$\phi \to 3\pi$ decay with Khuri-Treiman equations

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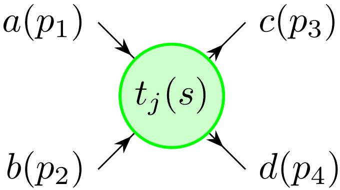
Excited QCD 2024

Necessity of Khuri-Treiman Equations



Only Right-Hand Cut:

- Bethe-Salpeter equation.
- Dispersive Relations.
- K-matrix formalism.

What we have...

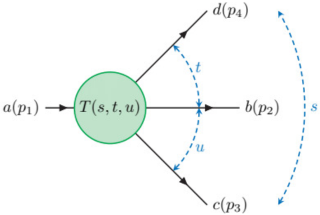
- Partial wave expansion **in the s-channel:**

$$T(s,t,u)=\sum_{j=0}^{\infty}(2j+1)P_j(z_s)t_j(s)$$

- Infinite number of partial waves $\Rightarrow$ requires truncation.
- Have both discontinuities: **RHC** and **LHC**.
- It can only reproduce poles in the s-channel.
- Crossing symmetry and unitarity are not fully recovered.

What we want...

- Handling of final states interactions (**FSI**).
- Simplified approach to discontinuities (with focus on **RHC**).
- Restoration of crossing symmetry.
- Analytic properties of amplitudes.
- Recover the unitarity of each channel (**t,u-channles**).
- Description of resonances.
- Versatility: be applied to a variety of processes, including three-body decays.



Limitations:

- High-energy limitations.
- Numerical challenges.
- Model dependence.

What we do...

Khuri-Treiman Equations: conceived to achieve the two-body unitarity in the three channels. — *[N. Khuri, S. Treiman, Phys. Rev. 119 , 1115 (1960)]*

- Consider three (s,t,u-channels) truncated **isobar** expansions:

$$T(s,t,u) = \sum_{j=0}^{\infty}(2j+1)P_j(z_s)t_j(s)$$

$$= \sum_{j=0}^{j_s}(2j+1)P_j(z_s)f_j^{(s)}(s) + \sum_{j=0}^{j_t}(2j+1)P_j(z_t)f_j^{(t)}(t) + \sum_{j=0}^{j_u}(2j+1)P_j(z_u)f_j^{(u)}($$

- s-channel singularities appear in $f_j^{(s)}(s)$.
- $\Rightarrow$ t,u-channel singularities are recovered in $f_j^{(t)}(t)$ and $f_j^{(u)}(u)$.
- **LHC** of $t_j(s)$ are given by the **RHC** of the crossed channel isobars:

$$t_j(s) = \frac{1}{2}\int dz\; P_j(z)T(s,t',u') = f_j^{(s)}(s) + \underbrace{\frac{1}{2}\int dz\; Q_{j,j'}(s,t')f_{j'}^{(t)}(t')}_{\widehat{f}_j^{(s)}(s)}$$

Our Decay ...

$$\phi(p_1) \to \pi^+(p_2)\,\pi^0(p_3)\,\pi^-(p_4)$$

... is the combination of these three channels:

- **s-channel:** $\phi(p_1)\pi^0(-p_3) \;\to\; \pi^+(p_2)\pi^-(p_4)$
- **t-channel:** $\phi(p_1)\pi^-(-p_2) \;\to\; \pi^0(p_3)\pi^-(p_4)$
- **u-channel:** $\phi(p_1)\pi^+(-p_4) \;\to\; \pi^+(p_2)\pi^0(p_3)$

- Initial state $\phi\,(1^{--}) \Rightarrow$ Final state $J = 1, 3, 5...$
- Three helicity states $\lambda = +1, 0, -1$.

- Parity: $\mathcal{H}_0 = 0$ y $\mathcal{H}_+ = \mathcal{H}_-$.
- Helicity amplitude:

$$\mathcal{M}_+ = \frac{\sqrt{\phi(s,t,u)}}{2} F(s,t,u)$$

Disclaimer!

The $\mathcal{H}_\lambda$ are not Lorentz invariant but...

$$|\mathcal{M}|^2 = \frac{1}{3} \sum_{\lambda=+1,0,-1} |\mathcal{H}_\lambda|^2$$

...is indeed.

