On exact controllability, optimal control and domain decomposition for degenerated wave equation on networks

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The 1-d model problem for a single interval

F. Alabau, P. Cannarsa and G.L SICON 2017

- We consider the 1-d-wave equation

$$u_{tt} - \big(a(x)u_x\big)_x = 0 \quad \text{in } \ ]0,\infty[\times]0,\ell[, \qquad \text{(PDE)}$$

where $a(x)$ satisfies (H1)

- $a \in \mathcal{C}([0,\ell]) \cap \mathcal{C}^1(]0,\ell])$ with:

$$\begin{cases} (i) & a(x) > 0 \ \ \forall x \in]0,\ell]\,, \ \ a(0) = 0\,, \\ (ii) & \mu_a := \sup_{0<x\leqslant\ell} \dfrac{x|a'(x)|}{a(x)} < 2\,, \ \ \text{and} \\ (iii) & a \in \mathcal{C}^{[\mu_a]}([0,\ell]), \end{cases} \qquad \text{(H1)}$$

where $[\cdot]$ stands for the integer part.

The 1-d model problem on a single interval

- We differentiate between two cases

$$\begin{cases} \mu_a \in [0,1) & \text{weak degeneration} \\ \mu_a \in [1,2) & \text{strong degeneration} \end{cases}$$

- We consider the control system

$$u_{tt} - \big(a(x)u_x\big)_x = 0 \quad \text{in } \;]0,\infty[\times]0,\ell[$$

with

$$\begin{cases} \text{boundary conditions } \; u(t,\ell) = f(t) \; \text{ and } \begin{cases} u(t,0) = 0 & \text{if } \; \mu_a \in [0,1[\\ \lim_{x\downarrow 0} a(x)\, u_x(t,x) = 0 & \text{if } \; \mu_a \in [1,2[\end{cases} & 0 < t < \infty \\ \text{initial conditions } \begin{cases} u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x) \end{cases} & x \in]0,\ell[. \end{cases} \tag{P}$$

Problem on a single interval: adjoint problem

- Lat a satisfy assumptions (H1), then we consider the adjoint problem

$$u_{tt} - \big(a(x)u_x\big)_x = 0 \quad \text{in } \;]0,\infty[\times]0,\ell[$$

with

$$\begin{cases} \text{boundary conditions } \; u(t,\ell) = 0 \; \text{ and } \; \begin{cases} u(t,0) = 0 & \text{if } \; \mu_a \in [0,\ell[\\ \lim_{x\downarrow 0} a(x)\, u_x(t,x) = 0 & \text{if } \; \mu_a \in [1,2[\end{cases} & 0 < t < \infty \\ \text{initial conditions } \; \begin{cases} u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x) \end{cases} & x \in]0,\ell[. \end{cases} \tag{adjP}$$

- **Theorem**: Let the minimal observation/control time be given by

$$T_a := \frac{4}{(2-\mu_a)\min\{1, a(1)\}} + 2\,\mu_a\sqrt{C_a},$$

where C_a is given by the Poincaré inequality. Then the system (adjP) is observable in time $T > T_a$.

Optimal control problem: penalization

We consider the final value optimal control problem

$$\min_{f\in L^2(0,T)} \int_0^T |f(t)|^2 dt + \frac{\kappa}{2}\left(\|u(T,\cdot)-z^0\|_H^2 + \|u_t(T,\cdot)-z^1\|_{V^*}^2\right)$$

subject to

$$u_{tt} - (au_x)_x = 0,\ (t,x)\in Q := (0,T)\times(0,\ell)$$
$$u(0,t) = 0,\ u(t,\ell) = f(t),\ t\in(0,T)$$
$$u(0,x) = u^0(x),\ u_t(0,x) = u^1(x),\ x\in(0,\ell).$$

The corresponding optimality system (in the strong formulation) is given by

$$u_{tt} - (au_x)_x = 0,\ p_{tt} - (ap_x)_x = 0,\ (t,x)\in Q$$
$$u(t,0) = 0,\ u(t,\ell) = p_x(t,\ell),\ p(t,0) = 0,\ p(t,\ell) = 0,\ t\in(0,T)$$
$$u(0,x) = u^0(x),\ u_t(0,x) = u^1(x), p(T,\cdot) = \kappa\mathcal{A}^{-1}(u_t(T,\cdot)-z^1),\ p_t(T,\cdot) = \kappa(u(T,\cdot)-z^0)\ x\in(0,\ell).$$

Limiting problem

We know from (ACL2017) that the system is exactly controllable (the adjoint observable) in time T_a. It follows then by a standard procedure that the optimal controls f^κ and the corresponding solutions $u(\cdot;\kappa), p(\cdot;\kappa)$ tend to the solution of the optimality system for the limiting norm-minimal controllability problem.

The corresponding optimality system (in the strong formulation) is given by

$$
\begin{aligned}
&u_{tt} - (au_x)_x = 0,\ p_{tt} - (ap_x)_x = 0,\ (x,t) \in Q \\
&u(t,0) = 0,\ u(\ell,t) = p_x(\ell,t),\ p(0,t) = 0,\ p(t,\ell) = 0,\ t \in (0,T) \\
&u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), p(\cdot,T) = p^0,\ p_t(\cdot,T) = p^1\ x \in (0,\ell),
\end{aligned}
$$

such that $(p^0, p^1) \in V \times H$ is given by the HUM eqation:

$$
< (p^0, p^1), (-z^1, z^0) >= \int_0^T |p_x(\ell,t)|^2 dt.
$$

In-span degeneration or two-link system

For simplicity, let c and d be a given pair of real numbers such that $0 \le c < 1 < d \le 2$. We set

$$\Omega_1 = (c, 1), \quad \Omega_2 = (1, d), \quad \Omega = (c, d), \quad \text{and} \quad \Omega_0 = \Omega \setminus \{1\} = \Omega_1 \cup \Omega_2.$$

Let $a : \overline{\Omega} \to \mathbb{R}$ be a given weight function with properties

(i) $a(1) = 0$, $a(x) > 0$ for all $x \in \Omega_0$, and there exists subintervals $(x_1^*, 1) \subset \Omega_1$ and $(1, x_2^*) \subset \Omega_2$ such that $a(\cdot)$ is monotonically decreasing on $(x_1^*, 1)$ and monotonically increasing on $(1, x_2^*)$;

(ii) $a \in C(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus \{1\})$;

(iii) $(\sqrt{a})_x \notin L^\infty(\Omega)$ whereas $(\sqrt{a})_x^{-1} \in L^\infty(\Omega)$.

