Global L^p Carleman estimates and applications

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Goal of the talk.

Discuss a global L^p Carleman estimates for the Laplace operator.

Keywords.
- Carleman estimates
- Laplace operator
- L^p estimates (Stein – Tomas thm)

I Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^d$ $(d \geq 3)$

$$\begin{cases} -\Delta u = f & \text{in } \Omega. \\ u = g & \text{on } \partial\Omega. \end{cases}$$

. $f \in H^{-1}(\Omega)$ & $g \in H^{1/2}(\partial\Omega) \Rightarrow u \in H^1(\Omega)$.

$f \in H^{-1}(\Omega)$? $\quad f = f_2 + f_{(2_*)'} + \operatorname{div} F$

with $f_2 \in L^2(\Omega)$, $\quad F \in L^2(\Omega; \mathbb{R}^d)$

$f_{(2_*)'} \in L^{(2_*)'}(\Omega)$

(Recall $H^1 \hookrightarrow L^{2*}$, $2* = \dfrac{2d}{d-2}$

$L^{(2*)'} \hookrightarrow H^{-1}$ $\quad (2*)' = \dfrac{2d}{d+2}$.)

Goal: Allow a source term $f = f_2 + f_{(2^*)'} + \mathrm{div}\, F$

with $f_2 \in L^2(\Omega)$, $f_{(2^*)'} \in L^{(2^*)'}(\Omega)$, $F \in L^2(\Omega; \mathbb{R}^d)$.

On u: Since $u \in H^1(\Omega)$, we want to estimate

$u \in L^2(\Omega)$, $\nabla u \in L^2(\Omega; \mathbb{R}^d)$ and $u \in L^{2^*}(\Omega)$.

$\rightsquigarrow$ Interest to develop L^p estimates

(when L^p estimates are better than the ones obtained by Sobolev's embedding).

Lower order perturbations of the Laplacian

$$\Delta u = \underbrace{Vu + W_1 \cdot Du + \operatorname{div}(W_2 u)}_{\text{Lower order if this is better than } H^{-1} \text{ for } u \in H^1(\Omega)}.$$

Critical scales.

$$V \in L^{d/2^+}(\Omega).$$

$$W_1 \in L^d(\Omega; \mathbb{R}^d).$$

$$W_2 \in L^d(\Omega; \mathbb{R}^d).$$

Unique continuation results for

$$-\Delta u = Vu + W_1 \cdot \nabla u + \operatorname{div}(W_2 u).$$

valid for $V \in L^{d/2^+}(\Omega)$, $W_1 \in L^{d^+}(\Omega;\mathbb{R}^d)$, $W_2 \in L^{d^+}(\Omega;\mathbb{R}^d)$.

$\rightsquigarrow$ [Koch - Tataru 2001.]

Key elements: . L^p Carleman estimates. [Jerison - Kenig 1985]

[Barcelo - Kenig - Ruiz Sogge 1987, 88].

....

. Wolff's argument [Wolff 1992]

$\simeq$ Choose appropriate weights depending on the solution.

Rk. Even strong unique continuation holds.

Rk There are counterexamples to unique continuation if $V \in L^p$, with $p < \frac{d}{2}$.

[Koch - Tataru 2002]

However Precise quantifications of smallness with respect to the norms of V, W_1, W_2 in sharp classes are, to our knowledge, missing.

L^2 Carleman estimates $\quad \Omega \subset \mathbb{R}^d$, $d \geq 3$, $\omega \subset \Omega$.