- Kibble function:

$$\phi(s,t,u) = 4sp^2(s)q^2(s)\sin^2\theta_s$$

- Decay width:

$$\frac{d^2\Gamma}{dsdt}(s,t) \sim \phi(s,t,u)\,|F(s,t,u)|^2$$

Kibble function

$\phi(s,t,u)$ determine the possible accessible states of the phase space:

- $\phi(s,t,u) > 0 \Rightarrow$ **Allowed**
- $\phi(s,t,u) < 0 \Rightarrow$ **Forbidden**

Function $F(s,t,u)$:

All the dynamics of the process is contained within this Lorentz invariant amplitude!!!

Why revisit $\phi \to 3\pi$?

- Studies of the same decay have been conducted before, but not like in this work. — *[F. Niecknig, B. Kubis y S. P. Schneider, Eur. Física. JC 72 (2012) 2014]*
- We can compare it to the decay $\omega \to 3\pi$.
- Recent measurements of the TFF (2015) by KLOE.

Solving Khuri-Treiman Equations

1. Start with partial wave decomposition:

$$F(s,t,u) = \sum_{j \text{ odd}}^{\infty} (p(s)q(s))^{j-1} P_j'(\cos\theta_s) f_j(s) = f_1(s) + \dots$$

2. Do the truncated isobar expansion:

Consider only $j = 1$ (ρ) isobar

$$F(s,t,u) = F^{(s)}(s) + F^{(t)}(t) + F^{(u)}(u)$$

Isospin Limit:

$$= F(s) + F(t) + F(u)$$

3. Make the PW projection of the KT decomposition:
   - Isobar: $f(s) = \mathbf{F(s)} + \hat{F}(s)$
   - Inhomogeneity: $\hat{\mathbf{F}}(\mathbf{s}) = \frac{3}{2}\int_{-1}^{1} dz_s\,(1 - z_s^2) F\left(t(s, z_s)\right)$
4. Impose the DR relation at discontinuity to preserve unitarity:

$$\Delta F(s) = F(s + i\epsilon) - F(s - i\epsilon) = 2i \underbrace{\rho(s)\left(t_I^{2\to2}(s)\right)^*}_{\sin\delta_I(s) e^{-i\delta_I(s)}\theta(s-4m^2)} \left(F(s) + \hat{F}(s)\right)$$

5. Subtract ... or not:

Unsubtracted DR

$$F(s) = \frac{1}{2\pi i}\int_{4m^2}^{\infty} ds' \,\frac{\Delta F(s')}{s'-s}$$

- $\mathbb{C}^1$ **space:** $F(s) = aF_0(s)$
- $F_0(s) =$ $\Omega(s)\left[1 + \frac{s}{\pi}\int_{4m^2}^{\infty} ds' \,\frac{\sin\delta(s')\hat{F}_0(s')}{s'|\Omega(s')|(s'-s)}\right]$

Once-subtracted DR

$$F(s) = F(0) + \frac{s}{2\pi i}\int_{4m^2}^{\infty} ds' \,\frac{\Delta F(s')}{s'(s'-s)}$$

- $\mathbb{C}^2$ **space:** $F(s) = a\left[F_a(s) + bF_b(s)\right]$
- $F_a(s) =$ $\Omega(s)\left[1 + \frac{s^2}{\pi}\int_{4m^2}^{\infty} ds' \,\frac{\sin\delta(s')\hat{F}_a(s')}{s'^2|\Omega(s')|(s'-s)}\right]$
- $F_b(s) =$ $\Omega(s)\left[s + \frac{s^2}{\pi}\int_{4m^2}^{\infty} ds' \,\frac{\sin\delta(s')\hat{F}_b(s')}{s'^2|\Omega(s')|(s'-s)}\right]$

Advantages of subtraction:

- Enhances accuracy with data.
- Manage kinematic singularities and inelastic contributions.

Disadvantages of subtraction:

- More parameters $\Rightarrow$ Additional uncertainties.

In principle, the two solutions are different, but there must be some value of b for which they coincide...