In-span degeneration

We follow the article by P. Kogut, O. Kupenko and G.L MMAS22.

We are concerned with the following controlled system

$$\begin{aligned} & u_{tt} - (a(x)u_x)_x = 0 \quad \text{in} \quad (0,T)\times\Omega, \\ & u(t,c) = f_1(t), \quad u(t,d) = f_2(t) \quad \text{on} \quad (0,T), \\ & u(0,\cdot) = y_0, \quad u_t(0,\cdot) = y_1 \quad \text{in} \quad \Omega, \\ & f_1, f_2 \in \mathcal{F}_{ad} = L^2(0,T). \end{aligned}$$

$$\mu_{1,a} := \sup_{x\in\Omega_1} \frac{(1-x)|a'(x)|}{a(x)}, \qquad \mu_{2,a} := \sup_{x\in\Omega_2} \frac{(x-1)|a'(x)|}{a(x)}.$$

Here, u_0, and u_1 are given functions, and $\mathcal{F}_{ad}$ stands for the class of admissible controls.

Se also BAI Jinyan and CHAI Shugen J Syst Sci Complex (2023) 36: 656-671.

Transmission conditions: weak degeneration

Let $a : \overline{\Omega} \to \mathbb{R}$ be a weight function satisfying properties (i)–(iii) and $1/a \in L^1(\Omega)$. Then $H^1_a(\Omega)$ is continuously embedded into the class of absolutely continuous functions on $\overline{\Omega}$, so

$$\lim_{x \nearrow 1} y(x) = \lim_{x \searrow 1} y(x), \quad |y(1)| < +\infty, \quad \forall\, y \in H^1_a(\Omega),$$

$$\lim_{x \nearrow 1} \sqrt{a(x)}y(x) = \lim_{x \searrow 1} \sqrt{a(x)}y(x) = 0, \quad \forall\, y \in H^1_a(\Omega).$$

In addition, if y is an arbitrary element of the space

$$H^2_a(\Omega) := \left\{ y \in H^1_a(\Omega) \ : \ a y_x \in W^{1,2}(\Omega) \right\},$$

then the following transmission condition

$$\lim_{x \nearrow 1} a(x) y_x(x) = \lim_{x \searrow 1} a(x) y_x(x) = L, \quad \text{with} \ \ |L| < +\infty,$$

holds true.

Transmission conditions: strong degeneration

Let $a : \overline{\Omega} \to \mathbb{R}$ be a weight function satisfying properties (i)–(iii) and $1/a \notin L^1(\Omega)$. Then the following assertions hold true:

$$\exists\, x_i \in \Omega_i,\ i = 1, 2,\ \text{such that}\ \ y(x) = o\left(|x-1|^{-\frac{1}{2}}\right)\ \text{for a.a.}\ \ x \in (x_1, x_2),$$

$$\lim_{x \nearrow 1} \sqrt{a(x)}y(x) = \lim_{x \searrow 1} \sqrt{a(x)}y(x) = 0, \quad \forall\, y \in H^1_a(\Omega),$$

$$\lim_{x \nearrow 1} a(x)y_x(x) = \lim_{x \searrow 1} a(x)y_x(x) = 0, \quad \forall\, y \in H^2_a(\Omega),$$

where the small symbol o stands for the Landau asymptotic notation.

We have a Poincaré inequality in the weighted Sobolev space $H^1_{a,0}$:

$$\|y\|_{L^2(\Omega)} \leq C_P \|y\|_{H^1_a(\Omega)}, \quad \forall\, y \in H^1_{a,0}(\Omega),$$

Poincaré inequality: strong degeneracy

Remark: Note that for Neumann control at $x \in \{c, d\}$ and strong degeneration $\mu_{i,a} \in [1, 2)$, we have to take the Neumann homogeneous condition $\lim_{x \to 1} a(x)u_x(x) = 0$. Therefore, the classical Poincaré inequality (as in ACL17) doesn't hold. In that case, we need to resort a more general Poincaré inequality and work in the quotient space and set

$$H^1_a(c, d) = \{u \in L^2(c, d) | au_x \text{ is absolutely cont.}, \sqrt{(}a)u_x \in L^2(0, 1), \int_c^d u(x)dx = 0\}$$

Such a (sharp) Poincaré inequality has been proven by Chua and Wheeden in 2000.

Poincaré inequality for general boundary conditions

Theorem[ChuaWheeden00]:
Let $C_{c,d}$ be defined as

$$C_{a,d} = \frac{1}{d-c}\left(\sup_{x\in(c,d)}\left\{(d-x)^{\frac{1}{2}}\int_c^x (t-c)^2 a(t)^{-1}dt\right\}^{\frac{1}{2}} + \sup_{x\in(c,d)}\left\{(x-c)^{\frac{1}{2}}\int_x^d (d-t)^2 a(t)^{-1}dt\right\}^{\frac{1}{2}}\right).$$

Then the weighted Poincaré inequality:

$$\left(\int_c^d |f(x) - \frac{1}{d-c}\int_c^d f(s)ds|^2 dx\right)^{\frac{1}{2}} \leq 2C_{c,d}\left(\int_c^d a(x)|f'(x)|^2 dx\right)^{\frac{1}{2}}$$

holds if and only if $C_{c,d} < \infty$.
Indeed, for the coeffients $a(x)$ as used here $C < \infty$.

The Theorem hold for more general situations, where dx is replaced by a measure $\nu(x)$ and $p = q = 2$ is $1 \leq p \leq p < \infty$.

