Double-heavy hadrons in the Born-Oppenheimer approximation and beyond

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Ruben Oncala, JS, Phys. Rev. D **96**, 014004 (2017)
JS, Jaume Tarrús Castellà, Phys. Rev. D **102**, 014012 (2020)
JS, Sandra Tomàs Valls, Phys. Rev. D **108**, 014025 (2023)

Heavy Flavors

- Heavy quarks: $Q = c, b, t,\ m_Q \gg \Lambda_{QCD}$
- Heavy hadrons: hadrons containing at least a heavy quark: $Q = b, c$
- In the hadron rest frame the heavy quark moves slowly $\Longrightarrow$ use a non-relativistic approximation
- A universal way to encode it together with relativistic correction is using Effective Field Theories
- NRQCD/HQET are the suitable ones
- They imply heavy quark spin symmetry at leading order.

NRQCD

W.E. Caswell and G.P. Lepage, Phys. Lett. **167B**, 437 (1986)
G. T. Bodwin, E. Braaten and G. P. Lepage, Phys. Rev. D **51** (1995) 1125

$$m_Q \quad >> \quad m_Q v \quad , \quad m_Q v^2 \quad , \quad \Lambda_{QCD}$$

$$\mathcal{L}_\psi = \psi^\dagger \Big\{ iD_0 + \frac{1}{2m_Q}\mathbf{D}^2 + \frac{1}{8m_Q^3}\mathbf{D}^4 + \frac{c_F}{2m_Q}\boldsymbol{\sigma}.g\mathbf{B} + \\ + \frac{c_D}{8m_Q^2}\left(\mathbf{D}.g\mathbf{E} - g\mathbf{E}.\mathbf{D}\right) + i\,\frac{c_S}{8m_Q^2}\boldsymbol{\sigma}.\left(\mathbf{D}\times g\mathbf{E} - g\mathbf{E}\times\mathbf{D}\right)\Big\}\psi$$

c_F, c_D and c_S are short distance matching coefficients calculable from QCD in powers of $\alpha_{\rm s}$. They depend on m_Q and μ (factorization scale) but not on the lower energy scales.

Exotic Hadrons

- Hadrons beyond mesons $q\bar{q}$ and baryons qqq
- QCD: any color singlet state made out of quarks and gluons may become a hadron
- I will restrict myself to discuss hadrons containing two heavy quarks
- The starting point can then be NRQCD

Hadrons with two heavy quarks

$$Q = b,\ c \quad , \quad q = u,\ d,\ s$$

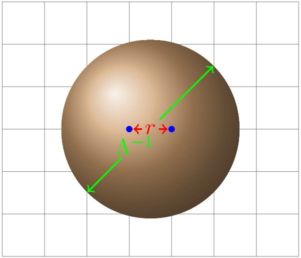
- $QQ+$ light quarks and gluons
  - Double Heavy Baryons: QQq
  - Tetraquarks: $QQ\bar{q}\bar{q}$
  - Pentaquarks: $QQqq\bar{q}$
  - Hybrids: $QQqg$
  - ...
- $Q\bar{Q}+$ light quarks and gluons
  - Heavy Quarkonium: $Q\bar{Q}$
  - Hybrids: $Q\bar{Q}g$
  - Tetraquarks: $Q\bar{Q}q\bar{q}$
  - Pentaquarks: $Q\bar{Q}qqq$
  - ...

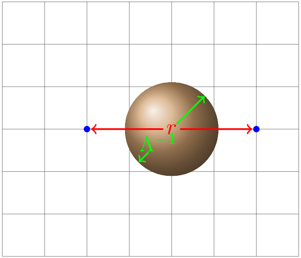
Heavy Quarkonium

$Q\bar{Q}$ bound state , $m_Q >> \Lambda_{QCD}$, $\alpha_s(m_Q) << 1$

- Heavy quarks move slowly $v << 1$
- Non-relativistic system $\rightarrow$ multiscale problem
  - $m_Q >> m_Q v$ (Relative momentum)
  - $m_Q v >> m_Q v^2$ (Binding energy)
  - $m_Q >> \Lambda_{QCD}$
- EFTs are useful (N. Brambilla, A. Pineda, JS and A. Vairo, Rev. Mod. Phys. 77, 1423 (2005))
  - NRQCD: $m_Q \gg m_Q v, m_Q v^2, \Lambda_{QCD}$ (W.E. Caswell and G.P. Lepage, Phys. Lett. **167B**, 437 (1986))
  - pNRQCD (weak coupling): $m_Q v \gg m_Q v^2, \Lambda_{QCD}$ (A. Pineda, JS, Nucl.Phys.Proc.Suppl.64:428-432,1998)
  - pNRQCD (strong coupling): $m_Q v, \Lambda_{QCD} \gg m_Q v^2$ (N. Brambilla, A. Pineda, JS, A. Vairo, Nucl.Phys.B566:275,2000)

How does the hadron look like ?





$m_Q v \sim 1/r \gg m_Q v^2 \gtrsim \Lambda_{QCD}$

weak coupling pNRQCD

$m_Q v \sim 1/r \gtrsim \Lambda_{QCD} \gg m_Q v^2$

strong coupling pNRQCD

|||

Born-Oppenheimer EFT

Figures: Najjar, Bali, 2009

pNRQCD weak coupling regime $\Lambda_{QCD} \lesssim m_Q v^2 \ll m_Q v$

$$
\begin{aligned}
\mathcal{L}_{\rm pNRQCD} = \int d^3\mathbf{r}\, \mathrm{Tr}\Big\{ & \mathrm{S}^\dagger \left(i\partial_0 - h_s(\mathbf{r},\mathbf{p},\mathbf{P_R},\mathbf{S}_1,\mathbf{S}_2,\mu)\right)\mathrm{S} + \\
& + \mathrm{O}^\dagger \left(iD_0 - h_o(\mathbf{r},\mathbf{p},\mathbf{P_R},\mathbf{S}_1,\mathbf{S}_2,\mu)\right)\mathrm{O} \Big\} \\
& + V_A(r,\mu)\mathrm{Tr}\left\{ \mathrm{O}^\dagger \mathbf{r}\cdot g\mathbf{E}\,\mathrm{S} + \mathrm{S}^\dagger \mathbf{r}\cdot g\mathbf{E}\,\mathrm{O} \right\} + \\
& + \frac{V_B(r,\mu)}{2}\mathrm{Tr}\left\{ \mathrm{O}^\dagger \mathbf{r}\cdot g\mathbf{E}\,\mathrm{O} + \mathrm{O}^\dagger \mathrm{O}\mathbf{r}\cdot g\mathbf{E} \right\} + \mathcal{O}(\mathbf{r}^2, \frac{1}{m_Q})
\end{aligned}
$$