Let $\varphi \in C^3(\overline{\Omega})$ s.t. $\varphi = 0$ on $\partial\Omega$, $\partial_n \varphi < 0$ on $\partial\Omega$

and $\exists \alpha, \beta > 0$, $\inf_{\Omega \setminus \overline{\omega}} |\nabla \varphi| \geq \alpha$

and $\forall x \in \overline{\Omega \setminus \omega}$, $\forall \xi \in \mathbb{R}^d$ with $|\nabla\varphi(x)| = |\xi|$ and $\nabla\varphi(x)\cdot\xi = 0$

$$D^2\varphi_x(\nabla\varphi(x), \nabla\varphi(x)) + D^2\varphi_x(\xi, \xi) \geq \beta |\nabla\varphi(x)|^2.$$

then $\exists C$, $\forall u \in H^1(\Omega)$ with $\Delta u = f_2 + \operatorname{div} F$, $f_2 \in L^2$, $F \in L^2$,

$\forall \tau \geq 1$,

$$\tau^{3/2} \|u e^{\tau\varphi}\|_{L^2} + \tau^{1/2} \|\nabla u\, e^{\tau\varphi}\|_{L^2} \lesssim \|e^{\tau\varphi} f_2\|_{L^2} + \tau \|e^{\tau\varphi} F\|_{L^2} + \tau^{3/4} \|e^{\tau\varphi} u\|_{H^{1/2}(\partial\Omega)} + \tau^{3/2} \|e^{\tau\varphi} u\|_{L^2(\omega)}$$

[Imanuvilov - Puel 2003].

- Subellipticity condition (Necessary & Sufficient Condit°, Hormander)

$\exists \alpha, \beta > 0, \quad \inf_{\Omega \setminus \overline{\omega}} |\nabla \varphi| \geq \alpha$

$\forall x \in \overline{\Omega \setminus \omega}, \quad \forall \xi \in \mathbb{R}^d$ with $|\nabla \varphi(x)| = |\xi|$ and $\nabla \varphi(x) \cdot \xi = 0$

$D^2 \varphi (\nabla \varphi(x), \nabla \varphi(x)) + D^2 \varphi_x (\xi, \xi) \geq \beta |\nabla \varphi(x)|^2.$

- Limitations with respect to lower order terms $\quad \Delta u = Vu.$

By Carleman $\tau^{\frac{3}{2} - \theta} \| u e^{\tau \varphi} \|_{H^\theta} \lesssim \| V u e^{\tau \varphi} \|_{L^2}$ + observations.

$\rightsquigarrow$ We can absorb $\| V u e^{\tau \varphi} \|_{L^2}$ by the left hand side if

$$\| V u e^{\tau \varphi} \|_{L^2} \leq C_V \| u e^{\tau \varphi} \|_{H^\theta} \quad \text{for some } \theta \in [0, \tfrac{3}{2}).$$

But $H^{3/2} \hookrightarrow L^p$ with $p = \dfrac{1}{\frac{1}{2} - \frac{3}{2d}} = \dfrac{2d}{d-3} \Rightarrow$ OK if $V \in L^{2d/3}(\Omega)$

Our result. [Dehman, Ervedoza, Thabouti 2023]

L^p Carleman estimates $\quad \Omega \subset \mathbb{R}^d$, $d \geq 3$, $\omega \subset \Omega$.

Same conditions on φ.

Then $\exists C$, $\forall u \in H^1(\Omega)$ with $\Delta u = f_2 + \operatorname{div} F + f_{(2_*)'}$

with $f_2 \in L^2(\Omega)$, $F \in L^2(\Omega;\mathbb{R}^d)$, $f_{(2_*)'} \in L^{(2_*)'}(\Omega)$.

$\forall \tau \geq 1$,

$$\tau^{3/2}\|u e^{\tau\varphi}\|_{L^2} + \tau^{1/2}\|\nabla u\, e^{\tau\varphi}\|_{L^2} \lesssim \|e^{\tau\varphi} f_2\|_{L^2} + \tau\|e^{\tau\varphi}F\|_{L^2} + \tau^{\frac{3}{4}-\frac{1}{2d}}\|e^{\tau\varphi} f_{(2_*)'}\|_{L^{(2_*)'}}$$
$$+\tau^{3/4}\|e^{\tau\varphi}u\|_{H^{1/2}(\partial\Omega)} + \tau^{3/2}\|e^{\tau\varphi}u\|_{L^2(\omega)}$$

and

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\|e^{\tau\varphi}u\|_{L^{2_*}} \lesssim \|e^{\tau\varphi} f_2\|_{L^2} + \tau\|e^{\tau\varphi}F\|_{L^2} + \tau^{\frac{3}{4}+\frac{1}{2d}}\|e^{\tau\varphi} f_{(2_*)'}\|_{L^{(2_*)'}}.$$
$$+\tau^{3/4}\|e^{\tau\varphi}u\|_{H^{1/2}(\partial\Omega)} + \tau^{3/2}\|e^{\tau\varphi}u\|_{L^2(\omega)}$$