Regularity and multiplier identity

For any mild solution $u(t,x)$, we have that $u_t(\cdot,c) \in L^2(0,T)$ and $u_t(\cdot,d) \in L^2(0,T)$ for any $T > 0$, and

$$\int_0^T a(c) y_x^2(t,c)\, dt \leq \frac{1}{1-c}\left[6T + \frac{4}{\min\{1, a(c), a(d)\}}\right] E_y(0),$$
$$\int_0^T a(d) y_x^2(t,d)\, dt \leq \frac{1}{d-1}\left[6T + \frac{4}{\min\{1, a(c), a(d)\}}\right] E_y(0).$$

Moreover,

$$\begin{aligned}(1-c)\int_0^T a(c) y_x^2(t,c)\, dt + (d-1)\int_0^T a(d) y_x^2(t,d)\, dt \\ = 2\left[\int_{\Omega_0} (x-1) u_x(t,x) u_t(t,x)\, dx\right]_{t=0}^{t=T} \\ + \int_0^T \int_{\Omega_0} \left(u_t^2(t,x) + \left[1 - \frac{(x-1)a_x(x)}{a(x)}\right] a(x) u_x^2(t,x)\right) dx\, dt.\end{aligned}$$

Observability/Controllability

Let $a : \overline{\Omega} \to \mathbb{R}$ be a weight function satisfying properties (i)–(iii), and let u be a mild solution of adjoint problem. Then, for every $T > 0$, the estimate

$$
\begin{aligned}
&(1-c)\int_0^T a(c)y_x^2(t,c)\,dt + (d-1)\int_0^T a(d)y_x^2(t,d)\,dt \\
&\quad \geq \left[(2-\max\{\mu_{1,a},\mu_{2,a}\})\,T - \frac{4}{\min\{1,a(c),a(d)\}} - 2C_P\right] E_y(0) \quad \text{(InvIneq)}
\end{aligned}
$$

holds true, where C_P is the constant in the Poincaré inequality.

$$
T_a := \frac{1}{(2-\max\{\mu_{1,a},\mu_{2,a}\})}\left[\frac{4}{\min\{1,a(c),a(d)\}} + 2C_P\right]. \qquad \text{(ContTime)}
$$

Penalization & convergence

We consider the final value optimal control problem

$$\min_{f_i \in L^2(0,T)} \int_0^T \sum_{i=1}^{2} |f_i(t)|^2 dt + \frac{\kappa}{2} \sum_{i=1}^{2} \left(\|u_i(T,\cdot) - z_i^0\|_{H_i}^2 + \|u_{i,t}(T,\cdot) - z_i^1\|_{V_i^*}^2 \right)$$

OCP^κ

subject to

$$u_{i,tt} - (au_{i,x})_x = 0, \; (t,x) \in Q := (0,T) \times (0,\ell)$$

$$u_1(c,t) = f_1(t), \; u_2(t,d) = f_2(t), \; t \in (0,T)$$

$$u_i(0,x) = u_i^0(x), \; u_{i,t}(0,x) = u_i^1(x), \; x \in (0,\ell).$$

relaxing the controllability constraint

In principle one needs to localize the term $\|u_t(T) - z^1\|_{V^*}^2$ more carefully!

According to the global observability estimate, we can conclude that the solutions $u_i(\cdot,\cdot;\kappa), p_i(\cdot,\cdot;\kappa)$ of the corresponding optimality system, as $\kappa \to \infty$ tend to the solutions u_i, p_i of the optimality system for the constraints optimal control problem.

Optimality system for finite κ

The corresponding optimality system (in the strong formulation) is given by

$$
\begin{aligned}
&u_{i,tt} - (a u_{i,x})_x = 0,\ p_{i,tt} - (a p_{i,x})_x = 0,\ (t,x) \in Q_i \\
&u_1(t,c) = p_{1,x}(t,c),\ u_2(t,d) = p_{2,x}(t,d),\ t \in (0,T) \\
&u_1(t,1) = u_2(t,1),\ \lim_{x\to 1_-} a(x) u_{1,x}(t,x) = \lim_{x \to 1_+} a(x) u_{2,x}(t,x) \\
&u(0,x) = u^0(x),\ u_t(0,x) = u^1(x), \\
&p(T) = \kappa \mathcal{A}^{-1}(u_t(T,\cdot) - z^1),\ p_t(T,\cdot) = \kappa(u(T,\cdot) - z^0)\ x \in (0,\ell).
\end{aligned}
\qquad OS^\kappa
$$

The idea is to decompose the global optimality system OS^κ iteratively into local optimality systems $lOS_i^{n,\kappa}$ or the global optimal control problem OCP^κ into $lOCP_i^{n,\kappa}$ ones on Ω_i

Optimality system for $\kappa = \infty$

The corresponding optimality system (in the strong formulation) for $'\kappa = \infty'$ is given by

$$
\begin{aligned}
& u_{i,tt} - (a u_{i,x})_x = 0,\ p_{i,tt} - (a p_{i,x})_x = 0,\ (t,x) \in Q_i \\
& u_1(t,c) = p_{1,x}(t,c),\ u_2(t,d) = p_{2,x}(t,d),\ t \in (0,T) \\
& u_1(t,1) = u_2(t,1),\ \lim_{x \to 1_-} a(x) u_{1,x}(t,x) = \lim_{x \to 1_+} a(x) u_{2,x}(t,x) \\
& u(0,x) = u^0(x),\ u_t(0,x) = u^1(x), \\
& p(T) = p^0,\ p_t(T) = p^1, (p^0, p^1) \in V \times H \text{ s.t.} \\
& \langle (p^0, p^1), (-z^1, z^0) \rangle = \int_0^T \left(p_{1,x}(t,c)^2 + p_{2,x}(t,d)^2 \right) dt
\end{aligned}
\qquad OS^\infty
$$

What happens if in $OS^{n,\kappa}$ $\kappa \to \infty$ for each n and otherwise if in $lOS_i^{n,\kappa}$ $n \to \infty$ for fixed κ ?

Domain decomposition and OCPs

$$
\begin{array}{ccccccc}
lOS_i^{n,\kappa} & \xrightarrow{n\to\infty} & OSP^{\kappa} & & lOCP_i^{n,\kappa} & \xrightarrow{n\to\infty} & OCP^{\kappa} \\
\Big\downarrow{\scriptstyle \kappa\to\infty} & & \Big\downarrow{\scriptstyle \kappa\to\infty} & & \Big\downarrow{\scriptstyle \kappa\to\infty} & & \Big\downarrow{\scriptstyle \kappa\to\infty} \\
lOS_i^{n} & \xrightarrow{n\to\infty} & OS^{\infty} & & lOCP_i^{n} & \xrightarrow{n\to\infty} & OCP^{\infty}
\end{array}
$$

What to do first? First optimize or decompose? First penalize or decompose?
And then the same questions with respect to discretization.....