- $h_{s,o} = \frac{\mathbf{p}^2}{m_Q} + V_{s,o}(r,\mu) + \mathcal{O}(\frac{1}{m_Q})$, quantum mechanical Hamiltonians with scale dependent potentials calculable in pertubation theory in $\alpha_{\rm s}(m_Q v)$ and $1/m_Q$ ($V_s \simeq -4\alpha_{\rm s}/3r$, $V_o \simeq \alpha_{\rm s}/6r$)
- Spin symmetry holds in $h_{s,o}$ up to $\mathcal{O}(\frac{1}{m_Q^2})$
- S=S($\mathbf{r}$, $\mathbf{R}$, t), O=O($\mathbf{r}$, $\mathbf{R}$, t) are the color singlet/octet wave function fields
- $\mathbf{E}$=$\mathbf{E}$($\mathbf{R}$, t) is the chromoelectric field

Born-Oppenheimer EFT $m_Q v^2 \ll \quad \Lambda_{QCD} \lesssim mv$

$$L_{\rm pNRQCD} = \int d^3\mathbf{x}_1 \int d^3\mathbf{x}_2 \; S^\dagger \big(i\partial_0 - h_s(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{S}_1, \mathbf{S}_2)\big) S,$$

$$h_s(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{S}_1, \mathbf{S}_2) = \frac{\mathbf{p}_1^2}{2m_Q} + \frac{\mathbf{p}_2^2}{2m_Q} + V_s(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{S}_1, \mathbf{S}_2),$$

$$V_s = V_s^{(0)} + \frac{V_s^{(1)}}{m_Q} + \frac{V_s^{(2)}}{m_Q^2} + \cdots ,$$

All V_ss can be, and most of them have been, calculated on the lattice

- $V_s^{(0)}$ and $V_s^{(1)}$ are central Spin Symmetry holds
- $V_s^{(2)}$ contains spin and velocity dependent terms

Born-Oppenheimer EFT at LO

- Matching to NRQCD in the static limit $\Rightarrow$ $V_s^{(0)}$ is the ground state energy of two static color sources separated at a distance r
- Can be extracted from lattice calculations of the Wilson loop

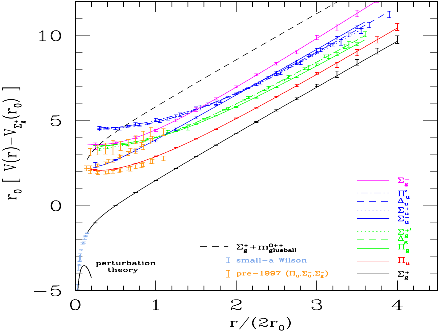


Figure: Meyer, Swanson, 2015

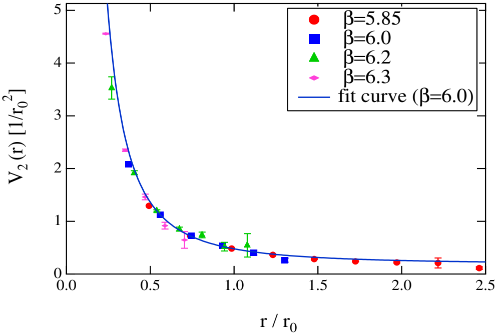
- Well fitted by the Cornell potential

$$V_s^{(0)} = V_{\Sigma_g^+}(r) \approx -\frac{k_g}{r} + \kappa r + E_g^{Q\bar{Q}} \quad , \; k_g = 0.489 \quad , \kappa = 0.187\,GeV^2$$

BOEFT beyond LO

Ex. at $\mathcal{O}(1/m_Q^2)$: $V_{L_2S_1}^{(1,1)}$ spin-orbit potential (Eichten, Feinberg, 79)

$$V_{L_2S_1}^{(1,1)}(r) = -\frac{c_F}{r^2} i\mathbf{r} \cdot \lim_{T\to\infty} \int_0^T dt\, t \,\langle\langle g\mathbf{B}(t,\mathbf{r}/2) \times g\mathbf{E}(0,-\mathbf{r}/2)\rangle\rangle$$



$V_2 = V_{L_2S_1}^{(1,1)}/c_F$, Koma, Koma, 09

pNRQCD strong coupling regime beyond LO

An example at $\mathcal{O}(1/m_Q^2)$: the $V_{L_2S_1}^{(1,1)}$ spin-orbit potential

- Short distance constraint: it must coincide with the perturbative evaluation,

$$V_{L_2S_1}^{(1,1)}(r) \sim c_F \frac{C_F \alpha_s}{r^3} \quad , \; r \to 0$$

(Gupta, Radford, 81)

- Long distance constraint: it must coincide with the QCD effective string theory result

$$V_{L_2S_1}^{(1,1)}(r) = -\frac{c_F g^2 \Lambda^2 \Lambda'}{\kappa \, r^2} \quad , \; r \to \infty$$

(Perez-Nadal, JS, 08)

