X Partial differential equations, optimal design and numerics

Random Batch Methods

for Efficient Simulation and Optimal Control of Networked 1D Hyperbolic Systems

Yue Wang, * joint work with D.W.M. Veldman, E. Zuazua

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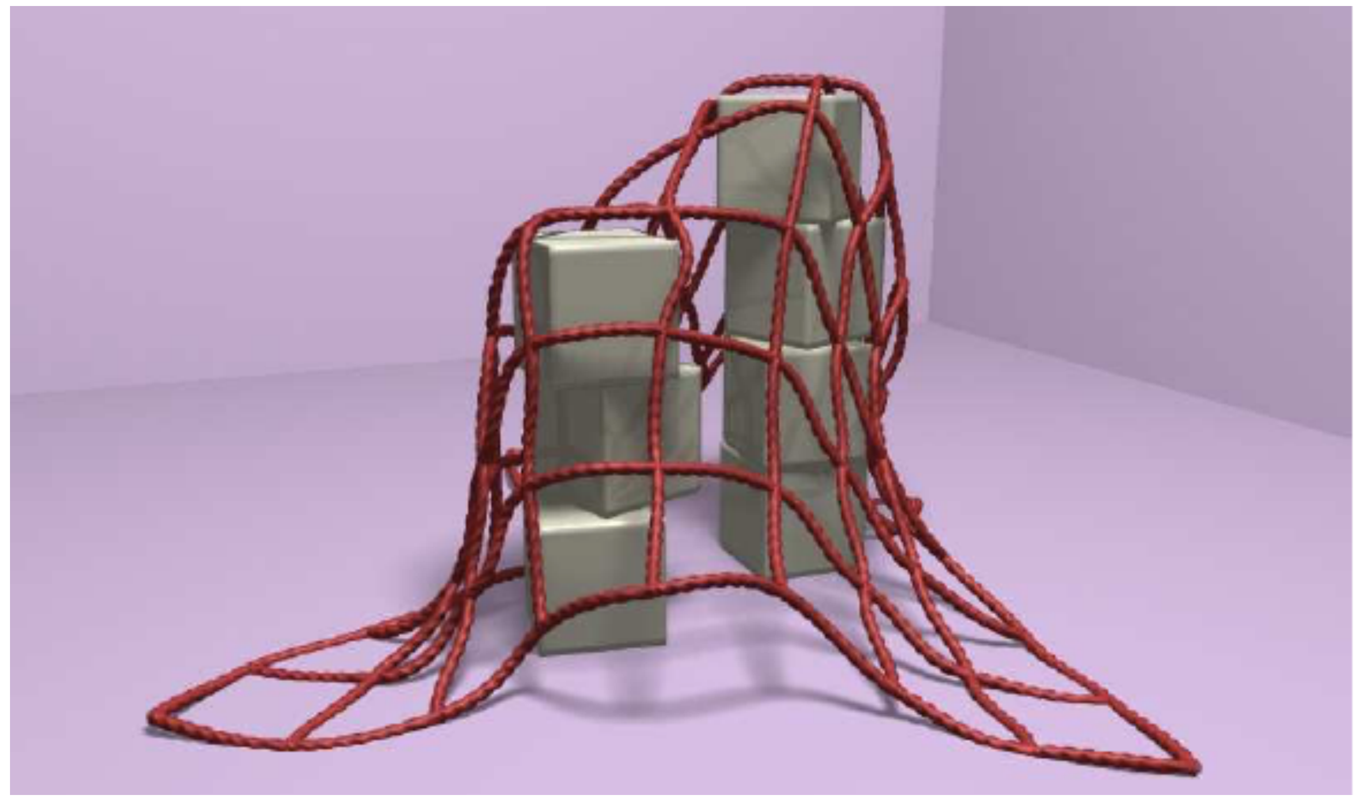
Friedrich-Alexander-Universität Erlangen-Nürnberg