Unsubtracted Sol. $\overset{b_{\text{sum}}}{\Longleftrightarrow}$ **Once-subtracted Sol.**

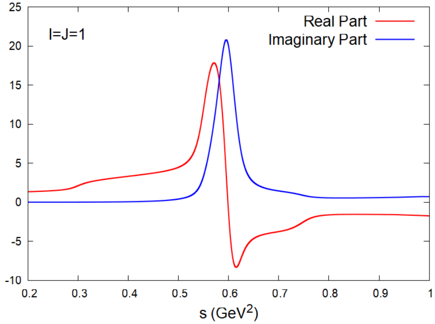
...which we call:

Sum Rule:

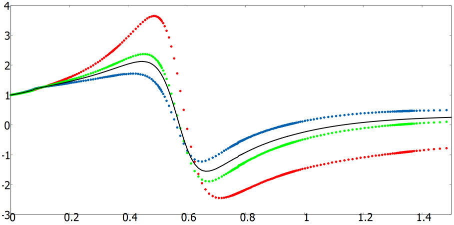
$$b_{\text{sum}} \equiv \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\sin\delta(s')\hat{F}(s')}{s'^2\,|\Omega(s')|}$$

Omnès Function:

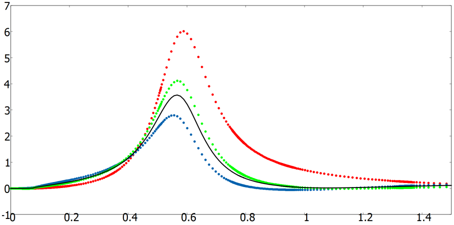
$$\Omega(s) = e^{\frac{s}{\pi}\int_{4m^2}^{\infty} ds' \, \frac{\delta(s')}{s'(s'-s)}}$$



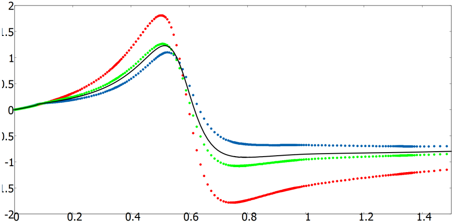
6. Iterate!



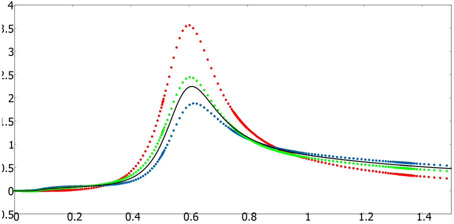
$\Re\{F_a(s)\}$ vs $s(GeV^2)$



$\Im\{F_a(s)\}$ vs $s(GeV^2)$



$\Re\{F_b(s)\}$ vs $s(GeV^2)$



$\Im\{F_b(s)\}$ vs $s(GeV^2)$

$\phi \to \gamma^*\pi^0$ Transition Form Factor

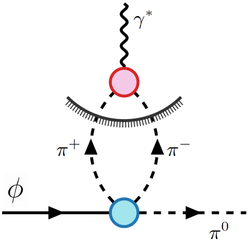
- The decays $\phi(\to \gamma^*\pi^0) \to \pi^0 l^+ l^-$ and $\phi \to \gamma\pi^0$ are governed by the **TFF** $f_{\phi\pi^0}(s)$:
  - Amplitude: $\mathcal{M} = f_{\phi\pi^0}(s)\epsilon_{\nu\mu\alpha\beta}\epsilon^{\mu}(p_\phi,\lambda)\,p^\nu q^\alpha \frac{ie^2}{s}\overline{u}(p_-)\gamma^\beta v(p_+)$
  - Decay width: $\Gamma = \left|f_{\phi\pi^0}(0)\right|^2 \frac{e^2\left(M_\phi^2 - m_{\pi^0}^2\right)^3}{96\pi M_\phi^3}$
- Dispersive representation:

$$\Delta f_{\phi\pi^0}(s) = i\frac{p^3(s)}{6\pi\sqrt{s}}F_\pi^{V*}(s) f_1(s)\theta(s-4m^2)$$

- Due to the low energy of the $\phi\pi^0$ system, we can take $F_\pi^{V*}(s) \approx \Omega(s)$.