First decompose then optimize or first optimize then decompopse?: virtual controls

$$\min_{f,g,u,z} \max_{\lambda} J(f,u,z,g;\lambda)^{\kappa,\rho} = \frac{1}{2}\int_0^T \sum_{i=1}^{2} |f_i(t)|^2 dt + \frac{\kappa}{2}\sum_{i=1}^{2}\Big(\|u_i(T,\cdot) - z_i^0\|^2_{H_i} + \|u_{i,t}(T,\cdot) - z_i^1\|^2_{V_i^*}\Big)$$

$$+\sum_{i=1}^{2}\int_0^T \lambda_i((u_i(t,1)-z) + \frac{\rho}{2}(u_i(t,1)-z)^2 dt$$

subject to

$$u_{i,tt} - (au_{i,x})_x = 0,\ (t,x) \in Q := (0,T)\times(0,\ell)$$

$$u_1(c,t) = f_1(t),\ u_2(t,d) = f_2(t),\ t \in (0,T)$$

$$u_{i,x}(t,1) = g(t),$$

$$u_i(0,x) = u_i^0(x),\ u_{i,t}(0,x) = u_i^1(x),\ x \in (0,\ell).$$

g is called virtual control

This problems can be seen as a relaxation of an exact synchronization problem at the interface

First decompose then optimize or first optimize then decompopse?: further relaxation

$$\min_{f,g,u,z,q} \max_{\lambda,\eta} \mathcal{L}(f,u,g,z,q;\lambda,\eta)^{\kappa,\rho,\sigma} = \frac{1}{2}\int_0^T \sum_{i=1}^{2} |f_i(t)|^2 dt + \frac{\kappa}{2}\sum_{i=1}^{2}\left(\|u_i(T,\cdot) - z_i^0\|_{H_i}^2 + \|u_{i,t}(T,\cdot) - z_i^1\|_{V_i^*}^2\right)$$

$$+ \sum_{i=1}^{2}\int_0^T \eta_i(g_i - z) + \frac{\sigma}{2}(g_i - z)^2 dt + \sum_{i=1}^{2}\int_0^T \lambda_i((u_i(t,1) - q) + \frac{\rho}{2}(u_i(t,1) - q)^2 dt$$

subject to

$$u_{i,tt} - (a u_{i,x})_x = 0,\ (t,x) \in Q := (0,T) \times (0,\ell)$$
$$u_1(c,t) = f_1(t),\ u_2(t,d) = f_2(t),\ t \in (0,T)$$
$$u_{i,x}(t,1) = g_i(t),$$
$$u_i(0,x) = u_i^0(x),\ u_{i,t}(0,x) = u_i^1(x),\ x \in (0,\ell).$$

We use the fractional step Uzawa-type saddle-point algorithm of Glowinski-LeTallec 89 (ALG3), as we did for parabolic problems with M.J. Gander 2024

Decomposed optimality system

The corresponding optimality system (in the strong formulation) for given κ is given by

$$
\begin{aligned}
&u_{i,tt}^{n+1} - (a u_{i,x}^{n+1})_x = 0,\ p_{i,tt}^{n+1} - (a p_{i,x}^{n+1})_x = 0,\ (t,x) \in Q_i \\
&u_i(t,v_i) = p_{i,x}(t,v_i),\ p_i^{n+1}(t,v_i) = 0, \\
&\epsilon_{i1} \lim_{x\to 1} a(x) u_{i,x}^{n+1}(t,x) + \beta p_i^{n+1}(t,1) = \lambda_i^n(t), \\
&\epsilon_{i1} \lim_{x\to 1} a(x) p_{i,x}^{n+1}(t,x) - \beta p_i^{n+1}(t,1) = \mu_i^n(t),\ t \in (0,T) \\
&u_i(0,x)^{n+1} = 0,\ u_{i,t}(0,x)^{n+1} = 0, \\
&p_i(T,x)^{n+1} = p_i^0(x),\ p_{i,t}^{n+1}(T,x) = p_i^1(x),\ x \in \Omega_i,
\end{aligned}
$$

with the iteration history

$$
\begin{aligned}
\lambda_i(t)^n &:= \beta \epsilon_{j1} p_j^n(t,1) - \epsilon_{j1} \lim_{x\to 1} a(x) p_j^n(t,x) \\
\mu_i(t)^n &:= -\beta \epsilon_{j1} u_j^n(t,1) - \epsilon_{j1} \lim_{x\to 1} a(x) u_j^n(t,x), \\
&\quad i = 1,2,\ t \in (0,T)
\end{aligned}
$$

Domain decomposition: the local problem

We consider the local final value optimal control problem $(v_1 := c, v_2 = d, v_0 = 1, \epsilon_{i1} = (-1)^i, \epsilon_{i0} = (-1)^{i+1})$ for the cost

$$J_i^\kappa(f_i, g_i, u_i) := \int_0^T |f_i(t)|^2 + \frac{1}{\beta}|g_i(t)|^2 dt + \frac{1}{2\beta}\int_0^T \left(\epsilon_{i1}\beta u_i(t,1) + \mu_i(t)|^2\right) dt$$
$$+ \frac{\kappa}{2}\left\|u_i(T) - z_i^0\|_H^2 + \|u_{i,t}(T) - z_i^1\|_{V_i^*}^2\right)$$

$$\min_{f_i \in L^2(0,T)^2, u_i} J_i^\kappa(f_i, g_i, u_i)$$

subject to

$$u_{i,tt} - (au_{i,x})_x = 0,\ (t,x) \in Q := (0,T) \times \Omega_i$$
$$u_i(v_i, t) = f_{1,i}(t),\ \lim_{x \to 1} \epsilon_{i1} a(x) u_{i,x}(t,x) = g_i(t) + \lambda_i(t),\ t \in (0,T)$$
$$u_i(0,x) = u_i^0(x),\ u_{i,t}(0,x) = u_i^1(x),\ x \in \Omega_i.$$