Friedrich-Alexander-Universität DYNAMICS, CONTROL, MACHINE LEARNING AND NUMERICS

Funded by DFG Deutsche Forschungsgemeinschaft German Research Foundation

Motivation: Networked 1D hyperbolic systems



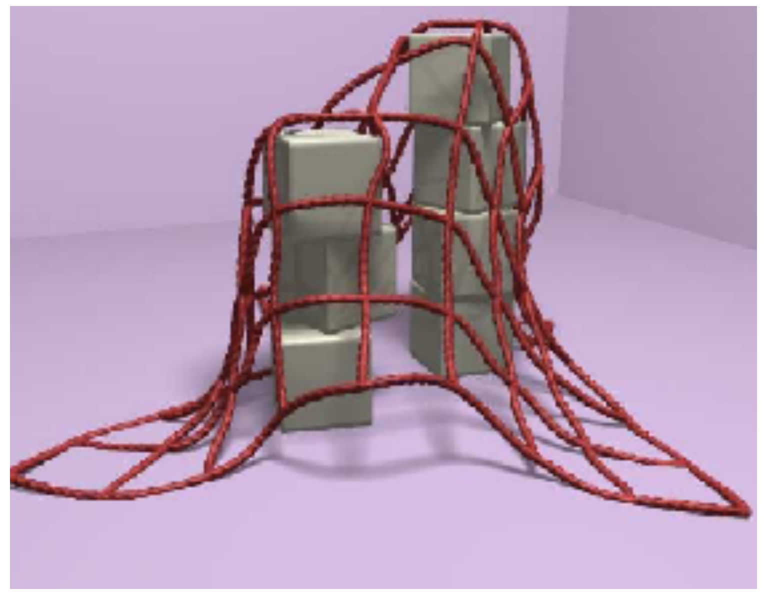
Network of large deflection strings
(Nonlinear vibrating strings)

NASA Flexible flight device
(Geometrically exact beams)

Gas transport networks
(Isothermal Euler equation)

Open canal
(Saint-Venant system)

Projects on Applied PDEs: Analysis, Control Theory and Numerics

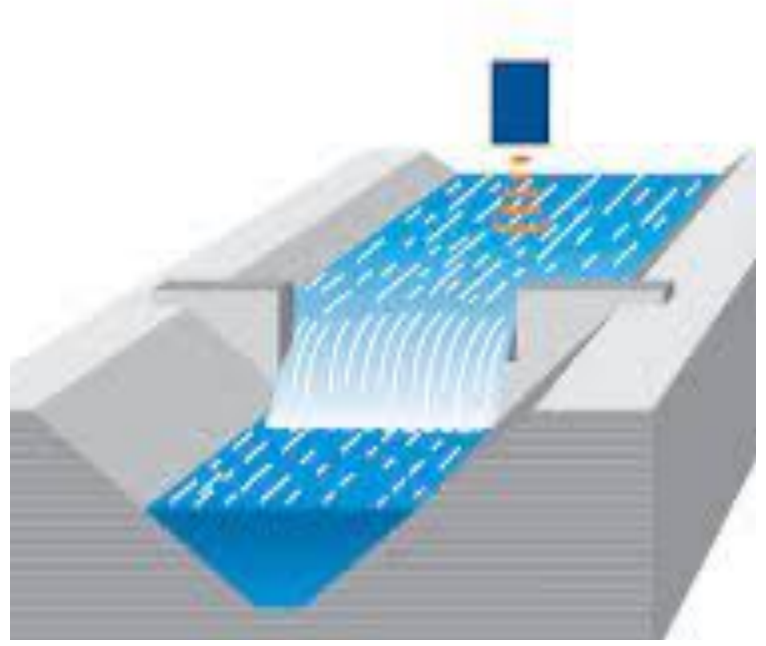


Network of Large Deflection Strings (Nonlinear coupled wave equations)

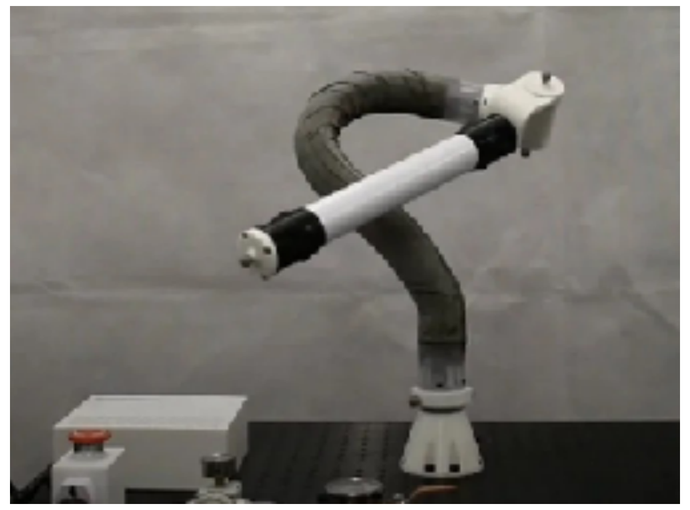
NASA Flexible Flight Device (Geometrically Exact Beams)

Wind Turbine

Gas transport networks (Isothermal Euler Equations)



Open Canal (Saint-Venant Equation)



Flexible Robotic Arm

- Project (2018-2019): Exact boundary controllability for coupled wave equations with dynamical boundary conditions
- Project (2017-2019): Control Theory on planar or spacial string networks: controllability of nodal profile for quasilinear hyperbolic systems
- Project DFG WA5144/1-1 (2022–2024): Analysis and Control of Nonlinear Hyperbolic Systems with Degeneration on Networks