Once-subtracted DR:

$$f_{\phi\pi^0}(s) = |f_{\phi\pi^0}(0)|e^{i\theta_{\phi\pi^0}(0)} + \frac{s}{12\pi^2}\int_{4m^2}^{\infty} ds' \, \frac{1}{\sqrt{(s')^3}} \frac{p^3(s')\Omega(s')f(s')}{(s'-s)}$$



From $\phi \to 3\pi$ amplitude $F(s,t,u)$

- A real global normalization $|a|$.
- A complex parameter $b = b_r + ib_i$ from the once-subtracted solution.

From $\phi \to \gamma^* \pi^0$ TFF $f_{\phi\pi^0}(s)$

- A real normalization of the TFF $\left|f_{\phi\pi^0}(0)\right|$.
- A relative phase $\theta_{\phi\pi^0}$ between $f_{\phi\pi^0}(0)$ and $f_{3\pi}(0)$.

ρ^0 and ω mixing

Due to the fact that they share the same quantum numbers 1^{--}, they can mix.

- A complex coupling constant A on the s-channel:

$$F(s) \;\to\; F(s) + A\frac{M_\omega^2}{M_\omega^2 - \sqrt{s}\Gamma_\omega i - s}$$

Global Parameters

- A real parameter N that serves as an adjustable parameter for the total number of events.

Input

Global Input

The phase-shift $\delta_i(s)$ parametrization of the $\pi\pi$ P-wave. (We take the 4 of them to compare).

— *J.R.Pelaez, F.J.Yndurain Phys.Rev.D69:114001,2004* and *J. Nebreda, JR Peláez y G. Ríos Física. Rev. D 83 , 094011*

To fit $\phi \to 3\pi$ amplitude $F(s,t,u)$

Dalitz Plot divided in bins.

$$(X_i, Y_i) \quad \leftrightarrow \quad (s_i, t_i)$$

— *KLOE collaboration*

To fit $\phi \to \gamma^*\pi^0$ TFF $f_{\phi\pi^0}(s)$

Measurement of the normalized TFF.

$$\left|\frac{f_{\phi\pi^0}(s)}{f_{\phi\pi^0}(0)}\right|^2$$

— *KLOE collaboration*

Particle Data Group

$$\Gamma_\phi, \mathcal{B}\left(\phi \to 3\pi\right), \mathcal{B}\left(\phi \to \gamma\pi^0\right), M_\omega, \Gamma_\omega$$

General Chi-Square χ^2

$$\chi^2 \equiv \chi^2_{DP} + \chi^2_{\Gamma_{3\pi}} + \chi^2_{TF} + \chi^2_{\Gamma_{\pi^0\gamma}}$$

- $\chi^2_{DP} = \sum_i \left(\frac{N^{\text{exp}}_{\text{ev},i} - N^{\text{th}}_{\text{ev},i}}{\sigma_{N^{\text{exp}}_{\text{ev},i}}} \right)^2$ with

$$N^{\text{exp}}_{\text{ev},i} = \frac{N}{(2\pi)^3 384 M_\phi^3} \frac{1}{2\Delta s \Delta t} \int_{s_i - \Delta s}^{s_i + \Delta s} \int_{t_i - \frac{(s-s_i)-\Delta t}{2}}^{t_i - \frac{(s-s_i)+\Delta t}{2}} \phi(s,t) \left| F(s,t) \right|^2 dtds$$

- $\chi^2_{\Gamma_{3\pi}} = \left(\frac{\Gamma^{exp}_{3\pi} - \Gamma^{th}_{3\pi}}{\sigma_{\Gamma^{exp}_{3\pi}}} \right)^2$ with $\Gamma^{th}_{3\pi} = \frac{1}{(2\pi^3)384 M_\phi^3} \int_{4m^2}^{\infty} \int_{t_-}^{t_+} \phi \left| F(s,t) \right|^2 dtds$