Domain decomposition: the local problem

At a given iteration index n, we consider the local final value optimal control problem ($v_1 := c, v_2 = d, v_0 = 1, \epsilon_{i1} = (-1)^i, \epsilon_{i0} = (-1)^{i+1}$) for the cost

$$J_i^{n,\kappa}(f_i, g_i, u_i) := \int_0^T |f_{1,i}(t)|^2 + \frac{1}{\beta}|g_i(t)|^2 dt + \frac{1}{2\beta}\int_0^T \left(\epsilon_{i1}\beta u_i^{n+1}(t,1) + \mu_i^n(t)|^2\right) dt$$

$$+ \frac{\kappa}{2}\left(\|u_i^{n+1}(T) - z_i^0\|_{H_i}^2 + \|u_i^{n+1} i, t(T) - z_i^1\|_{V_i^*}^2\right)$$

$$\min_{f_i \in L^2(0,T), u_i} J_i^{n,\kappa}(f_i, g_i, u_i) \qquad (lOCP_i^{n,\kappa})$$

subject to

$$u_{i,tt}^{n+1} - (a u_{i,x}^{n+1})_x = 0,\ (t,x) \in Q := (0,T) \times \Omega_i$$

$$u_i^{n+1}(v_i, t) = f_i(t),\ \epsilon_{i1} \lim_{x \to 1} a(x) u_{i,x}^{n+1}(t,x) = g_i(t) + \lambda_i(t)^n,\ t \in (0,T)$$

$$u_i^{n+1}(0) = 0,\ u_{i,t}^{n+1}(0) = 0,\ x \in \Omega_i.$$

Convergence: $(lOCP_i^{n,\kappa}) \rightarrow (OCP^{\kappa})$

It has been shown in G.L. SICON99 for the non-degenerated wave equation that, for a given κ, the solutions $(u_i^{n+1}(\cdot;\kappa), p_i^{n+1}(\cdot;\kappa))$ of the optimnality system for $(lOCP_i^{n,\kappa})$ converge, as $n \to \infty$, to the solutions $(u_i(\cdot;\kappa), p_i(\cdot;\kappa))$ of the optimality system for the global optimal control problem (OCP^{κ}).
The proof can be extended to the degenerated wave equation.

Indeed,

$$(u_i^n(\cdot;\kappa), u_{i,t}^n(\cdot;\kappa))) \to (u_i(\cdot;\kappa), u_{i,t}(\cdot;\kappa)) \quad \text{in } C([0,T]; H_i \times V_i^*), i = 1,2$$

$$(p_i^n(\cdot;\kappa), p_{i,t}^n(\cdot;\kappa))) \to (p_i(\cdot;\kappa, p_{i,t}(\cdot;\kappa)) \quad \text{in } C([0,T]; V_i \times H_i), i = 1,2$$

$$p_i^n(\cdot, v_i;\kappa) \to p_i(\cdot, v_i;\kappa) \quad \text{in } L^2(0,T).$$

Relation between OCPs and DDM

$$
\begin{array}{ccc}
lOCP_i^{n,\kappa} & \xrightarrow{n\to\infty} & OCP^{\kappa} \\
\kappa\to\infty \Big\downarrow & & \Big\downarrow \kappa\to\infty \\
lOCP_i^{n} & \xrightarrow{n\to\infty} & OCP^{\infty}
\end{array}
$$

G.L SICON99

J. Lagnese and G.L SICON99

Results for non-degenrate wave equation

Results for the degenrate wave equation 2024

Also: First optimize then decompose or first decompose then optimize!
Discussion with M. Gander 2024

Convergence: $(lOCP_i^{n,\kappa}) \to (lOCP_i^n)$

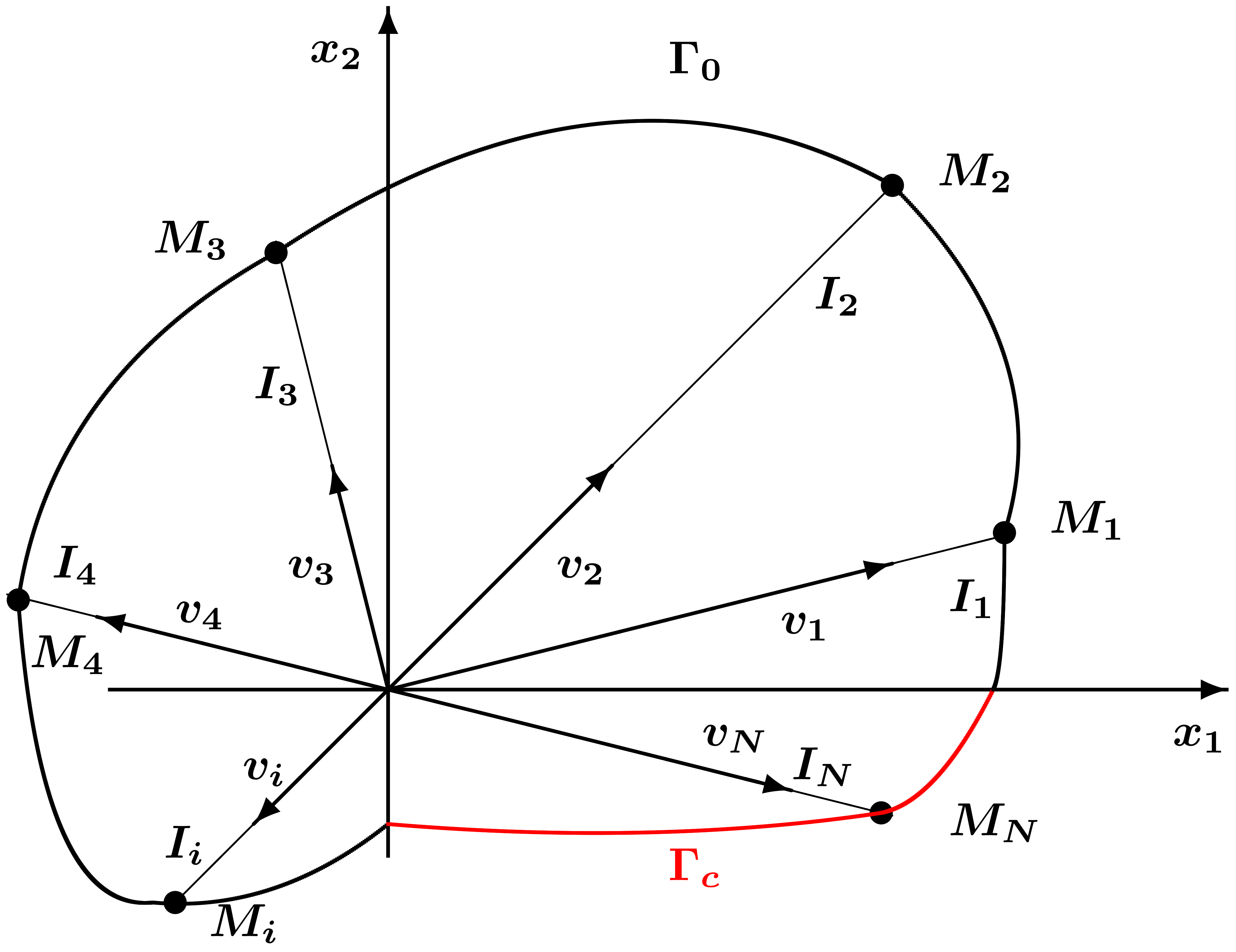
We need a local controllability result:

$$
\begin{aligned}
& u_{i,tt}^{n+1} - (a u_{i,x}^{n+1})_x = 0, \ (t,x) \in Q := (0,T) \times \Omega_i \\
& u_i^{n+1}(v_i, t) = f_i(t), \ \epsilon_{i1} \lim_{x \to 1} a(x) u_{i,x}^{n+1}(t,x) = g_i(t) + \lambda_i(t)^n, \ t \in (0,T) \\
& u_i^{n+1}(0) = 0, \ u_{i,t}^{n+1}(0) = 0, \ x \in \Omega_i. \\
& u_i^{n+1}(T) = z_i^0, \ u_i^{n+1}(T) = z^1, \ x \in \Omega_i.
\end{aligned}
$$

... and, in fact, we do have a corresponding observability inequality!

The proof then follows as in the article Lagnese and G.L. SICON99.

Problem on a planar network: singular measures



P.I. Kogut, O. Kupenko and G.L., PAFA 2022

We consider a planar star graph.
To I_i we associate a singular measure μ_i s.t. that μ_i is uniformly distributed on I_i and coincides with Lebesgue measure $\mathcal{L}^1$.
Setting $d\mu = \sum_{i=1}^N d\mu_i$,
we see that μ is a singular measure with respect to the Lebesgue measure $\mathcal{L}^2$, and $\mu\left(\Omega \setminus \cup_{i=1}^N I_i\right) = 0$.

I_i can be parametrized as a function of its length by means of the function $z_i : [0, \ell_i] \to I_i$, i.e.,

$$z_i(\xi) = \xi \frac{v_i}{|v_i|_{\mathbb{R}^2}}, \quad \forall\, \xi \in [0, \ell_i], \quad |z_i(\xi)| = \sqrt{z_{i,1}^2 + z_{i,2}^2} = \xi, \text{ and } z_i(\ell_i) = M_i.$$

Star graph representation via singular measures

We consider the problem in 2-d as follows

$$\begin{aligned}
&u_{tt} - \mathrm{div}^\mu(a\nabla^\mu u) + qu = 0 \quad \text{in } (0,T)\times\Omega,\\
&u(t,M_i) = f_i(t) \quad \text{for a.a. } t\in(0,T \text{ and } i=1,\dots,N-1,\\
&u(0,x) = y_0(x), \quad u_t(0,x) = y_1(x) \quad \text{for } \mu\text{-a.a. } x\in\Omega,
\end{aligned} \tag{2-d-P}$$

where (y_0, y_1) is a given initial state and ∇^μ the tangential gradient, div^μ the divergence wrt μ.

Local interpretation: If $u \in W^{1,2}_{a,0}(\Omega,\Gamma_0,d\mu)$ then its restriction $u_i = u|_{I_i} \in H^1_{a_i}$-function of a single variable. Namely,

$$u_i \in H^1_{a_i,0}(0,\ell_i),\ i=1,\dots,N-1, \quad u_N \in H^1_{a_N}(0,\ell_N),$$

$$\frac{du_i}{d\xi} = \left(z, \frac{v_i}{|v_i|}\right)\Big|_{x=\xi\frac{v_i}{|v_i|}} \text{ for a.a. } \xi\in I_i, \text{ i=1,...,N}, \quad \forall\, z\in\Gamma^\mu(u),$$

where $a_i = a\left(\xi\frac{v_i}{|v_i|}\right)$, and $\frac{du_i}{d\xi}$ stands for the weak derivative of $u_i = u\left(\xi\frac{v_i}{|v_i|}\right)$.

Problem on a planar network: observation inequality

Theorem: (KKL-PAFA22) Let $a : \Omega \to \mathbb{R}$ be a given weight function, and let u be a mild solution. Then, for every $T > 0$, the estimate

$$\sum_{i=1}^{N} \ell_i a(M_i) \int_0^T \left| \frac{\partial u(t, M_i)}{\partial v_i} \right|^2 dt \geq C^* E_u(y_0, y_1, 0),$$

holds true with

$$\begin{aligned} C^* &= (2 - \max\{\eta_{1,a}, \dots, \eta_{N,a}\})\, T \\ &\quad - 4 \sum_{i=1}^{N} \max\left\{1, \frac{\ell_i^2}{a(M_i)}\right\} - 2 \max\{\eta_{1,a}, \dots, \eta_{N,a}\} \sum_{i=1}^{N} \sqrt{C_{i,a}} \end{aligned}$$