Corollary. $\Omega \subset \mathbb{R}^d$, $d \geq 3$, $\omega \subset \Omega$

$\exists C > 0$, if $u \in H^1_0(\Omega)$, $\Delta u = Vu + W_1 \cdot \nabla u$ in Ω

with $V \in L^{q_0}(\Omega)$, $W_1 \in L^{q_1}(\Omega; \mathbb{R}^d)$

$q_0 > \frac{d}{2}$ and $q_1 > \frac{3d-2}{2}$

then

$$\|u\|_{L^2(\Omega)} \leq C e^{C\left(\|V\|_{L^{q_0}}^{\gamma(q_0)} + \|W_1\|_{L^{q_1}}^{\delta(q_1)}\right)} \|u\|_{L^2(\omega)}.$$

with

$$\gamma(q_0) = \begin{cases} \dfrac{1}{\frac{3}{2}\left(1 - \frac{d}{2q_0}\right) + \frac{1}{2q_0}} & \text{if } q_0 \geq d \\ \dfrac{1}{\left(\frac{3}{4} + \frac{1}{2d}\right)\left(2 - \frac{d}{q_0}\right)} & \text{if } q_0 \in \left(\frac{d}{2}, d\right] \end{cases}$$

and $\delta(q_1) = \dfrac{2}{1 - \frac{3d-2}{q_1}}$

Rk. 1. Similar result for $\Delta u = Vu + \mathrm{div}(W_2 u)$.

Mixing W_1 and W_2 gives unexpected complicated estimates due to the method.

Rk 2. $W_1 \in L^{q_1}(\Omega; \mathbb{R}^d)$

$q_1 > \frac{3d-2}{2}$ $\longrightarrow$ $q_1 > d$ is the expected critical scale

but it is known from [Kenig Ruiz Sogge '87] that one cannot do better using L^p Carleman estimate

$\rightsquigarrow$ Wolff's argument is needed

$\rightsquigarrow$ Ongoing work. [Caro - Ervedoza - Thebouti]

II Strategy of the proof.

- Consider the conjugate operator.
- Localize at a rate depending on τ
- Change of variables $\rightsquigarrow$ the case of a strip with a linear weight
- Write the parametrix
- Estimates the operators appearing in the parametrix
- Glue all the estimates.

The conjugate operator.

$$\Delta u = f + \operatorname{div} F \quad \text{in } \Omega.$$

$\rightsquigarrow$ $w = e^{\tau\varphi} u$ satisfies

$$\Delta w - 2\tau \nabla\varphi \cdot \nabla w + \tau^2 |\nabla\varphi|^2 w - \tau \Delta\varphi\, w$$

$$= \tilde{f} + \operatorname{div} \tilde{F} \quad \text{in } \Omega$$

where $\begin{cases} \tilde{f} = e^{\tau\varphi}(f - \tau\nabla\varphi \cdot F) \\ \tilde{F} = e^{\tau\varphi} F. \end{cases}$

Local estimate.

For $x_0 \in \overline{\Omega} \setminus \omega$, set $w_{x_0}(x) = \eta_{x_0}(x)\, w(x)$

with $\eta_{x_0}(x) = \eta\left(\tau^{\alpha}(x - x_0)\right)$,

η radial, $\mathcal{C}_c^\infty(B(1))$, $= 1$ in $B(\frac{1}{2})$

Localization in $B(x_0, \tau^{-\alpha})$.