$QQ/Q\bar{Q}$ + light quarks and gluons ($m_Q v$, $\Lambda_{QCD} \gg m_Q v^2$)

$$\mathcal{L}_{\rm HEH} = \sum_{\kappa^p} \Psi^\dagger_{\kappa^p} \left[i\partial_t - h_{\kappa^p} \right] \Psi_{\kappa^p}$$

$$h_{\kappa^p} = \frac{\boldsymbol{p}^2}{m_Q} + \frac{\boldsymbol{P}^2}{4m_Q} + V^{(0)}_{\kappa^p}(\boldsymbol{r}) + \frac{1}{m_Q} V^{(1)}_{\kappa^p}(\boldsymbol{r},\, \boldsymbol{p}) + \mathcal{O}\left(\frac{1}{m_Q^2}\right)$$

- LDF$\equiv$ light quarks + gluons, characterized by their quantum numbers (κ, p ...)
  - $\kappa \equiv$ total angular momentum, $p \equiv$ parity (P/CP)
  - Quantum numbers not explicitely displayed: baryon number (B), isospin (I), strangeness (S), principal quantum number
- $V^{(0)}_{\kappa^p}$, $V^{(1)}_{\kappa^p}$, ... must be calculated non-perturbatively
- A truncation of $\mathcal{L}_{\rm HEH}$ needed for practical calculations $\Longrightarrow$ keep a limited number of lower lying κ^p

- $V^{(0)}_{\kappa^p}$ is a $(2\kappa+1)\times(2\kappa+1)\times\mathbb{I}_{2\,Q_1}\times\mathbb{I}_{2\,Q_2}$ matrix, which can be decomposed into irreducible representations of $D_{\infty\,h}$, the symmetry group of a diatomic molecule

$$V^{(0)}_{\kappa^p}(\boldsymbol{r})=\sum_{\Lambda}V^{(0)}_{\kappa^p\Lambda}(r)\mathcal{P}_{\kappa\Lambda}$$

$\mathcal{P}_{\kappa\Lambda}$ projects onto LDF angular momenta $\pm\Lambda$ in the direction joining the two heavy quarks, $\Lambda=\kappa,\ \kappa-1,\ \dots,\ \kappa-[\kappa]$

$$\begin{aligned}
\mathcal{P}_{\frac{1}{2}\frac{1}{2}}&=\mathbb{I}_2^{\rm lq}\\
\mathcal{P}_{\frac{3}{2}\frac{1}{2}}&=\frac{9}{8}\mathbb{I}_4^{\rm lq}-\frac{1}{2}\left(\hat{\boldsymbol{r}}\cdot\boldsymbol{S}_{3/2}\right)^2\\
\mathcal{P}_{\frac{3}{2}\frac{3}{2}}&=-\frac{1}{8}\mathbb{I}_4^{\rm lq}+\frac{1}{2}\left(\hat{\boldsymbol{r}}\cdot\boldsymbol{S}_{3/2}\right)^2\\
\mathcal{P}_{10}&=\mathbb{I}_3^{\rm lq}-(\hat{\boldsymbol{r}}\cdot\boldsymbol{S}_1)^2\\
\mathcal{P}_{11}&=(\hat{\boldsymbol{r}}\cdot\boldsymbol{S}_1)^2\\
&\dots
\end{aligned}$$

- $V^{(1)}_{\kappa^p} = V^{(1)}_{\kappa^p \mathrm{SI}} + V^{(1)}_{\kappa^p \mathrm{SD}}$
- $V^{(1)}_{\kappa^p \mathrm{SI}}$ does not depend on the spin or orbital angular momentum of the heavy quarks $\Longrightarrow$ admits the same decomposition as $V^{(0)}_{\kappa^p}$
- $V^{(1)}_{\kappa^p \mathrm{SD}}$ depends on the spin and orbital angular momentum of the heavy quarks

$$V^{(1)}_{\kappa^p \mathrm{SD}}(\boldsymbol{r}) = \sum_{\Lambda\Lambda'} \mathcal{P}_{\kappa\Lambda} \left[V^{sa}_{\kappa^p \Lambda\Lambda'}(r) \boldsymbol{S}_{QQ} \cdot (\mathcal{P}_{10} \cdot \boldsymbol{S}_\kappa) + V^{sb}_{\kappa^p \Lambda\Lambda'}(r) \boldsymbol{S}_{QQ} \cdot (\mathcal{P}_{11} \cdot \boldsymbol{S}_\kappa) + V^{l}_{\kappa^p \Lambda\Lambda'}(r) \left(\boldsymbol{L}_{QQ} \cdot \boldsymbol{S}_\kappa \right) \right] \mathcal{P}_{\kappa\Lambda'}$$

$$2\boldsymbol{S}_{QQ} = \boldsymbol{\sigma}_{QQ} = \boldsymbol{\sigma}_{Q_1} \mathbb{I}_{2\,Q_2} + \mathbb{I}_{2\,Q_1} \boldsymbol{\sigma}_{Q_2} \quad , \quad \mathcal{P}^{ij}_{10} = \hat{\boldsymbol{r}}^i \hat{\boldsymbol{r}}^j \quad , \quad \mathcal{P}^{ij}_{11} = \delta^{ij} - \hat{\boldsymbol{r}}^i \hat{\boldsymbol{r}}^j$$

Matching to NRQCD

- Build an NRQCD operator with the quantum numbers of Ψ_{κ^p}

$$\mathcal{O}^{Q\bar{Q}}_{\kappa^p}(t,\, \boldsymbol{r},\, \boldsymbol{R}) = \chi_c^\top(t,\, \boldsymbol{x}_2)\phi(t,\, \boldsymbol{x}_2,\, \boldsymbol{R})\mathcal{Q}_{Q\bar{Q}\kappa^p}(t,\, \boldsymbol{R})\phi(t,\, \boldsymbol{R},\, \boldsymbol{x}_1)\psi(t,\, \boldsymbol{x}_1)$$
$$\mathcal{O}^{QQ}_{\kappa^p}(t,\, \boldsymbol{r},\, \boldsymbol{R}) = \psi^\top(t,\, \boldsymbol{x}_2)\phi^\top(t,\, \boldsymbol{R},\, \boldsymbol{x}_2)\mathcal{Q}_{QQ\kappa^p}(t,\, \boldsymbol{R})\phi(t,\, \boldsymbol{R},\, \boldsymbol{x}_1)\psi(t,\, \boldsymbol{x}_1)$$