- $\chi^2_{TF} = \sum_i \left(\frac{\left| \frac{f_{\phi\pi^0}(s_i)}{f_{\phi\pi^0}(0)} \right|^2_{exp} - \left| \frac{f_{\phi\pi^0}(s_i)}{f_{\phi\pi^0}(0)} \right|^2_{exp}}{\sigma_i^{TF}} \right)^2$ with

$$\left| \frac{f_{\phi\pi^0}(s_i)}{f_{\phi\pi^0}(0)} \right|^2_{th} = \frac{1}{4\sqrt{s_i}\Delta\sqrt{s_i}} \int_{(\sqrt{s_i} - \Delta\sqrt{s_i})^2}^{(\sqrt{s_i} + \Delta\sqrt{s_i})^2} \left| \frac{f_{\phi\pi^0}(s_i)}{f_{\phi\pi^0}(0)} \right|^2 ds$$

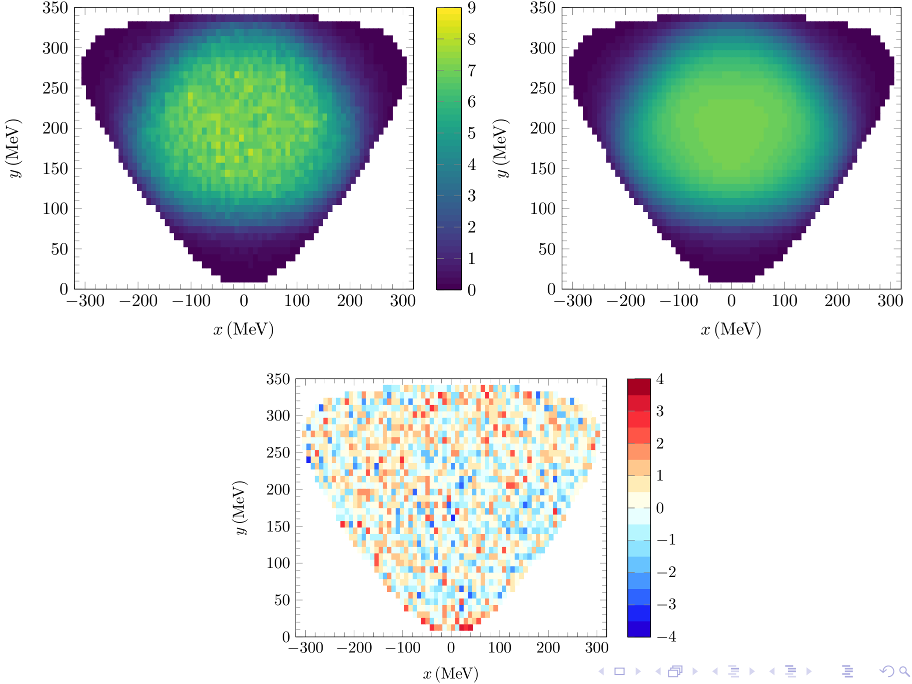
- $\chi^2_{\Gamma_{\phi\pi^0}} = \left(\frac{\Gamma^{exp}_{\phi\pi^0} - \Gamma^{th}_{\phi\pi^0}}{\sigma_{\Gamma^{exp}_{\phi\pi^0}}} \right)^2$

	δ_1	δ_2	δ_3	δ_4
$\lvert a\rvert$ $\left[\mathrm{GeV}^{-3}\right]$	15.60(23)	15.16(22)	15.23(22)	15.07(22)
$\Re(b)$ $\left[\mathrm{GeV}^{-2}\right]$	0.690(19)	0.810(17)	0.788(17)	0.819(16)
$\Im(b)$ $\left[\mathrm{GeV}^{-2}\right]$	0.312(30)	0.570(40)	0.545(38)	0.594(40)
$\Re(A)$ $\left[\mathrm{GeV}^{-3}\right]$	0.1251(98)	0.1208(97)	0.1208(97)	0.1212(97)
$\Im(A)$ $\left[\mathrm{GeV}^{-3}\right]$	0.025(12)	0.053(12)	0.050(12)	0.056(12)
$\lvert f_{\phi\pi^0}(0)\rvert$ $\left[\mathrm{GeV}^{-1}\right]$	0.1351(26)	0.1351(26)	0.1351(26)	0.1351(26)
$\theta_{\phi\pi^0}(0)$	0.40(17)	0.50(17)	0.48(17)	0.50(17)
$N_{ev}[10^6]$	6.60(10)	6.60(10)	6.60(10)	6.60(10)
χ^2/d	1.012	1.029	1.026	1.0309