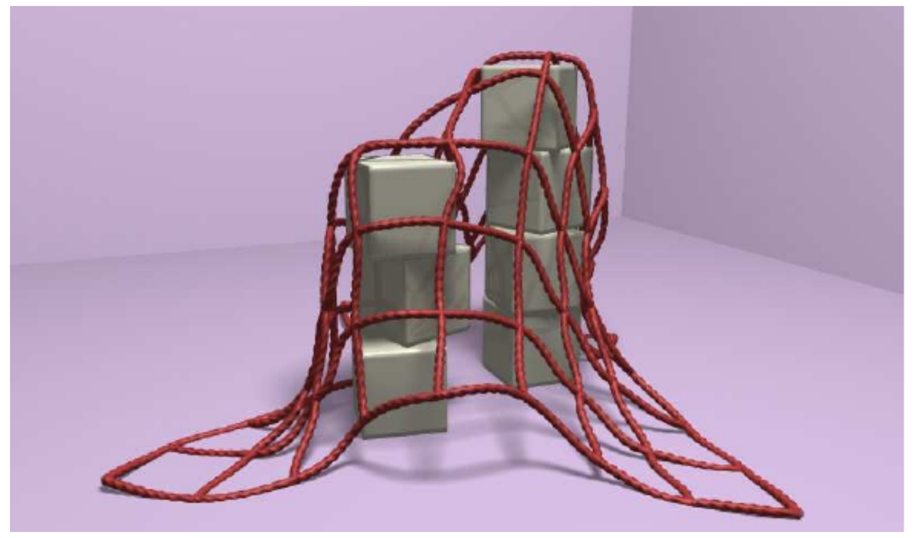
ConFlex

- Project ConFlex (2017–2022) and ModConFlex (2023–2027): Modeling and control for flexible structures interacting with fluids

TRR 154

- Project SFB TRR154 (2018–2026) Mathematical modelling, simulation and optimization using the example of gas networks