- Examples:
  - Hybrid

$$\mathcal{Q}^{\alpha}_{1^{+-}}(t, \boldsymbol{x}) = \left(\boldsymbol{e}^{\dagger}_{\alpha} \cdot \boldsymbol{B}(t, \boldsymbol{x})\right)$$

  - $Q\bar{Q}q\bar{q}$ tetraquark

$$\mathcal{Q}_{0^{++}}(t, \boldsymbol{x}) = [\bar{q}(t, \boldsymbol{x})T^a q(t, \boldsymbol{x})]\, T^a$$

  - Doubly heavy baryons

$$\mathcal{Q}^{\alpha}_{(1/2)^{+}}(t, \boldsymbol{x}) = \underline{T}^l \left[P_+ q^l(t, \boldsymbol{x})\right]^{\alpha}$$

  - $QQ\bar{q}\bar{q}$ tetraquark

$$\mathcal{Q}_{0^-}(t, \boldsymbol{x}) = \left[\bar{q}(t, \boldsymbol{x})\underline{T}^l\gamma^2 q^*(t, \boldsymbol{x})\right] \underline{T}^l$$

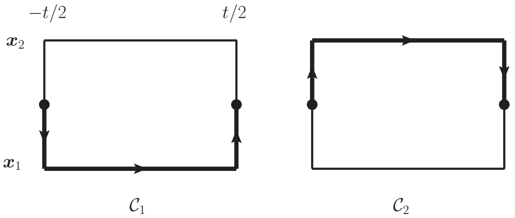
Matching to NRQCD

- Impose $\mathcal{O}^{h}_{\kappa^p}(t,\, \boldsymbol{r},\, \boldsymbol{R}) = \sqrt{Z_{h\kappa^p}}\Psi_{h\kappa^p}(t,\, \boldsymbol{r},\, \boldsymbol{R})\,, \quad h = QQ\,, Q\bar{Q}.$

$$\langle 0|\, T\{\mathcal{O}^{h}_{\kappa^p}(t/2)\mathcal{O}^{h\dagger}_{\kappa^p}(-t/2)\}|0\rangle = \sqrt{Z_{h\kappa^p}}\langle 0|\, T\{\Psi_{h\kappa^p}(t/2)\Psi^{\dagger}_{h\kappa^p}(-t/2)\}|0\rangle\sqrt{Z^{\dagger}_{h\kappa^p}}$$

  - Then at $\mathcal{O}(1)$

$$V^{(0)}_{h\kappa^p\Lambda}(\boldsymbol{r}) = \lim_{t\to\infty}\frac{i}{t}\log\left(\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\langle 1\rangle^{h\kappa^p}_{\Box}\right]\right)$$



- At $\mathcal{O}\left(\frac{1}{m_Q}\right)$, for instance,

$$V^{sb}_{\kappa^p\Lambda\Lambda'} = -\, c_F \lim_{t\to\infty} \sqrt{\frac{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\right] \mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda'}\right]}{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\langle 1\rangle^{h\kappa^p}_{\Box}\right] \mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda'}\langle 1\rangle^{h\kappa^p}_{\Box}\right]}}$$

$$\times \frac{\ln\left(\frac{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\langle 1\rangle^{h\kappa^p}_{\Box}\right]\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda'}\right]}{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda'}\langle 1\rangle^{h\kappa^p}_{\Box}\right]\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\right]}\right)}{2t\sinh\left(\ln\sqrt{\frac{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\langle 1\rangle^{h\kappa^p}_{\Box}\right]\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda'}\right]}{\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda'}\langle 1\rangle^{h\kappa^p}_{\Box}\right]\mathrm{Tr}\left[\mathcal{P}_{\kappa\Lambda}\right]}}\right)}$$

$$\times \int_{-t/2}^{t/2} dt' \frac{\mathrm{Tr}\left[\left(\boldsymbol{S}_\kappa \cdot \mathcal{P}_{11}\right) \cdot \left(\mathcal{P}_{\kappa\Lambda}\langle g\boldsymbol{B}(t',\boldsymbol{x}_1)\rangle^{h\kappa^p}_{\Box}\mathcal{P}_{\kappa\Lambda'}\right)\right]}{\mathrm{Tr}\left[\left(\boldsymbol{S}_\kappa \cdot \mathcal{P}^{\mathrm{c.r.}}_{11}\right) \cdot \left(\mathcal{P}_{\kappa\Lambda}\boldsymbol{S}_\kappa\mathcal{P}_{\kappa\Lambda'}\right)\right]}$$

Applications

- Doubly Heavy Baryons: QQq (JS, Tarrús Castellà, 20, 21)
- Hyperfine splittings of Heavy Quarkonium Hybrids: $Q\bar{Q}g$ (JS, Tomàs Valls, 23)

Disclamer:

- Interactions with heavy-light meson/baryon pairs neglected
- They have been recently addressed in the BOEFT for heavy quarkonium (Tarrús Castellà, 22; Bruschini, 23)
- It can be easily generalized to heavy exotics

Heavy Quarkonium Hybrids

- Spin average spectrum (BO approximation; Braaten, Langmack, Hudson Smith, 2014; Berwin, Brambilla, Tarrús Castellà, Vairo, 15; Oncala, Soto, 17)
  - Based on lattice data (Juge, Kuti, Morningstar, 02; Bali, Pineda, 03)
  - More recent and accurate lattice data available (Capitani, Philipsen, Reisinger, Riehl, Wagner, 18; Schlosser, Wagner, 21)
- Inclusive decay width to heavy quarkonium (Oncala, JS, 17)
  - Revisited in (Brambilla, Lai, Mohapatra, Vairo, 22)
    - Improved the $\Delta S = 0$ transitions $\mathcal{O}(1/m_Q^0)$
    - Calculated the $\Delta S = 1$ transitions $\mathcal{O}(1/m_Q^2)$
- Mixing with heavy quarkonium ($\mathcal{O}(1/m_Q)$, spin-dependent; Oncala, JS, 17)
  - Important effects when a quarkonium state and a hybrid state with the same quantum numbers have similar masses
  - Leads to violations of spin conservation
  - Likely to increase the estimates of the $\Delta S = 1$ transitions above