$\alpha < \frac{1}{2} \longrightarrow$ Allows to absorb the terms coming from the localization

$\alpha > \frac{1}{4} \longrightarrow$ Allows to approximate φ by its 2nd order approximation.

A suitable change of coordinates.

When localized in $B(x_0, \tau^{-\alpha})$ with $\frac{1}{4} < \alpha < \frac{1}{2}$,

$$\Delta w - 2\tau \nabla \varphi \cdot \nabla w + \tau^2 |\nabla \varphi|^2 w - \tau \Delta \varphi \, w$$
$$= f + \operatorname{div} F \quad \text{in } \Omega.$$

$$\updownarrow$$

$$\partial_{11} \tilde{w} - 2\tau \partial_1 \tilde{w} + \tau^2 \tilde{w}$$
$$+ \sum_{j=2}^{d} (1 - \lambda_j x_1) \partial_{x_j x_j} \tilde{w} = \tilde{f} + \operatorname{div} \tilde{F}$$

" Canonical Form ".

in the strip $(-1, 1) \times \mathbb{R}^{d-1}$.

with $\lambda_j = \frac{2}{|\nabla \varphi(x_0)|^2} \left(D^2 \varphi_{x_0} (\nabla \varphi(x_0), \nabla \varphi(x_0)) + D^2 \varphi_{x_0} (L_j, L_j) \right) > 0$

$L_j \cdot \nabla \varphi(x_0) = 0$, $|L_j| = |\nabla \varphi(x_0)|$ and $(L_2, \dots, L_d)$ basis of $\{\nabla \varphi(x_0)\}^{\perp}$.

Construction of a parametrix

$$\partial_{11} w - 2\tau \partial_1 w + \tau^2 w + \sum_{j=2}^{d} (1 - \lambda_j x_1) \partial_{jj} w = f + \operatorname{div} F$$

in $(-1,1) \times \mathbb{R}^{d-1}$.

$\rightarrow$ Fourier transform $x' \rightarrow \xi'$.

$$\left[(\partial_1 - \tau)^2 - \Psi(x_1, \xi')^2 \right] \hat{w} = \hat{f} + \partial_1 \hat{F}_1 + \sum_{j \geq 2} i \xi_j \hat{F}_j$$

with $\Psi(x_1, \xi')^2 = \sum_{j \geq 2}^{d} (1 - \lambda_j x_1) \xi_j^2$.

$$(\partial_1 - \tau)^2 - \Psi(x_1, \xi')^2 = (\partial_1 - \tau - \Psi(x_1, \xi')) (\partial_1 - \tau + \Psi(x_1, \xi')) + \text{l.o.t}$$

$$\rightsquigarrow \quad w = K_{\tau,0} f + K_{\tau,1} F_1 + \sum_{j\geq 2}^{d} K_{\tau,j} F_j + \underbrace{R_\tau w}_{\text{Remainder.}}$$

with

$$\widehat{K_{\tau,0} f}(x_1,\xi') = \int_{y_1\in(-1,1)} k_{\tau,0}(x_1,y_1,\xi')\,\hat f(y_1,\xi')\,dy_1 .$$

$$k_{\tau,0}(x_1,y_1,\xi') = -\mathbb{1}_{\{\psi(x_1,\xi')>\tau\}} \int_{x_0}^{\min\{x_1,y_1\}} e^{-\tau(y_1-x_1) - \int_{x_1}^{\tilde x_1}\psi(\tilde y_1,\xi')\,d\tilde y_1 - \int_{\tilde x_1}^{y_1}\psi(\tilde y_1,\xi')\,d\tilde y_1}\, d\tilde x_1$$

$$+ \mathbb{1}_{\{\psi(x_1,\xi')<\tau\}} \int_{x_1}^{y_1} e^{-\tau(y_1-x_1) + \int_{x_1}^{\tilde x_1}\psi(\tilde y_1,\xi')\,d\tilde y_1 - \int_{\tilde x_1}^{y_1}\psi(\tilde y_1,\xi')\,d\tilde y_1}\, d\tilde x_1 .$$

... $\rightsquigarrow$ Remains to estimate these operators ...