and with $C_{i,a}$ given by

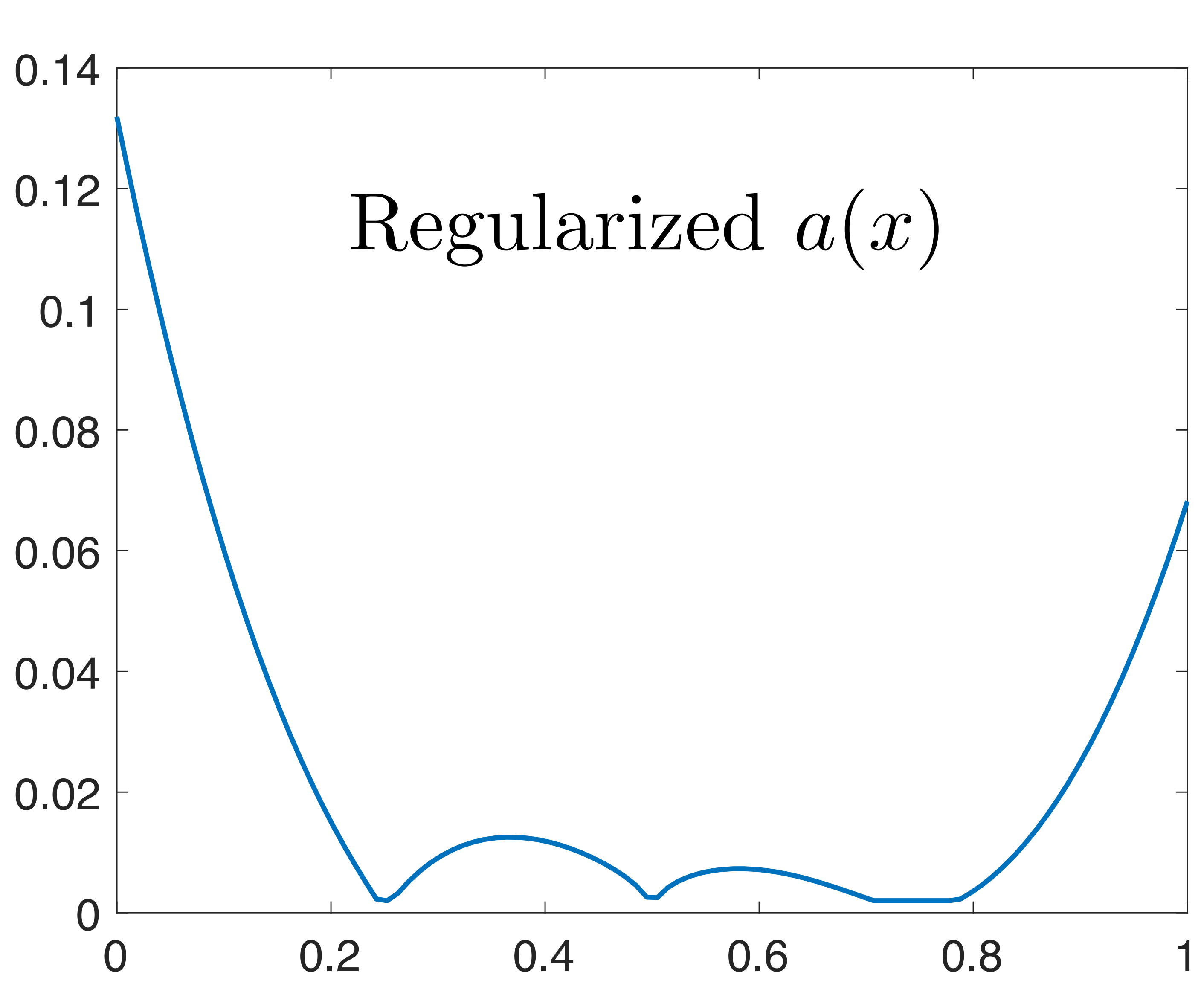
$$C_{i,a} = \frac{\ell_i^2}{a(M_i)} \min\left\{4, \frac{1}{2 - \eta_{i,a}}\right\}, \quad \forall\, i = 1, \dots, N.$$

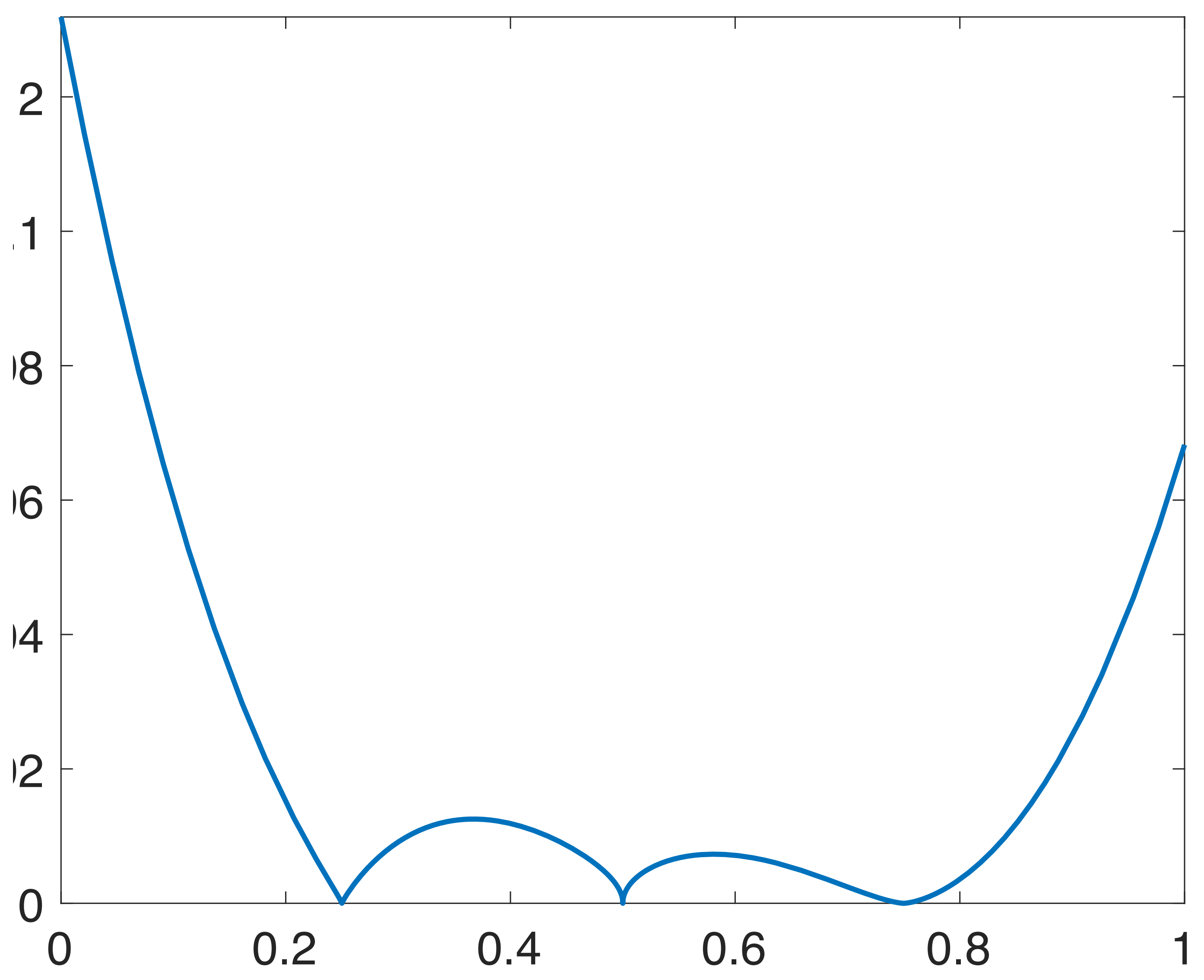
Open questions: damage sensitivity

- What happens if there are more points of degeneracy?