Results for charm

NL_J	w-f	$c\bar{c}$	Hybrid	$\mathcal{J}^{PC}$ $(S=0)$	$\mathcal{J}^{PC}$ $(S=1)$	Λ_η^ϵ
$1s$	S	3068		0^{-+}	1^{--}	Σ_g^+
$2s$	S	3678		0^{-+}	1^{--}	Σ_g^+
$3s$	S	4131		0^{-+}	1^{--}	Σ_g^+
$1p_0$, (H_3)	P^+		4486	0^{++}	1^{+-}	Σ_u^-
$4s$	S	4512		0^{-+}	1^{--}	Σ_g^+
$2p_0$	P^+		4920	0^{++}	1^{+-}	Σ_u^-
$3p_0$	P^+		5299	0^{++}	1^{+-}	Σ_u^-
$4p_0$	P^+		5642	0^{++}	1^{+-}	Σ_u^-
$1p$	S	3494		1^{+-}	$(0,1,2)^{++}$	Σ_g^+
$2p$	S	3968		1^{+-}	$(0,1,2)^{++}$	Σ_g^+
$1(s/d)_1$, (H_1)	$P^\pm$		4011	1^{--}	$(0,1,2)^{-+}$	$\Pi_u\Sigma_u^-$
$1p_1$, (H_2)	P^0		4145	1^{++}	$(0,1,2)^{+-}$	Π_u
$2(s/d)_1$	$P^\pm$		4355	1^{--}	$(0,1,2)^{-+}$	$\Pi_u\Sigma_u^-$
$3p$	S	4369		1^{+-}	$(0,1,2)^{++}$	Σ_g^+
$2p_1$	P^0		4511	1^{++}	$(0,1,2)^{+-}$	Π_u
$3(s/d)_1$	$P^\pm$		4692	1^{--}	$(0,1,2)^{-+}$	$\Pi_u\Sigma_u^-$
$4(s/d)_1$	$P^\pm$		4718	1^{--}	$(0,1,2)^{-+}$	$\Pi_u\Sigma_u^-$
$4p$	S	4727		1^{+-}	$(0,1,2)^{++}$	Σ_g^+
$3p_1$	P^0		4863	1^{++}	$(0,1,2)^{+-}$	Π_u
$5(s/d)_1$	$P^\pm$		5043	1^{--}	$(0,1,2)^{-+}$	$\Pi_u\Sigma_u^-$
$5p$	S	5055		1^{+-}	$(0,1,2)^{++}$	Σ_g^+
$1d$	S	3793		2^{-+}	$(1,2,3)^{--}$	Σ_g^+
$2d$	S	4210		2^{-+}	$(1,2,3)^{--}$	Σ_g^+
$1(p/f)_2$, (H_4)	$P^\pm$		4231	2^{++}	$(1,2,3)^{+-}$	$\Pi_u\Sigma_u^-$
$1d_2$, (H_5)	P^0		4334	2^{--}	$(1,2,3)^{-+}$	Π_u
$2(p/f)_2$	$P^\pm$		4563	2^{++}	$(1,2,3)^{+-}$	$\Pi_u\Sigma_u^-$
$3d$	S	4579		2^{-+}	$(1,2,3)^{--}$	Σ_g^+
$2d_2$	P^0		4693	2^{--}	$(1,2,3)^{-+}$	Π_u
$3(p/f)_2$	$P^\pm$		4886	2^{++}	$(1,2,3)^{+-}$	$\Pi_u\Sigma_u^-$
$4d$	S	4916		2^{-+}	$(1,2,3)^{--}$	Σ_g^+
$4(p/f)_2$	$P^\pm$		4923	2^{++}	$(1,2,3)^{+-}$	$\Pi_u\Sigma_u^-$
$3d_2$	P^0		5036	2^{--}	$(1,2,3)^{-+}$	Π_u