Fourier restrictions theorems and applications.

Thm (Stein-Tomas) Let $n \geq 2$, $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$. Then the map

$$\begin{aligned} L^1(\mathbb{R}^n) &\longrightarrow L^2(\mathbb{S}^{n-1}) \\ f &\longmapsto \hat{f}\,\big|_{\mathbb{S}^{n-1}} \end{aligned}$$

can be extended by continuity on $L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n)$

and

$$\|\hat{f}\|_{L^2(\mathbb{S}^{n-1})} \leq C \|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n)}.$$

What matters.

* Curvature of S^{n-1}.

$\rightsquigarrow$ Can be adapted to the ellipsoids

$$\Sigma_a = \{ \xi \in \mathbb{R}^n, \ \Psi(a, \xi) = 1 \}$$

where $\Psi(a,\xi)^2 = \sum_{j=1}^{n} (1 - a\lambda_j) \xi_j^2$.

* $n \simeq d-1$

$$\rightsquigarrow \quad \frac{2(n+1)}{n+3} = \frac{2d}{d+2} = (2_*)'.$$

Based on this, if we consider an operator

$K_{a,k} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ given by

$$\widehat{K_{a,k} f}\,(\xi) = k\left(\psi(a,\xi), \frac{\xi}{\psi(a,\xi)}\right)\hat{f}(\xi),$$

we have

$\approx$ radial decomposition

$$\| K_{a,k} \|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \| k \|_{L^\infty(\mathbb{R}_+, L^\infty(\Sigma_a))}$$

and

$$\| K_{a,k} \|_{\mathcal{L}\left(L^{2\left(\frac{n+1}{n+3}\right)}(\mathbb{R}^n), L^{2\left(\frac{n+1}{n-1}\right)}(\mathbb{R}^n)\right)} \leq C \int_0^\infty \| k(\lambda, \cdot) \|_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}} \, d\lambda .$$

$\rightsquigarrow$ Allows to estimate the operators

$K_{\tau,0}$, $K_{\tau,j}$ as operators

$$\mathcal{L}\left(L^{(2*)'}(\mathbb{R}^d), L^{(2*)}(\mathbb{R}^d)\right)$$

$$\mathcal{L}\left(L^{(2*)'}(\mathbb{R}^d), L^{2}(\mathbb{R}^d)\right).$$

$$\mathcal{L}\left(L^{2}(\mathbb{R}^d), L^{(2*)}(\mathbb{R}^d)\right)$$

Rk. This step also use the (1d) Hardy - Littlewood - Sobolev thm.

$$\|K_{\tau,0} f\|_{L^{2*}} \leq \Big\| \|K_{\tau,0} f(x_1,\cdot)\|_{L^{2*}_{x'}} \Big\|_{L^{2*}_{x_1}}$$

$$\leq \Big\| \int_{-1}^{1} \Big(\int_{\lambda>0} \|k_{\tau,0}(x_1,y_1,\lambda,\cdot)\|_{L^\infty(\Sigma_{x_1})} \lambda^{1-2/d} \, d\lambda \Big) \|f(y_1,\cdot)\|_{L^{(2*)'}_{y'}} \, dy_1 \Big\|_{L^{2*}_{x_1}}$$

$$\leq C \Big\| \int_{-1}^{1} \frac{1}{|x_1-y_1|^{1-2/d}} \|f(y_1)\|_{L^{(2*)'}_{y'}} \, dy_1 \Big\|_{L^{2*}_{x_1}}$$

$$\leq C \Big\| \|f(y_1,\cdot)\|_{L^{(2*)'}_{y'}} \Big\|_{L^{(2*)'}_{y_1}}$$

$$\leq C \|f\|_{L^{(2*)'}} .$$

Last Step: A Gluing argument.