...and if there are, can we handle different degrees of degeneracy?

- How about the sensitivity with respect to
  - the α_is,
  - the locations x_i?
  - shape/topological variations for





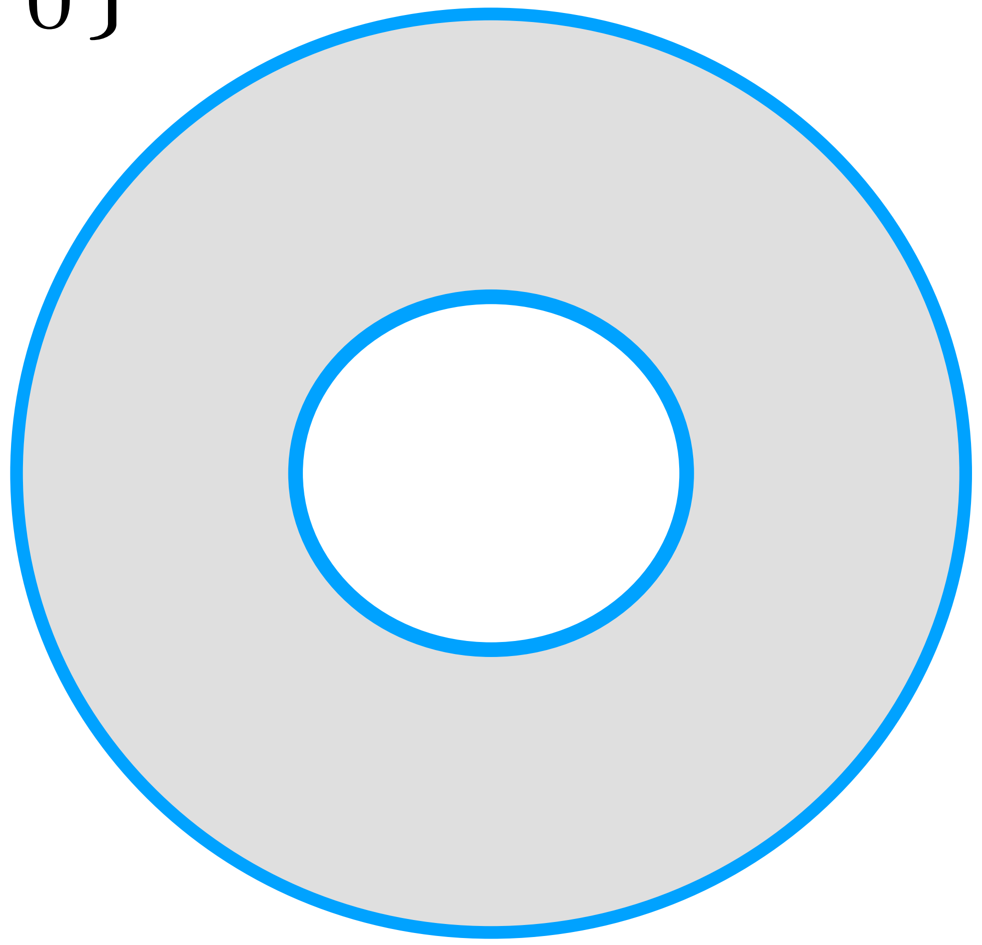
$$a(x) = \chi_\omega(x)a_1(x) + (1 - \chi(x)a_0(x)$$

$$a(x) = |x - x_0|^{\alpha_0}|x - x_1|^{\alpha_1}|x - x_2|^{\alpha_2}$$

Open questions: higher dimensions?

• What can be done for the wave equation on a, say, ring-like domain, where the coefficient degenerates at the inner circle? I.e. $a(x) = ((x_1^2 + x_2^2) - r_0^2)^{\frac{\alpha}{2}}$ and $\Omega = \{x \in \mathbb{R} : r_0 \leq \|x\| \leq R\}$, $\Gamma_1 = \{x : \|x\| = R\}$, $\Gamma_0 = \{x : \|x\| = r_0\}$

$$u_{tt} - \operatorname{div}(a(x)\nabla u) = 0, \quad \text{in } Q$$
$$u = f \text{ on } (0,T) \times \Gamma_1, \; u = 0 \text{ on } (0,T) \times \Gamma_0$$
$$u(0) = u^0, \; u_t(0) = u^1 \text{ in } \Omega.$$



• What if the degeneration is strong: $\alpha \to 2$? There is a strong connection with problem of cloaking! (Uhlmann, Lassas,.....G.L. et al.)

Open questions: damage evolution

- Evolution of internal damage: we consider

$$min J^{\kappa,\nu}(u,a,f) := \int_0^T f^2 dt + \frac{\nu}{2}\int_0^T\int_0^\ell |a-1|^2 dxdt + \frac{\kappa}{2}\left(\|u(0)-u^0\|_V^2 + \|u_t(0)-u^1\|_H^2\right)$$

s.t.

$$u_{tt} - ((a(t,x)u_x)x = 0,\ (t,x) \in Q$$

$$a_t = \omega a_{xx} - \Phi(\int_0^\ell \chi(x,s)u_x(t,s)^2 ds) - \gamma)_+,\ t \in (0,T)$$

$$u(t,0) = 0,\ u(t,\ell) = f(t),\ t \in (0,T)$$

$$a(t,0) = 1,\ a(t,\ell) = 1, t \in (0,T)$$

$$u(0,x) = u^0(x),\ u_t(0,x) = u^1(x),\ x \in (0,\ell),$$

$$a(0,x) = 1,\ x \in (0,\ell)$$

$$0 \le a(t,x) \le 1,\ (t,x) \in Q.$$

This, or ODE variants of the damage evolution equation my lead to non-local in space and time coefficients.

There is a lot of literature on damage evolution: Fremont, Kuttler and Shillor 1999, Bouchitté and Roubicek 2007 but nothing on control!

Thank you for your attention!