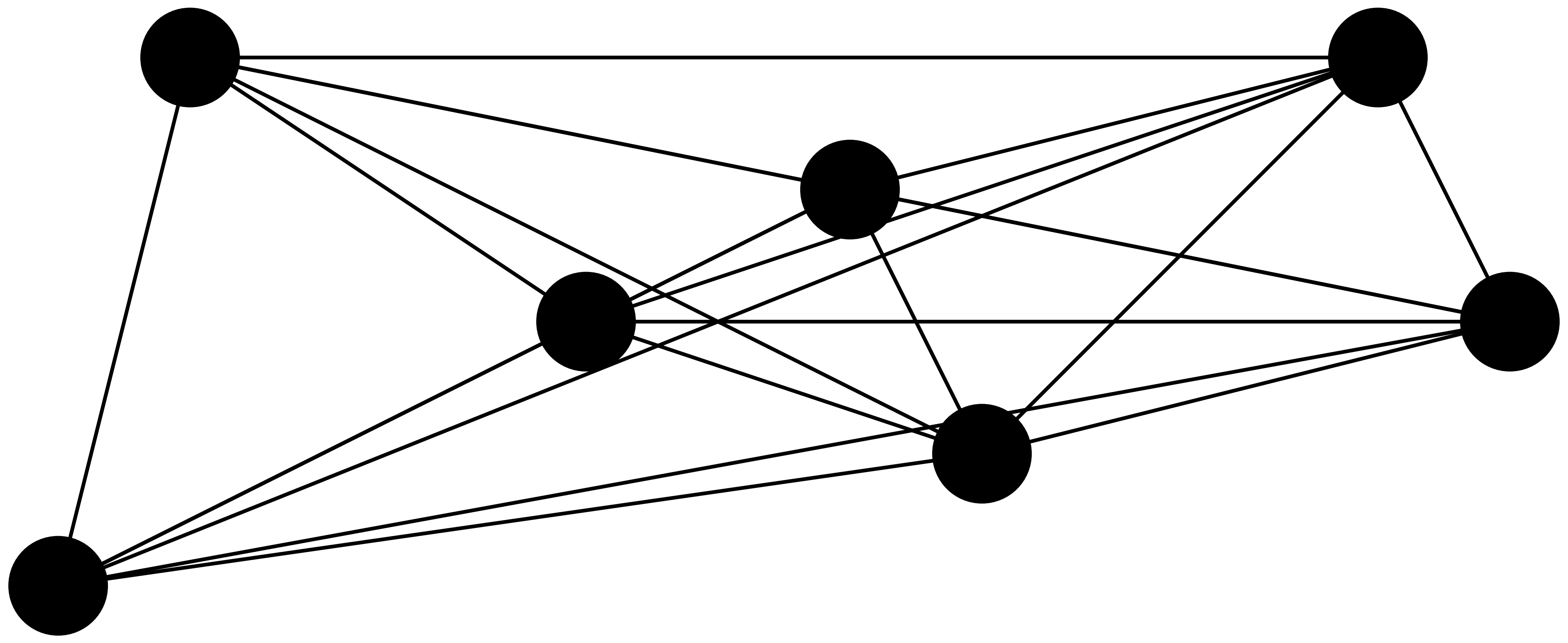
Motivation

- Accurate and Fast Prediction of Numerical Solutions/ Optimal Control for Networked PDEs is of significant interest for many scientific applications, say, real-time capable methods and algorithms.
- Large scale networks may contain more than 20K edges, many nonlinear elements, and complex topological structure, e.g. circles/loops inside.
- Recent success of stochastic methods (e.g. stochastic gradient descent) in optimization and training large (neural) networks.

Origins of the Random Batch Method

Initial Motivation: Simulation and control of large *interacting particle systems* can be computationally demanding.

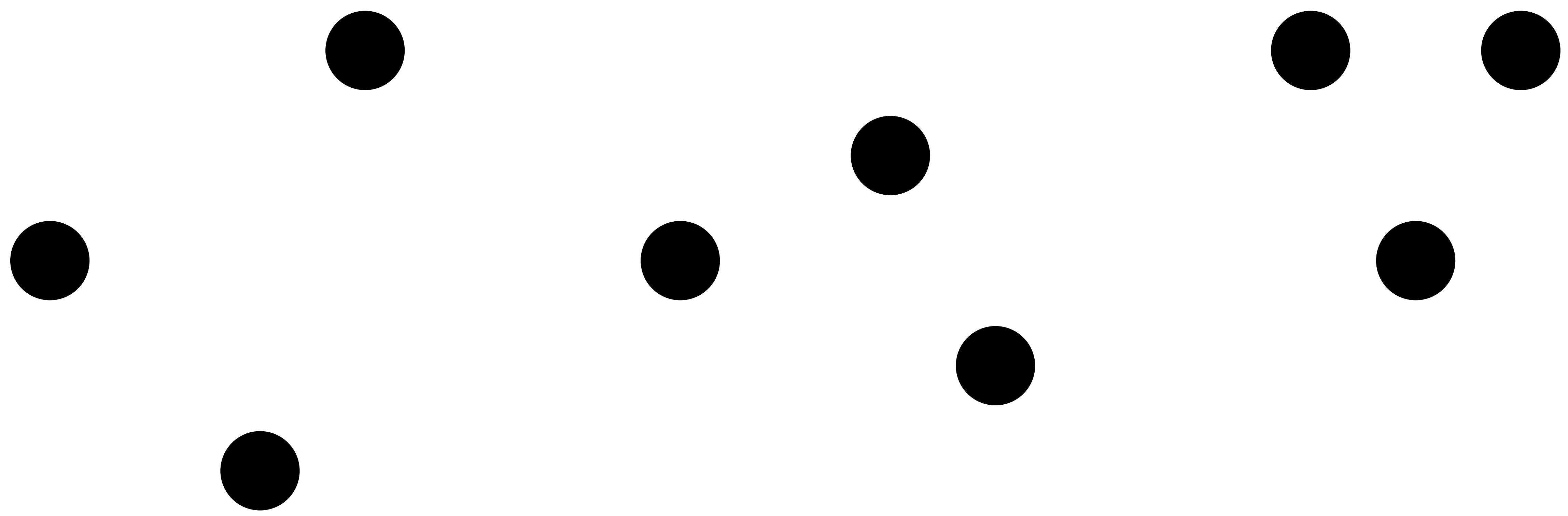


There are $N(N-1)/2$ interaction forces between N particles.
$\Rightarrow$ Computational cost grows rapidly when N is large.

Origins of the Random Batch Method

Proposed simulation method: The Random Batch Method

[Shi Jin, Lei Li, Jian-Guo Liu, J. of Computational Physics, 2020]