Results for bottom

NL_J	w-f	$b\bar{b}$	Hybrid	$\mathcal{J}^{PC}$ ($S=0$)	$\mathcal{J}^{PC}$ ($S=1$)	Λ_η^ϵ
1s	S	9442		0^{-+}	1^{--}	Σ_g^+
2s	S	10009		0^{-+}	1^{--}	Σ_g^+
3s	S	10356		0^{-+}	1^{--}	Σ_g^+
4s	S	10638		0^{-+}	1^{--}	Σ_g^+
$1p_0$, (H_3)	P^+		11011	0^{++}	1^{+-}	Σ_u^-
$2p_0$	P^+		11299	0^{++}	1^{+-}	Σ_u^-
$3p_0$	P^+		11551	0^{++}	1^{+-}	Σ_u^-
$4p_0$	P^+		11779	0^{++}	1^{+-}	Σ_u^-
1p	S	9908		1^{+-}	$(0,1,2)^{++}$	Σ_g^+
2p	S	10265		1^{+-}	$(0,1,2)^{++}$	Σ_g^+
3p	S	10553		1^{+-}	$(0,1,2)^{++}$	Σ_g^+
$1(s/d)_1$, (H_1)	$P^\pm$		10690	1^{--}	$(0,1,2)^{-+}$	$\Pi_u\Sigma_u^-$
$1p_1$, (H_2)	P^0		10761	1^{++}	$(0,1,2)^{+-}$	Π_u
4p	S	10806		1^{+-}	$(0,1,2)^{++}$	Σ_g^+
$2(s/d)_1$	$P^\pm$		10885	1^{--}	$(0,1,2)^{-+}$	$\Pi_u\Sigma_u^-$
$2p_1$	P^0		10970	1^{++}	$(0,1,2)^{+-}$	Π_u
5p	S	11035		1^{+-}	$(0,1,2)^{++}$	Σ_g^+
$3(s/d)_1$	$P^\pm$		11084	1^{--}	$(0,1,2)^{-+}$	$\Pi_u\Sigma_u^-$
$4(s/d)_1$	$P^\pm$		11156	1^{--}	$(0,1,2)^{-+}$	$\Pi_u\Sigma_u^-$
$3p_1$	P^0		11175	1^{++}	$(0,1,2)^{+-}$	Π_u
6p	S	11247		1^{+-}	$(0,1,2)^{++}$	Σ_g^+
$5(s/d)_1$	$P^\pm$		11284	1^{--}	$(0,1,2)^{-+}$	$\Pi_u\Sigma_u^-$
1d	S	10155		2^{-+}	$(1,2,3)^{--}$	Σ_g^+
2d	S	10454		2^{-+}	$(1,2,3)^{--}$	Σ_g^+
3d	S	10712		2^{-+}	$(1,2,3)^{--}$	Σ_g^+
$1(p/f)_2$, (H_4)	$P^\pm$		10819	2^{++}	$(1,2,3)^{+-}$	$\Pi_u\Sigma_u^-$
$1d_2$, (H_5)	P^0		10870	2^{--}	$(1,2,3)^{-+}$	Π_u
$4d$	S	10947		2^{-+}	$(1,2,3)^{--}$	Σ_g^+
$2(p/f)_2$	$P^\pm$		11005	2^{++}	$(1,2,3)^{+-}$	$\Pi_u\Sigma_u^-$
$2d_2$	P^0		11074	2^{--}	$(1,2,3)^{-+}$	Π_u
5d	S	11163		2^{-+}	$(1,2,3)^{--}$	Σ_g^+
$3(p/f)_2$	$P^\pm$		11197	2^{++}	$(1,2,3)^{+-}$	$\Pi_u\Sigma_u^-$
$3d_2$	P^0		11275	2^{--}	$(1,2,3)^{-+}$	Π_u
$4(p/f)_2$	$P^\pm$		11291	2^{++}	$(1,2,3)^{+-}$	$\Pi_u\Sigma_u^-$

The hyperfine splitting of heavy quarkonium hybrids

- The lower lying hybrid potentials correspond to $\kappa^p = 1^+$
- This leads to two spin projections on the direction Q-$\bar{Q}$, $\Lambda = 0, 1$
- The general formulas above imply that there are two independent potentials: $V^{sa}_{1^+11}(r)$, $V^{sb}_{1^+10}(r)$ (Oncala, JS, 17; Brambilla, Lai, Segovia, Tarrús Castellà, Vairo, 18, 19)
- No lattice calculation available for them. How to estimate them?
  - Brambilla et al. used weak coupling pNRQCD short distance expressions to estimate them which hold for $r \ll 1/\Lambda_{QCD}$. The $1/m_Q^2$ spin dependent potentials were also included.
  - We (JS, Tomàs Valls, 23) use an interpolation between the short distance expressions and long distance ones calculated in the QCD effective string theory (Pérez-Nadal, JS, 08; Brambilla, Groher, Martinez, Vairo, 14).
- Typical values: $\langle 1/r \rangle \sim 0.17 - 0.42$ GeV for $c\bar{c}g$, $\langle 1/r \rangle \sim 0.22 - 0.53$ GeV for $b\bar{b}g$ (Berwein, Brambilla, Tarrús Castellà, Vairo, 14)

Hyperfine Splittings

JS, 17

- They appear at $\mathcal{O}(1/m_Q)$ ($\mathcal{O}(1/m_Q^2)$) in hybrids (quarkonium)
- They lead to the following mass formulae

$$\frac{M_{1\,J+1} - M_{0\,J}}{M_{1\,J} - M_{0\,J}} = -J \qquad \frac{M_{1\,J-1} - M_{0\,J}}{M_{1\,J} - M_{0\,J}} = J + 1$$

$$\begin{aligned} (s/d)_1: &\quad M_{2^{-+}} + M_{0^{-+}} &= M_{1^{-+}} + M_{1^{--}} \\ p_1: &\quad M_{2^{+-}} + M_{0^{+-}} &= M_{1^{+-}} + M_{1^{++}} \\ (p/f)_2: &\quad M_{3^{+-}} + M_{1^{+-}} &= M_{2^{+-}} + M_{2^{++}} \\ d_2: &\quad M_{3^{-+}} + M_{1^{-+}} &= M_{2^{-+}} + M_{2^{--}} \end{aligned}$$

  - Consistent with the values of the lattice HSC
- Induces mixing between different hybrid states

The short distance potentials

- The two independent potentials are rearranged in

$$\begin{aligned} V_{hf}(r) &= \frac{1}{6} V^{sa}_{1^+11}(r) - \frac{1}{3} V^{sa}_{1^+10}(r) \quad (spin - spin) \\ V_{hf2}(r) &= -\frac{1}{2} \left(V^{sa}_{1^+11}(r) + V^{sb}_{1^+10}(r) \right) \quad (tensor) \end{aligned}$$

- At short distances:

$$V_{hf}(r)/m_Q = A + \mathcal{O}(r^2) \qquad , \qquad A \sim c_F \Lambda^2_{QCD}/m_Q$$
$$V_{hf2}(r)/m_Q = Br^2 + \mathcal{O}(r^4) \qquad , \qquad B \sim c_F \Lambda^4_{QCD}/m_Q$$

The short distance potentials depend on two unknown non-perturbative parameters.