We have estimates on each

$$w_{x_0}(x) = \eta(\tau^{\alpha}(x - x_0))\, w(x), \quad x_0 \in \overline{\Omega} \setminus \omega$$

We then integrate in $x_0 \in \overline{\Omega} \setminus \omega$.

$\leadsto$ Yield the global estimate.

N.B. The boundary term does not yield any specific difficulty in this argument since the boundary is a level set of the weight function.

III Application Quantification of Unique Continuation with respect to lower order terms.

$(W_2 = 0)$

$$V \in L^{q_0} \longrightarrow V = \underset{\substack{\cap \\ L^{d/2}}}{V_{d/2}} + \underset{\substack{\cap \\ L^{d}}}{V_{d}} + \underset{\substack{\cap \\ L^{\infty}}}{V_{\infty}}$$

$q_0 > d/2$

$$W_1 \in L^{q_1} \longrightarrow W_1 = \underset{\substack{\cap \\ L^{d}}}{W_{1,d}} + \underset{\substack{\cap \\ L^{\infty}}}{W_{1,\infty}}$$

$$q_1 > \frac{3d-2}{2}$$

$$\Delta u = Vu + W_1 \cdot \nabla u, \qquad u \in L^2 \cap L^{2*}, \; \nabla u \in L^2.$$

$$\rightsquigarrow \quad f_{(2*)'} = V_{d/2}\, u + V_d\, u + W_{1,d}\, \nabla u$$

$$f_2 = V_\infty u + W_{1,\infty} \cdot \nabla u.$$

$$\Rightarrow \tau^{3/2} \| u e^{\tau\varphi} \|_{L^2} + \tau^{1/2} \| \nabla u e^{\tau\varphi} \|_{L^2}$$

$$\lesssim \|V_\infty\|_{L^\infty} \|u e^{\tau\varphi}\|_{L^2} + \|W_{1,\infty}\|_{L^\infty} \|\nabla u e^{\tau\varphi}\|_{L^2}$$

$$+ \tau^{\frac{3}{4} - \frac{1}{2d}} \|V_d\|_{L^d} \|u e^{\tau\varphi}\|_{L^2} + \tau^{\frac{3}{4} - \frac{1}{2d}} \|W_{1,d}\|_{L^d} \|\nabla u e^{\tau\varphi}\|_{L^2}.$$

$$+ \tau^{\frac{3}{4} - \frac{1}{2d}} \|V_{d/2}\|_{L^{d/2}} \|u e^{\tau\varphi}\|_{L^{2*}} \quad + \text{observations}.$$

Provided

$$\begin{cases} \|V_\infty\|_{L^\infty} + \tau^{\frac{3}{4}-\frac{1}{2d}} \|V_d\|_{L^d} \ll \tau^{3/2} \\ \|W_{1,\infty}\|_{L^\infty} + \tau^{\frac{3}{4}-\frac{1}{2d}} \|W_{1,d}\|_{L^d} \ll \tau^{1/2} \end{cases}$$

$$\tau^{3/2} \|u e^{\tau\varphi}\|_{L^2} + \tau^{\frac{1}{2}} \|\nabla u\, e^{\tau\varphi}\|_{L^2}$$

$$\lesssim \tau^{3/4 - 1/2d} \|V_{d/2}\|_{L^{d/2}} \|u e^{\tau\varphi}\|_{L^{2^*}} + \text{observations.}$$

Similarly,

$$\tau^{\frac{3}{4}+\frac{1}{2d}} \| u e^{\tau\varphi} \|_{L^{2^*}}$$

$$\lesssim \|V_\infty\|_{L^\infty} \|u e^{\tau\varphi}\|_{L^2} + \|W_{1,\infty}\|_{L^\infty} \|\nabla u e^{\tau\varphi}\|_{L^2}$$

$$+ \|V_d\|_{L^d} \|u e^{\tau\varphi}\|_{L^{2^*}}$$

$$+ \tau^{\frac{3}{4}+\frac{1}{2d}} \left(\|V_{d/2}\|_{L^{d/2}} \|u e^{\tau\varphi}\|_{L^{2^*}} + \|W_{1,d}\|_{L^d} \|\nabla u e^{\tau\varphi}\|_{L^2} \right)$$