Origins of the Random Batch Method

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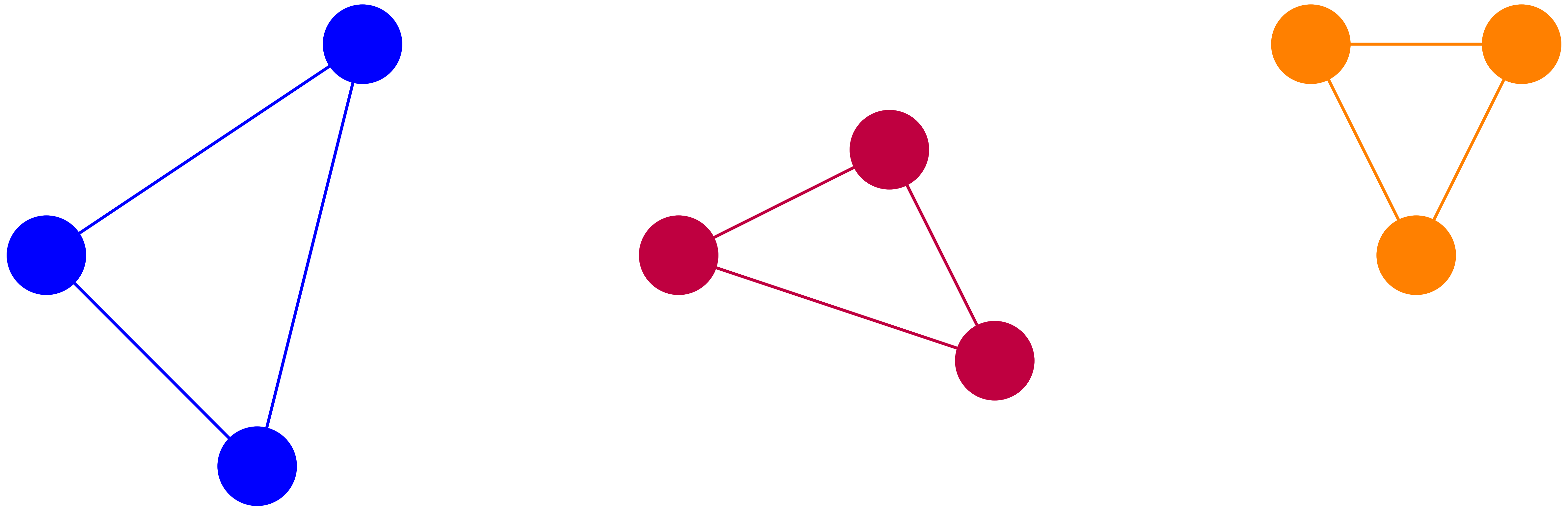
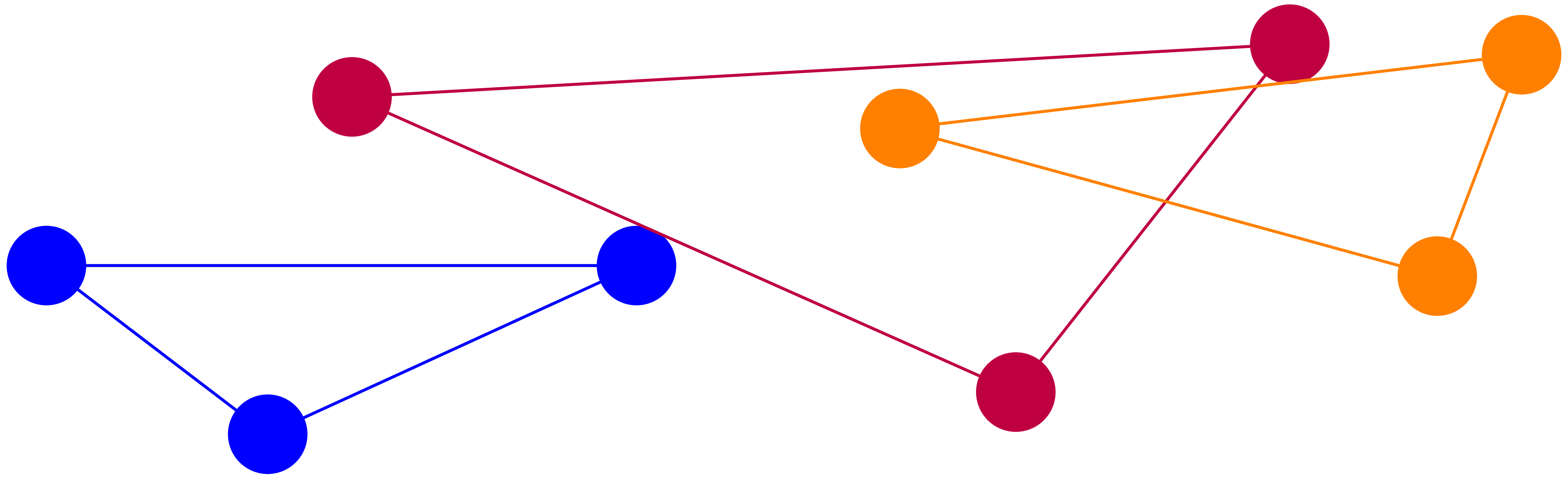


- Divide the N particles randomly into batches of size $P \geq 2$.
- Consider only interactions between particles in the same batch.
- Do a simulation over a short time interval of length h.