Fit utilitites

- We make use of the library Minuit2 of C++, with Midgrad, Minos and Covariance as subroutines.
- Montecarlo Resampling (M.C.) of 10^5 for each $\delta_i(s)$.
- Total number of points (with $\phi(s_i, t_i) > 0$ criteria):

$$N = N_{DP} + N_{\Gamma_{3\pi}} + N_{TF} + N_{\Gamma_{\pi^0\gamma}} = 1860 + 1 + 15 + 1 = 1877$$

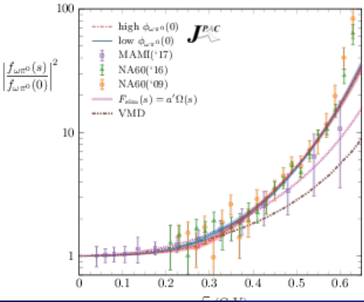

y (MeV)
x (MeV)
y (MeV)
x (MeV)
y (MeV)
x (MeV)

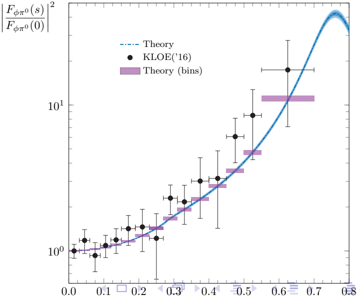
	δ_1	δ_2	δ_3	δ_4
$\Re(b_{\text{sum}})$	0.61279	0.65841	0.62613	0.62081
$\Im(b_{\text{sum}})$	0.50581	0.52535	0.52288	0.52413
$\Re(b)$	0.690(19)	0.810(17)	0.788(17)	0.819(16)
$\Im(b)$	0.312(30)	0.570(40)	0.545(38)	0.594(40)

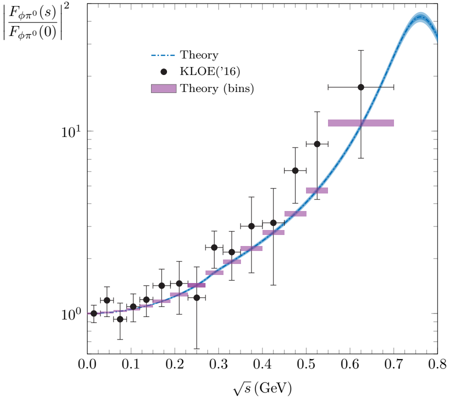
Sum rule for $\omega \to 3\pi$

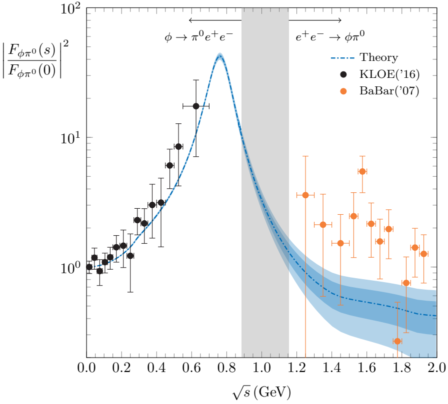
$$b_{sum} = 0.54382 + 0.08219i \qquad b_{fit} = -0.341(40) + 2.62(79)i$$

— *JPAC collaboration Eur. Phys. J. C (2020) 80:1107*









Disclaimer!

- We haven't fitted the data from the BaBar experiment.
- KT formalism is not prepared for high energies.
- We can still observe a good trend in them!!

Conclusions

- Difference between the results of the fit and the same values computed through the sum rule $\Rightarrow$ Justify the need of the extra subtraction.
- The values of the transition form factor given by the Khuri-Treiman approach with the additional subtraction used for our process are in very good agreement with the experimental data.
- For high energies, there is a good trend of what happens in the dispersive region.
- A different trend compared to the ω decay, there must be physics behind that process that has not been taken into account.