+ observations

We require

$$\|V_d\|_{L^d} \ll \tau^{\frac{3}{4}+\frac{1}{2d}} \quad \text{and} \quad \|V_{d/2}\|_{L^{d/2}} \ll 1$$

$$\leadsto \tau^{\frac{3}{4}+\frac{1}{2d}} \| u e^{\tau\varphi} \|_{L^{2^*}}$$

$$\lesssim \| V_\infty \|_{L^\infty} \| u e^{\tau\varphi} \|_{L^2} + \| W_{1,\infty} \|_{L^\infty} \| \nabla u \, e^{\tau\varphi} \|_{L^2} + \tau^{\frac{3}{4}+\frac{1}{2d}} \| W_{1,d} \|_{L^d} \| \nabla u \, e^{\tau\varphi} \|_{L^2} + \text{obs}$$

Combine with

$$\tau^{3/2} \| u e^{\tau\varphi} \|_{L^2} + \tau^{\frac{1}{2}} \| \nabla u \, e^{\tau\varphi} \|_{L^2} \lesssim \tau^{3/4 - 1/2d} \| V_{d/2} \|_{L^{d/2}} \| u e^{\tau\varphi} \|_{L^{2^*}} + \text{observations.}$$

Provided

$\| V_\infty \|_{L^\infty} \ll \tau^{3/2}$, $\| V_d \|_{L^d} \ll \tau^{\frac{3}{4}+\frac{1}{2d}}$, $\| W_{1,\infty} \|_{L^\infty} \ll \tau^{1/2}$, $\| W_{1,d} \|_{L^d} \ll \tau^{-\frac{1}{4}+\frac{1}{2d}}$

$\| V_{d/2} \|_{L^{d/2}} \ll 1$.

$$W_{1,\infty} = W_1 \, \mathbb{1}_{|W_1| \leq \lambda_1}, \qquad W_{1,d} = \mathbb{1}_{|W_1| > \lambda_1} W_1 .$$

$$V_\infty = V \, \mathbb{1}_{|V| \leq \lambda_0}, \quad V_d = V \, \mathbb{1}_{|V| > \lambda_0}, \quad V_{d/2} = 0 \quad \text{if } q_0 \geq d$$

OR

$$V_\infty = 0, \quad V_d = V \, \mathbb{1}_{|V| \leq \lambda_0}, \quad V_{d/2} = V \, \mathbb{1}_{|V| \geq \lambda_0} \quad \text{if } q_0 \in \left(\frac{d}{2}, d\right).$$

$+$ Optimize in λ_0, λ_1.

IV Further comments.

- Adapt Wolff's argument to get quantitative estimates for $W_1, W_2 \in L^{d+}(\Omega; \mathbb{R}^d)$.

  $\rightsquigarrow$ Work in progress Caro, SE, Thabouti

- These L^p Carleman estimates can in principle be adapted to the case of an interface [Le Rousseau - Robbiano, Le Rousseau - Lerner].

- What about elliptic metric? $\operatorname{div}(A(x)\nabla \cdot\,)$ instead of Δ?

- Parabolic equations? [Koch - Tataru].

- Discussing the optimality of the integrability conditions for the unique continuation property for the stationary Stokes pb

$$\begin{cases} -\Delta u + \bar{y} \cdot \nabla u + (u \cdot \nabla) \bar{y} + \nabla p = 0. \\ \operatorname{div} u = 0 \end{cases}$$

[Fabre - Lebeau '96].

Condition on $\bar{y}$?

Thank you for your attention!

Ref: L^p Carleman estimates for elliptic boundary value problems and applications to the quantification of unique continuation,
Belhassen Dehman, Sylvain Ervedoza, Lotfi Thabouti,
to appear in Annales Henri Lebesgue.