Origins of the Random Batch Method

Proposed simulation method: The Random Batch Method

[Shi Jin, Lei Li, Jian-Guo Liu, J. of Computational Physics, 2020]



- Divide the N particles randomly into batches of size $P \geq 2$.
- Consider only interactions between particles in the same batch.
- Do a simulation over a short time interval of length h.
- Repeat.

Origins of the Random Batch Method

Proposed simulation method: The Random Batch Method

[Shi Jin, Lei Li, Jian-Guo Liu, J. of Computational Physics, 2020]

In formulas:

First-order particle dynamics: For each particle $i \in \{1, 2, \dots, N\}$

$$\dot{x}_i(t) = \frac{1}{N-1} \sum_{\substack{j=1 \\ j \neq i}}^{N} f_{ij}(x_j(t) - x_i(t)).$$

Random Batch Method.

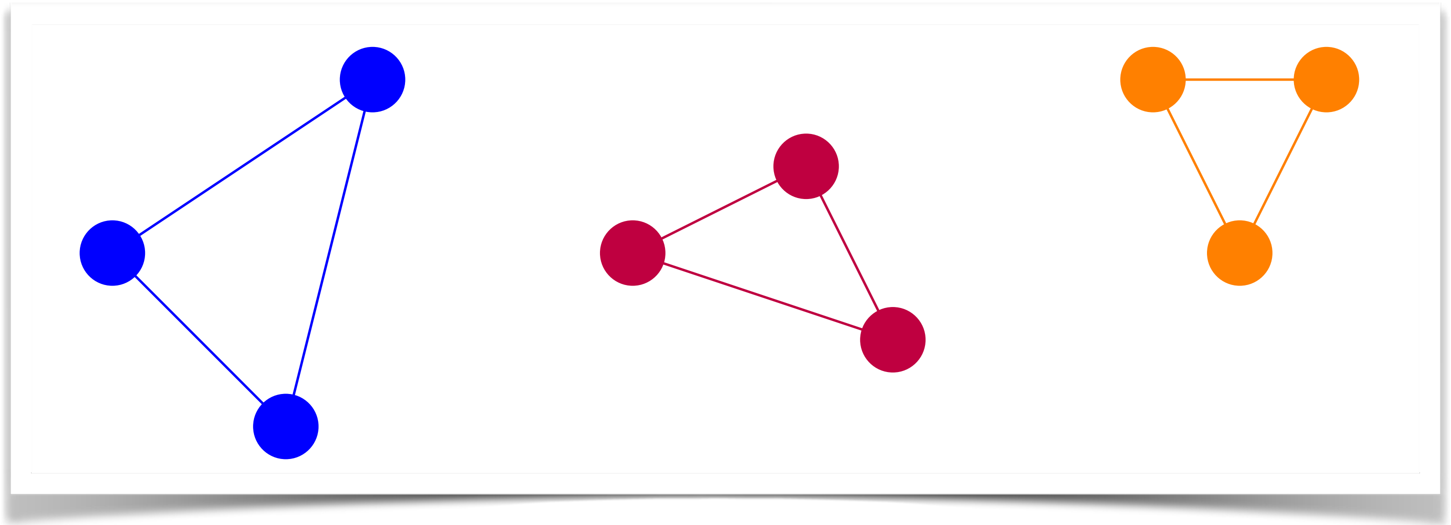
1. Set $k = 0$ and set $\tilde{x}_i(0)$ for all $i \in \{1, 2, \dots, N\}$.
2. Partition $\{1, 2, \dots, N\}$ into batches of size $P \geq 2$, i.e.

$$\{1, 2, \dots, N\} = \dot{\bigcup}_{b=1}^{N/P} \mathcal{B}_b^k, \qquad |\mathcal{B}_b^k| = P.$$

3. For each i, solve on $[kh, (k+1)h]$

$$\dot{x}_{h,i}(t) = \frac{1}{P-1} \sum_{j \in \mathcal{B}_{b(i)}^k,\ j \neq i} f_{ij}(x_{h,j}(t) - x_{h,i}(t)), \qquad b(i) \text{ s.t. } i \in \mathcal{B}_{b(i)}^k.$$

4. Set $k \leftarrow k+1$ and go to step 2.



N=9, P=3

Main Results:

- the RBM-solution converges to the solution of the original problem as $h \to 0$.
- the RBM reduces the computational cost from $O(N^2)$ to $O(PN)$.

RBM for Optimal Control

- The RBM can speed up the solution of optimal control problems governed by interacting particles systems [D. Ko, E. Zuazua, 2021] (only numerical experiments).

