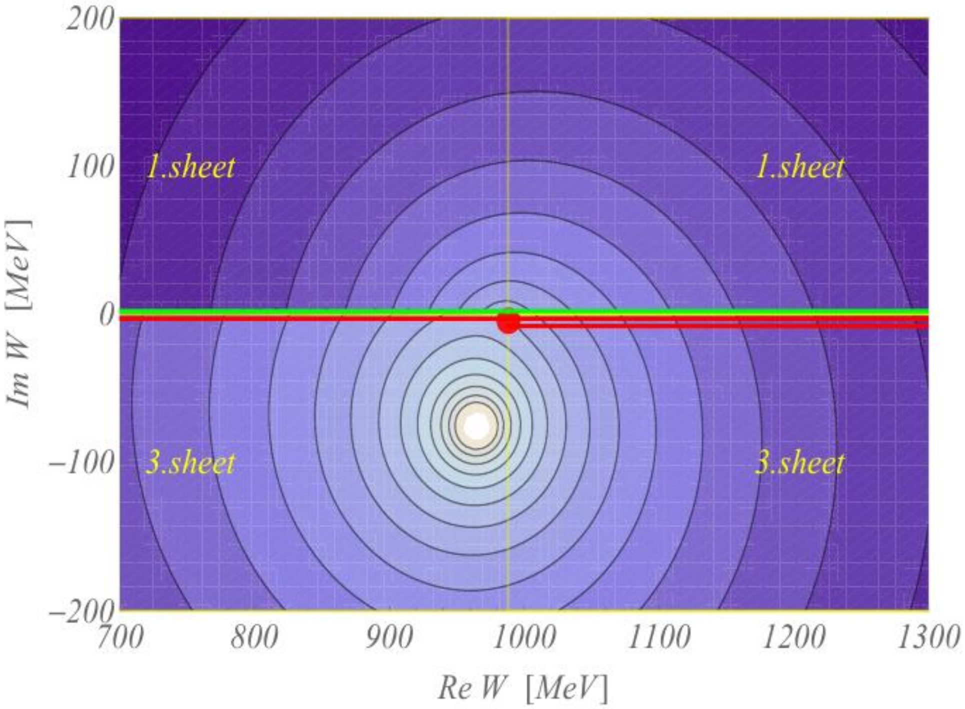
Two poles:

2nd RS pole $(1007 - i\,25)$ MeV

3rd RS pole $(967 - i\,77)$ MeV







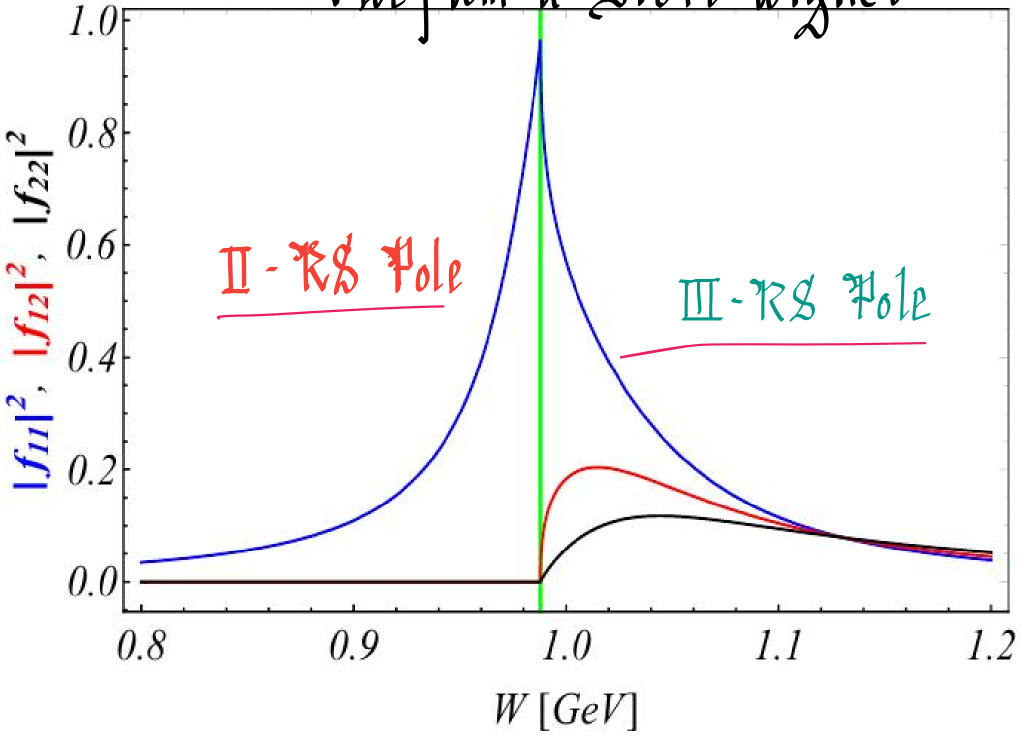
Toy Model: $g_1^2 = 5\ \text{GeV}^2$

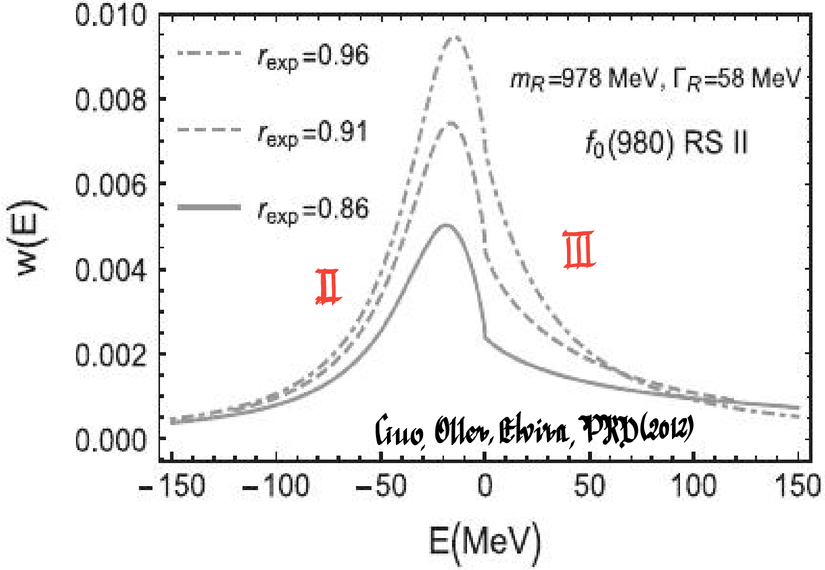
$g_2^2 = 10\ \text{GeV}^2$

2nd RS pole: $(1007 - i\,25)$ MeV

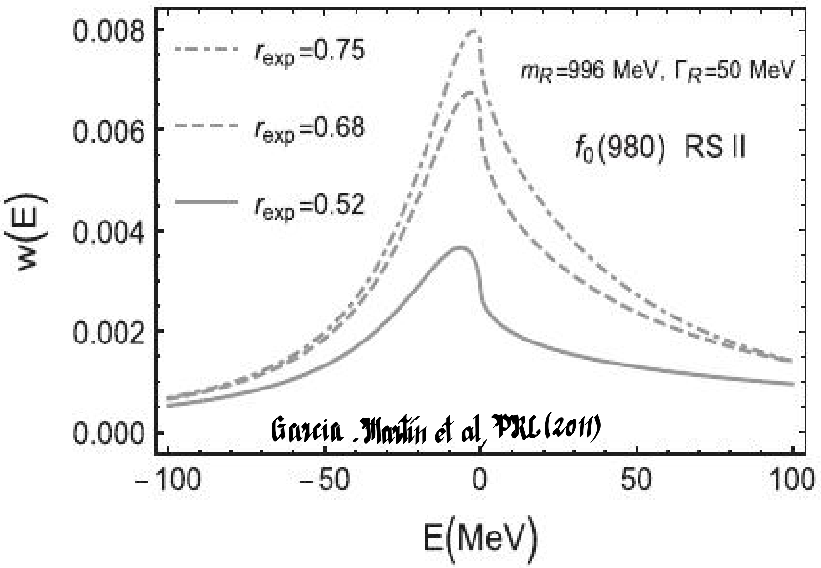
3rd RS pole: $(967 - i\,77)$ MeV

Far from a Breit-Wigner

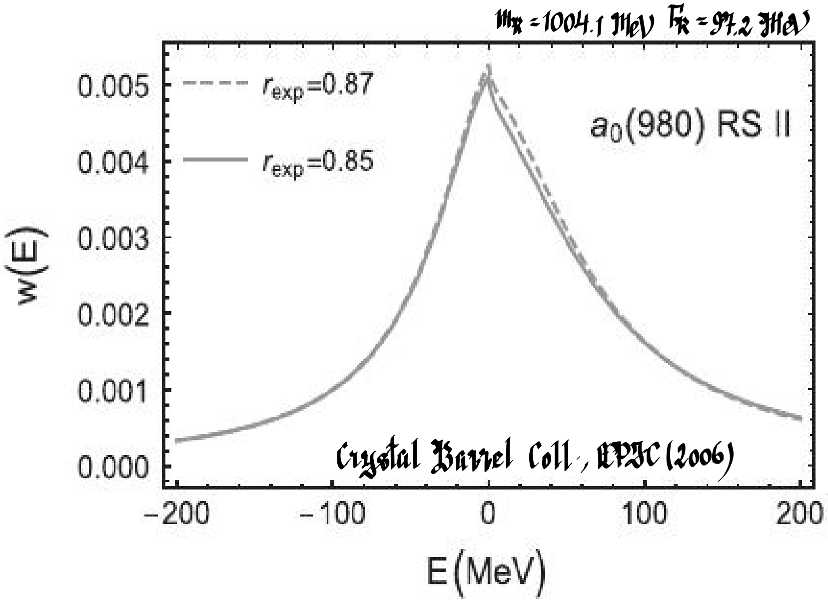



0.010