The long distance potentials

- At long distances:

$$\frac{V^{sa}_{1^+11}(r)}{m_Q} = -\frac{2c_F\pi^2 g\Lambda'''}{m_Q\kappa r^3} \equiv V^{sa}_{ld}(r)$$

$$\frac{V^{sb}_{1^+10}(r)}{m_Q} = \mp\frac{c_F g\Lambda'\pi^2}{m_Q\sqrt{\pi\kappa}}\frac{1}{r^2} \equiv V^{sb}_{ld}(r)$$

  - κ is the string tension $\sim \Lambda^2_{QCD}$
  - $g\Lambda'$, $g\Lambda''' \sim \Lambda_{QCD}$ also enter the spin dependent potentials for heavy quarkonium
  - They can be extracted from lattice calculations of those potentials (Koma, Koma, 09)

$$g\Lambda' \sim -59\,\text{MeV} \quad ; \quad g\Lambda''' \sim \pm 230\,\text{MeV}$$

(Oncala, JS, 17)

The interpolating potentials

- We use the following interpolation

$$\frac{V_{hf}(r)}{m_Q} = \frac{A + \left(\frac{r}{r_0}\right)^2 \left(\frac{1}{6} V_{ld}^{sa}(r_0) - \frac{r}{3r_0} V_{ld}^{sb}(r_0)\right)}{1 + \left(\frac{r}{r_0}\right)^5}$$

$$\frac{V_{hf2}(r)}{m_Q} = \frac{Br^2 - \left(\frac{r}{r_0}\right)^5 \left(\frac{r_0}{2r} V_{ld}^{sa}(r_0) + \frac{1}{2} V_{ld}^{sb}(r_0)\right)}{1 + \left(\frac{r}{r_0}\right)^7} .$$

- $r_0 \simeq 3.96 \text{ GeV}^{-1} \sim 1/\Lambda_{QCD}$

Charmonium Hybrids HFS

- We use lattice data of the HSC for charmonium to fix A and B (relativistic charm, $m_\pi \sim 240$ MeV, Cheung, O'Hara, Moir, Peardon, Ryan, Thomas, Tims, 16)
- We focus on hyperfine splittings not on spin averages
- Same strategy as Brambilla et al. , 19
  - We have a 2 parameter fit and get $A = 0.115 \pm 0.034$ GeV, $B = 0.0038 \pm 0.0154$ GeV^3 with a $\chi^2/\mathrm{dof} \sim 0.64$
  - Brambilla et al. , 19 have an 8 parameter fit with a $\chi^2/\mathrm{dof} \sim 0.99$
  - Including long distance information from the QCD string improves the description of lattice data
- Once A and B are fixed, we can predict the bottomonium hyperfine splittings

Charmonium Hybrids HFS

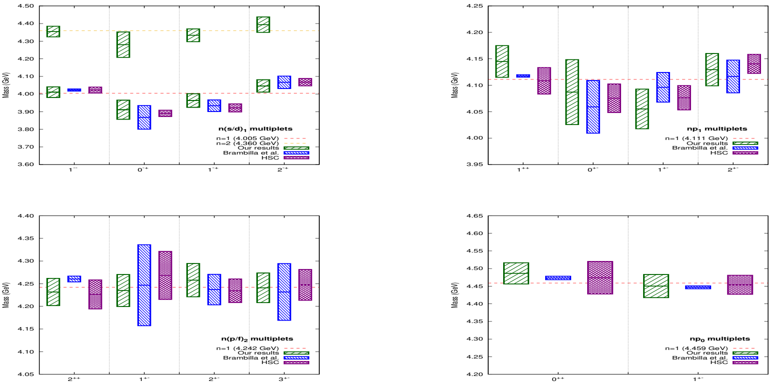


Figure: The spectrum of the lower-lying $n\,(s/d)_1$ (H_1), $n\,p_1$ (H_2), $n\,(p/f)_2$ (H_4) and $n\,p_0$ (H_3) charmonium hybrids

Bottomonium Hybrids HFS

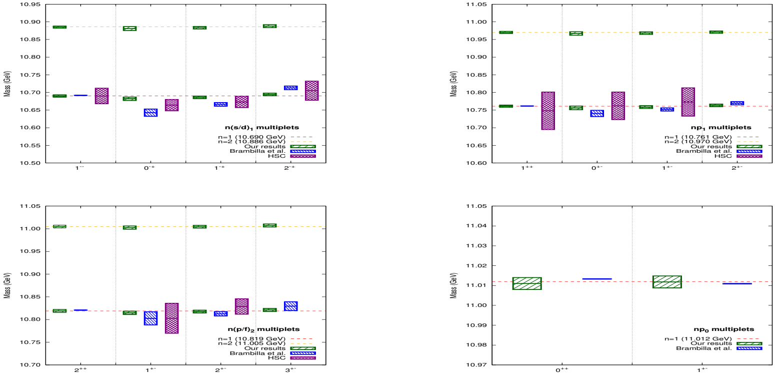
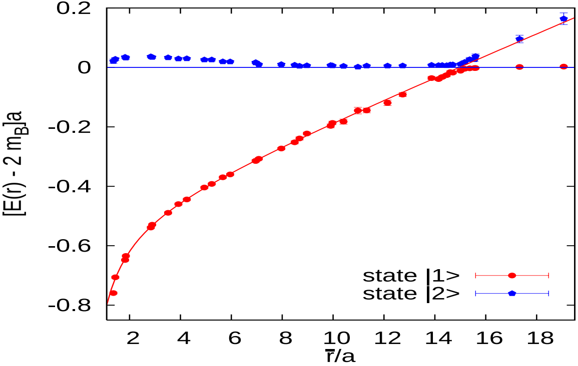


Figure: The spectrum of the lower-lying $n\,(s/d)_1$ (H_1), $n\,p_1$ (H_2), $n\,(p/f)_2$ (H_4) and $n\,p_0$ (H_3) bottomonium hybrids

Conclusions

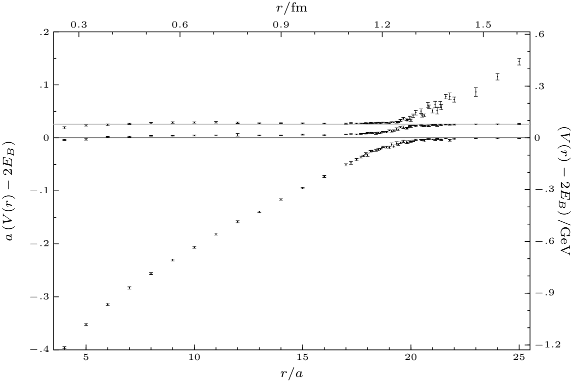
- The BOEFT provides a QCD based framework to address doubly-heavy exotics systematically
- It requires non-perturbative potentials as an input
- When those potentials are not available, a Cornell-like approach of interpolating between the short distance QCD (pNRQCD) calculation and a long distance string calculation appears to be promising