Instead of computing the minimizer $u^*(t)$ of

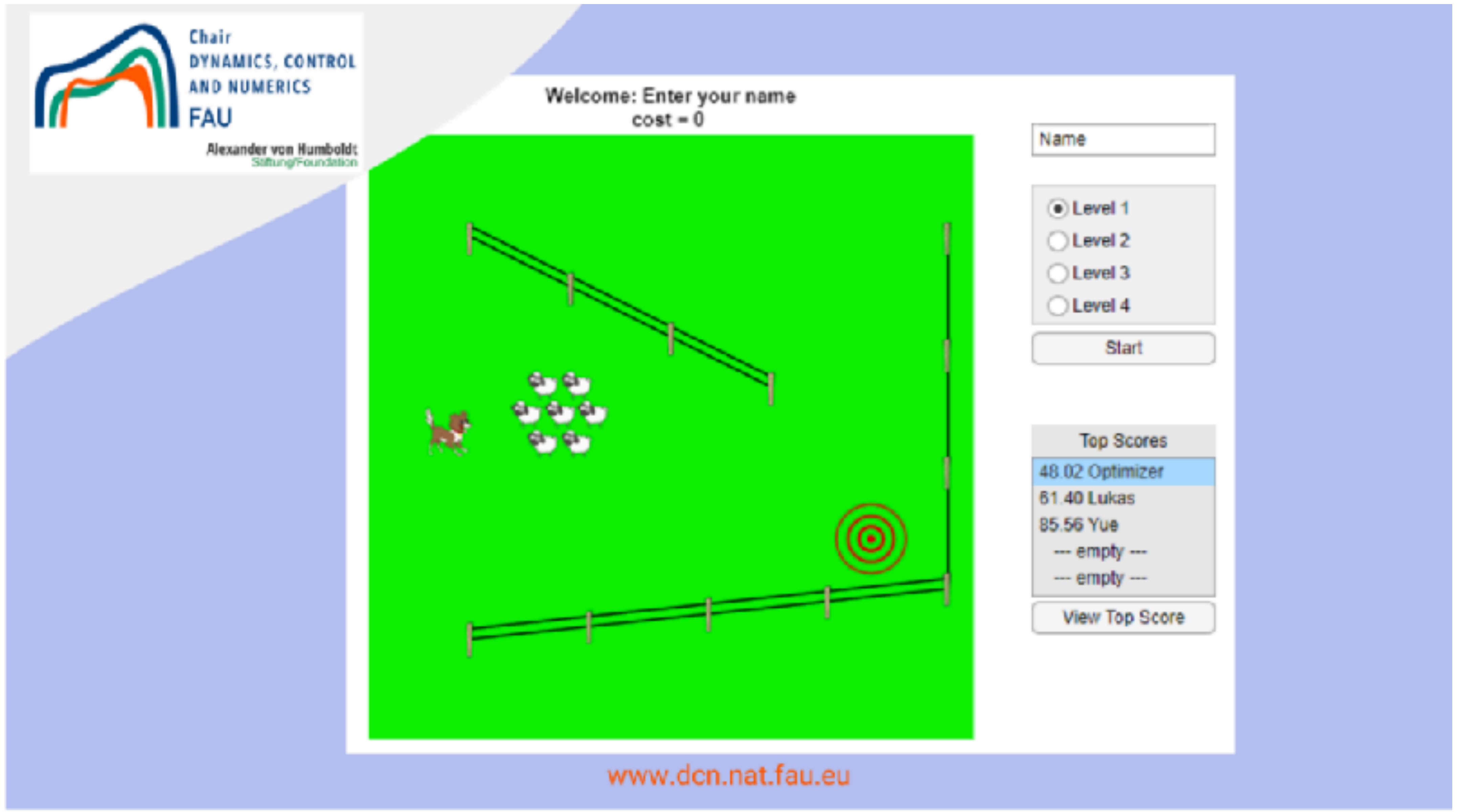
$$J = \int_0^T f_0(x(t), u(t))\,\mathrm{d}t,$$

subject to

$$\dot{x}_i(t) = \frac{1}{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N} f_{ij}(x_j(t) - x_i(t)) + \sum_{k=1}^{M} g_{ik}(x_i(t))u_k(t),$$

it is faster to compute the minimizer $u_h^*(t)$ of J subject to

$$\dot{x}_{h,i}(t) = \frac{1}{P-1}\sum_{\substack{j\in\mathcal{B}^k_{b(i)}\\ j\neq i}} f_{ij}(x_{h,j}(t) - x_{h,i}(t)) + \sum_{k=1}^{M} g_{ik}(x_i(t))u_{h,k}(t).$$



https://github.com/danielveldman/sheep_herding_game

“Lange Nacht der Wissenschaften” 2023 in Erlangen, Fürth and Nürnberg

RBM for Optimal Control

- The first convergence proof is given in [D.Veldman, E.Zuazua, 2022] for finite dimensional linear-quadratic optimal control in the operator-splitting setting.