0.008
0.006
0.004
0.002
0.000
w(E)
r_{exp}=0.96
r_{exp}=0.91
r_{exp}=0.86
m_R=978 MeV, Γ_R=58 MeV
f_0(980) RS II
II
III
Guo, Oller, Elvira, PRD(2012)
-150
-100
-50
0
50
100
150
E(MeV)

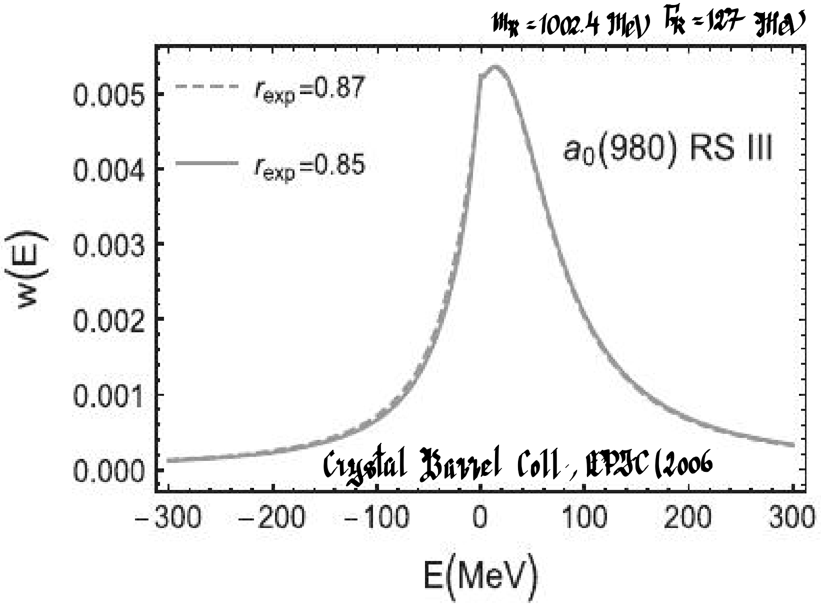
(a)


0.008
0.006
0.004
0.002
0.000
w(E)
r_{exp}=0.75
r_{exp}=0.68
r_{exp}=0.52
m_R=996 MeV, Γ_R=50 MeV
f_0(980) RS II
Garcia-Martin et al, PRL(2011)
-100
-50
0
50
100
E(MeV)