String breaking



Bali, Neff, Duessel, Lippert, Schilling, 2005

String breaking



Bulava, Hörz, Knechtli, Koch, Moir, Morningstar, Peardon, 2019

XYZ identification ∼ 2019

Resonance	J^{PC}	Assignement	Mass (MeV)	Observations
X(3823)	2^{--}	$1d$	3792	
X(3860)	$0 \text{ or } 2^{++}$	$2p$	3968	
X(3872)	1^{++}	$2p$	3967	
X(3915)	$0 \text{ or } 2^{++}$	$2p$	3968	
X(3940)	$?^{??}$	$2p$	3968	
Y(4008)	1^{--}	$1(s/d)_1$	4004	mixing
X(4140)	1^{++}	??	??	$1p_1$ does not decay to quarkonium
X(4160)	$?^{??}$	$1p_1$	4146	
Y(4220)	1^{--}	$2d$	4180	$Y(4260) \rightarrow Y(4220)$, mixing
X(4230)	1^{--}	$2d$	4180	$X(4230) = Y(4220)$, mixing
X(4350)	$?^{?+}$	$2(s/d)_1$ or 3p	4355 or 4369	
Y(4320)	1^{--}	$2(s/d)_1$	4366	mixing
Y(4360)	1^{--}	$2(s/d)_1$	4366	$Y(4360) = Y(4320)$?
X(4390)	1^{--}	$2(s/d)_1$	4366	$Y(4390) = Y(4360)$?
X(4500)	0^{++}	$1p_0$	4566	not enough mixing
Y(4630)	1^{--}	$3d$	4559	
Y(4660)	1^{--}	$3(s/d)_1$	4711	mixing
X(4700)	0^{++}	$4p$	4703	
$\Upsilon(10860)$	1^{--}	$5s$	10881	mixing
$\mathrm{Y_b}(10890)$	1^{--}	$2(s/d)_1$	10890	mixing
$\Upsilon(11020)$	1^{--}	$4d$	10942	

XYZ identification ~ 2024

Resonance	J^{PC}	Assignement	Mass (MeV)	Observations
$\psi_2(3823)$	2^{--}	$1d$	3792	
$\psi_3(3842)$	3^{--}	$1d$	3792	
$\chi_{c0}(3860)$	0^{++}	??	??	
$\chi_{c1}(3872)$	1^{++}	??	??	
$\chi_{c0}(3915)$	0^{++}	$2p$	3968	
$\chi_{c2}(3930)$	2^{++}	$2p$	3968	
$X(3940)$	$?^{??}$	$2p$	3968	
$\psi(4040)$	1^{--}	$1(s/d)_1$	4004	mixing
$\chi_{c1}(4140)$	1^{++}	$1p_1$	4146	not enough mixing
$X(4160)$	$?^{??}$	$1p_1$	4146	
$\psi(4230)$	1^{--}	$2d$	4180	mixing
$\chi_{c1}(4272)$	1^{++}	$3p$	4369	
$X(4350)$	$?^{?+}$	$2(s/d)_1$ or 3p	4355 or 4369	
$\psi(4360)$	1^{--}	$2(s/d)_1$	4366	mixing
$\psi(4415)$	1^{--}	$4s$	4513	
$\chi_{c0}(4500)$	0^{++}	$1p_0$	4566	not enough mixing
$X(4630)$	1^{--}	$3d$	4559	
$\psi(4660)$	1^{--}	$3(s/d)_1$	4711	mixing
$\chi_{c1}(4685)$	1^{++}	$4p$	4727	
$\chi_{c0}(4700)$	0^{++}	$4p$	4703	
$\Upsilon(10753)$	1^{--}	$3d$	10712	mixing
$\Upsilon(10860)$	1^{--}	$5s$	10881	mixing
$\Upsilon(11020)$	1^{--}	$4d$	10942	

1^{--} bottomonium spectrum $\sim$ 2019

NL_J	$\lambda = 0.6$	Hybrid %	PDG
$1s$	9.441	0	$\Upsilon(1S)$
$2s$	10.000	2	$\Upsilon(2S)$
$1d$	10.133	2	$\Upsilon(1D)$
$3s$	10.352	0	$\Upsilon(3S)$
$2d$	10.440	2	$\Upsilon(10.520)$? (Belle, 19)
$4s$	10.635	1	$\Upsilon(4S)$
$1(s/d)_1$	10.688	79	??
$3d$	10.713	56	$\Upsilon(10.750)$ (Belle, 19)
$5s$	10.881	17	$\Upsilon(10860)$
$2(s/d)_1$	10.886	75	$Y_b(10890)$
$4d$	10.942	11	$\Upsilon(11020)$

- The 17% hybrid component of $\Upsilon(10860)$ may explain the observed spin symmetry violating decays to h_b

1^{--} bottomonium spectrum $\sim$ 2024

NL_J	$\lambda = 0.6$	Hybrid %	PDG
$1s$	9.441	0	$\Upsilon(1S)$
$2s$	10.000	2	$\Upsilon(2S)$
$1d$	10.133	2	$\Upsilon(1D)$
$3s$	10.352	0	$\Upsilon(3S)$
$2d$	10.440	2	??
$4s$	10.635	1	$\Upsilon(4S)$
$1(s/d)_1$	10.688	79	??
$3d$	10.713	56	$\Upsilon(10.750)$
$5s$	10.881	17	$\Upsilon(10860)$
$2(s/d)_1$	10.886	75	??
$4d$	10.942	11	$\Upsilon(11020)$

- The 17% hybrid component of $\Upsilon(10860)$ may explain the observed spin symmetry violating decays to h_b