(b)


m_R = 1004.1 MeV Γ_R = 97.2 MeV
0.005
0.004
0.003
0.002
0.001
0.000
w(E)
r_{exp}=0.87
r_{exp}=0.85
a_0(980) RS II
Crystal Barrel Coll., EPJC(2006)
-200
-100
0
100
200
E(MeV)

(c)


m_R = 1002.4 MeV Γ_R = 127 MeV
0.005
0.004
0.003
0.002
0.001
0.000
w(E)
r_{exp}=0.87
r_{exp}=0.85
a_0(980) RS III
Crystal Barrel Coll., EPJC(2006
-300
-200
-100
0
100
200
300
E(MeV)

(d)

Introduction :

Around September 2021 I contacted J.R.Peláez to discuss about the coupling of the $f_0(980)$ to $\pi\pi$ provided by his group in

García-Martín, Kaminski, Peláez, Ruiz de Elvira, Phys. Rev. Lett. 107, 072001 (2011)

-benchmark calculation of several $f_0(980)$ pole properties-

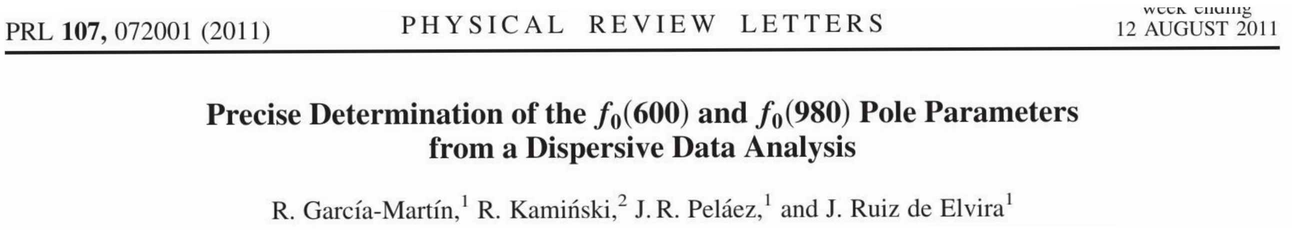
$$\frac{g^2_{\pi\pi}\, p_\pi}{8\pi M^2} \gg \Gamma_{pole} \;?! \qquad \sim 200 \;\&\; 50 \text{ MeV}$$

★

In November 2021 I contacted him back because I had understood what was happening Z.-Q. Wang, X.-W. Kang, J.A. Oller, L. Zhang, Phys. Rev. D105, 074016 (2022)

In April 2022 I wento to Madrid and discussed with him this solution.

Then, he also told me about E. Klempt, U. Thoma, et al. they also found this problem, and we prepared a joint paper Phys. Lett. B 844, 138070 (2023)



PRL 107, 072001 (2011) PHYSICAL REVIEW LETTERS week ending 12 AUGUST 2011

Precise Determination of the $f_0(600)$ and $f_0(980)$ Pole Parameters from a Dispersive Data Analysis

R. García-Martín,[1] R. Kamiński,[2] J. R. Peláez,[1] and J. Ruiz de Elvira[1]

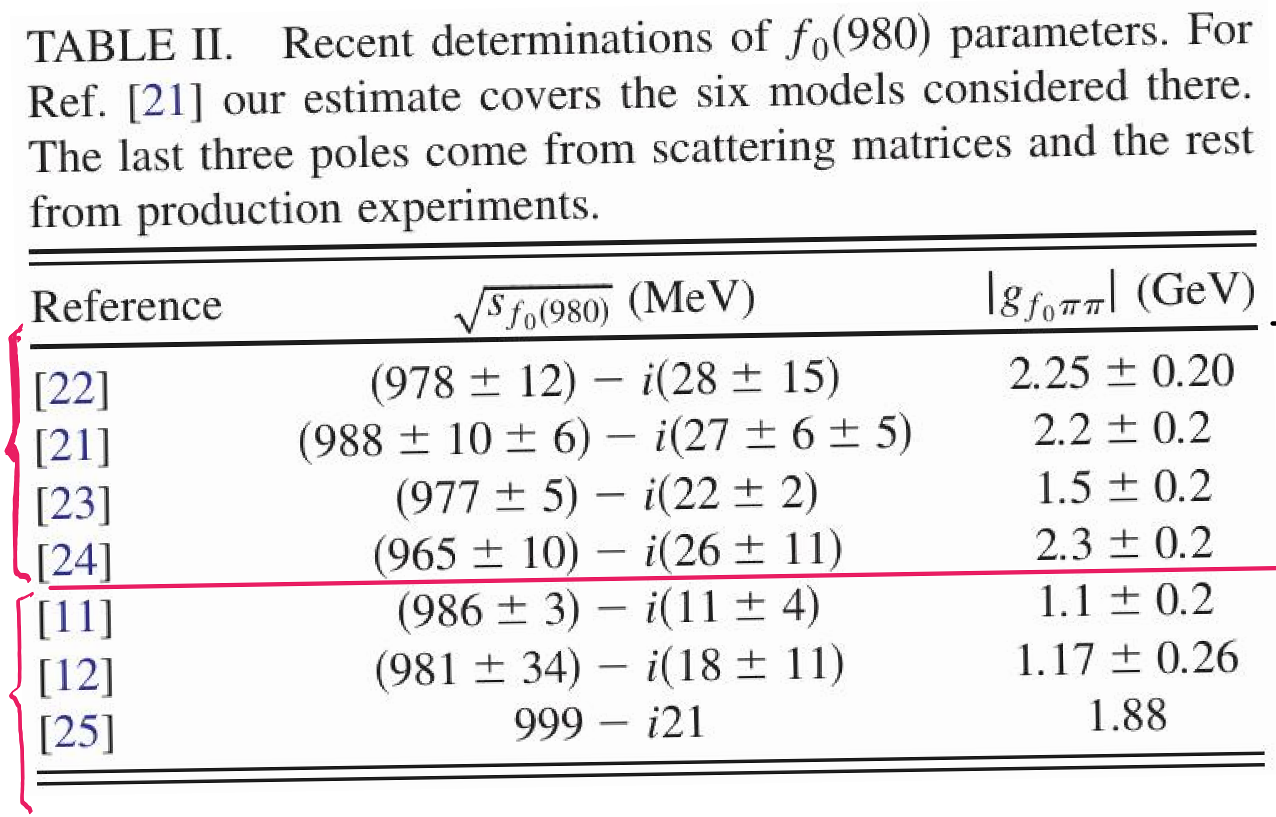
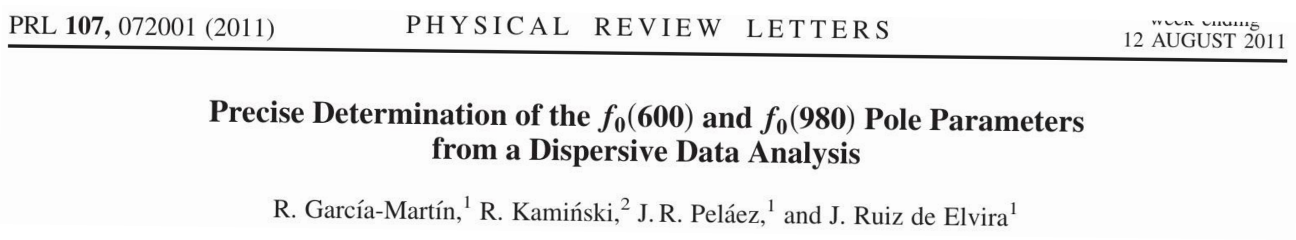


TABLE II. Recent determinations of $f_0(980)$ parameters. For Ref. [21] our estimate covers the six models considered there. The last three poles come from scattering matrices and the rest from production experiments.

	Reference	$\sqrt{s_{f_0(980)}}$ (MeV)	$\lvert g_{f_0\pi\pi}\rvert$ (GeV)	$\frac{g_{\pi\pi}^2 p_\pi}{8\pi M^2} = \Gamma_{\pi\pi}$
Flatté or BW bare couplings	[22]	$(978 \pm 12) - i(28 \pm 15)$	2.25 ± 0.20	99 (MeV)
	[21]	$(988 \pm 10 \pm 6) - i(27 \pm 6 \pm 5)$	2.2 ± 0.2	94
	[23]	$(977 \pm 5) - i(22 \pm 2)$	1.5 ± 0.2	44
	[24]	$(965 \pm 10) - i(26 \pm 11)$	2.3 ± 0.2	104
Residue	[11]	$(986 \pm 3) - i(11 \pm 4)$	1.1 ± 0.2	23
	[12]	$(981 \pm 34) - i(18 \pm 11)$	1.17 ± 0.26	27
	[25]	$999 - i21$	1.88	68



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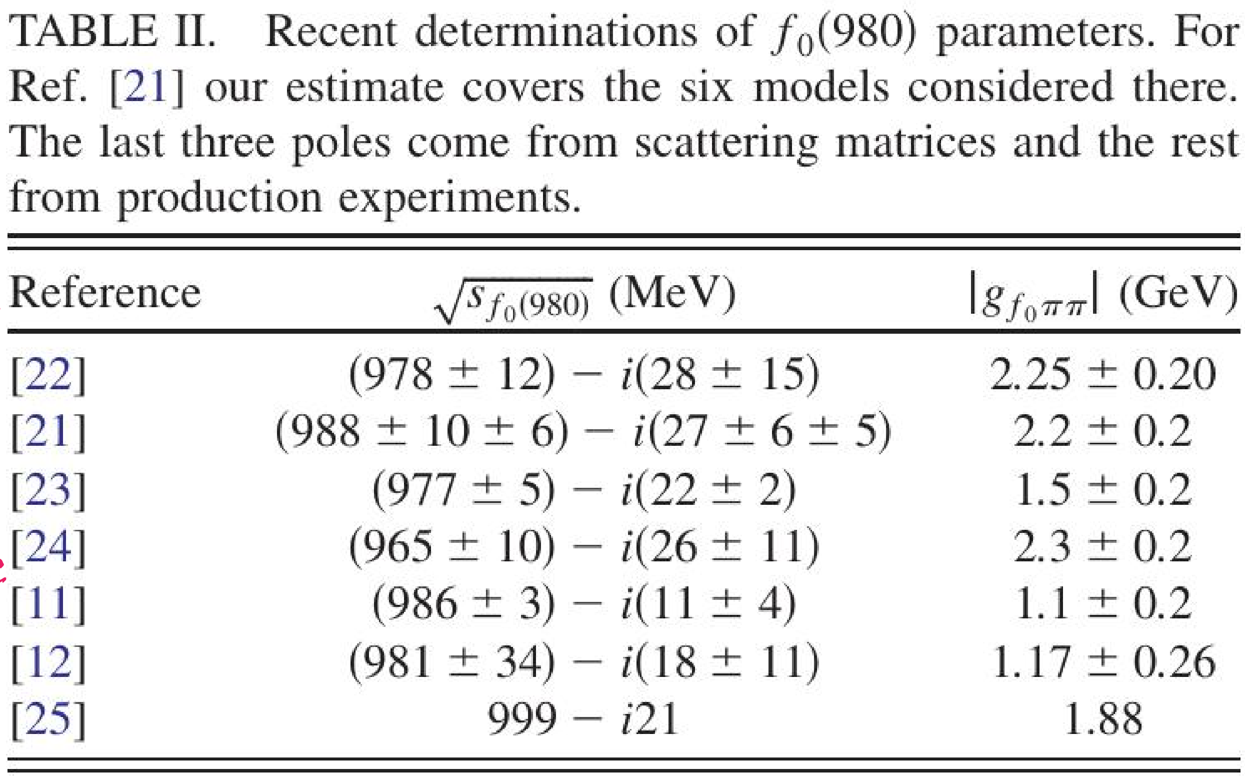

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[25]	$999 - i21$	1.88

Flatté or energy-dependent

-Bare coupling- BW's

Residue

-Renormalized coupling