$$\min_u \int_0^T (|x(t)-x_d(t)|^2) + |u(t)|^2 dt,$$
$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

- Convergence Results

  - *For a deterministic control $u(t)$*
  $$\mathbb{E}[|x_h(t) - x(t)|^2] \le h\mathrm{Var}[A_h](\|A\|t^2 + 2t)(|x_0| + |Bu|_{L^1})^2.$$
  - *For a stochastic control $u_h(\boldsymbol{\omega}, t)$ satisfying $|Bu_h(\boldsymbol{\omega})|_{L^2} \le U$*
  $$\mathbb{E}[|x_h(t) - x(t)|^2] \le C_{[T,\|A\|]} h\mathrm{Var}[A_h](x_0 + U\sqrt{t}).$$
  - *Optimality gap*
  $$\mathbb{E}[|J_h(u_h^*) - J(u^*)|] \le C\left(\sqrt{h\mathrm{Var}[A_h]} + h\mathrm{Var}[A_h]\right).$$
  - *Convergence in the controls*
  $$\mathbb{E}[|u_h^* - u^*|_{L^2}^2] \le Ch\mathrm{Var}[A_h].$$

where $\mathbb{E}[A_h(t)] = A$ (ensured) and $\mathrm{Var}[A_h] = \mathbb{E}[\|A_h(t) - A\|^2] = \sum_{\omega=1}^{2^M} \left\|\sum_{m\in S_\omega} \frac{A_m}{\pi_m} - A\right\|^2 p_\omega.$

Step 1 Split the matrix A as
$$A = \sum_{m=1}^{M} A_m.$$

Step 2 Enumerate the 2^M subsets of $\{1, 2, \dots, M\}$ as $S_1, S_2, \dots S_{2^M}$. Assign to each subset S_ω a probability p_ω.

Step 3 Divide $[0, T]$ into K subintervals $[t_{k-1}, t_k)$ of length $\le h$. For each $[t_{k-1}, t_k)$, randomly choose an index $\omega_k \in \{1, 2, \dots, 2^M\}$ according to the probabilities p_ℓ. Set $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_K)$.

Step 4 Define the matrix $A_h(\boldsymbol{\omega}, t)$
$$A_h(\boldsymbol{\omega}, t) = \sum_{m\in S_{\omega_k}} \frac{A_m}{\pi_m}, \qquad t \in [t_{k-1}, t_k),$$
where π_m is the probability that m is an element of the selected subset, i.e.
$$\pi_m = \sum_{\omega\in\{\omega | m\in S_\omega\}} p_\omega.$$

Step 5 Compute the minimizer $u_h^*(\boldsymbol{\omega}, t)$ of the 'simpler' LQR problem
$$\min_{u\in L^2(0,T;\mathbb{R}^q)} J_h(\boldsymbol{\omega}, u) = \int_0^T (|x_h(\boldsymbol{\omega}, t) - x_d(t)|_Q^2 + |u(t)|_R^2)\, dt,$$
$$\dot{x}_h(\boldsymbol{\omega}, t) = A_h(\boldsymbol{\omega}, t) x_h(\boldsymbol{\omega}, t) + Bu(t), \qquad x_h(\boldsymbol{\omega}, 0) = x_0.$$

RBM for PDEs?

- Whether this algorithm can accelerate the simulation and optimization of nonlinear dynamics and **networked infinite-dimensional systems** (A is unbounded operator), its convergence theory and applications are still rather open! e.g. see RBM for abstract evolution equations of parabolic type in *A randomized operator splitting scheme inspired by stochastic optimization methods*, [M. Eisenmann, T. Stillfjord, Numerisch Mathematik, 2024]

RBM for Hyperbolic equations: Toy Example

Consider the transport equation

$$\begin{aligned} &y_t(t,x) + v(x) y_x(t,x) = 0, && t \in (0,T), x \in \mathbb{R}, \\ &y(0,x) = y_0(x), && x \in \mathbb{R}, \end{aligned}$$

where $v(x)$ is bounded and Lipschitz, y_0 is globally Lipschitz.

- We split the generator of the semi-group as

$$\underbrace{-v(x)\frac{\partial}{\partial x}}_{A} = \sum_{m=1}^{M} \underbrace{-v_m(x)\frac{\partial}{\partial x}}_{A_m},$$

  where the $v_m(x)$ are Lipschitz and bounded.
- In each time step, we randomly choose batch B_k, subset of $\{1, .., M\}$, of size P and consider the velocity field as

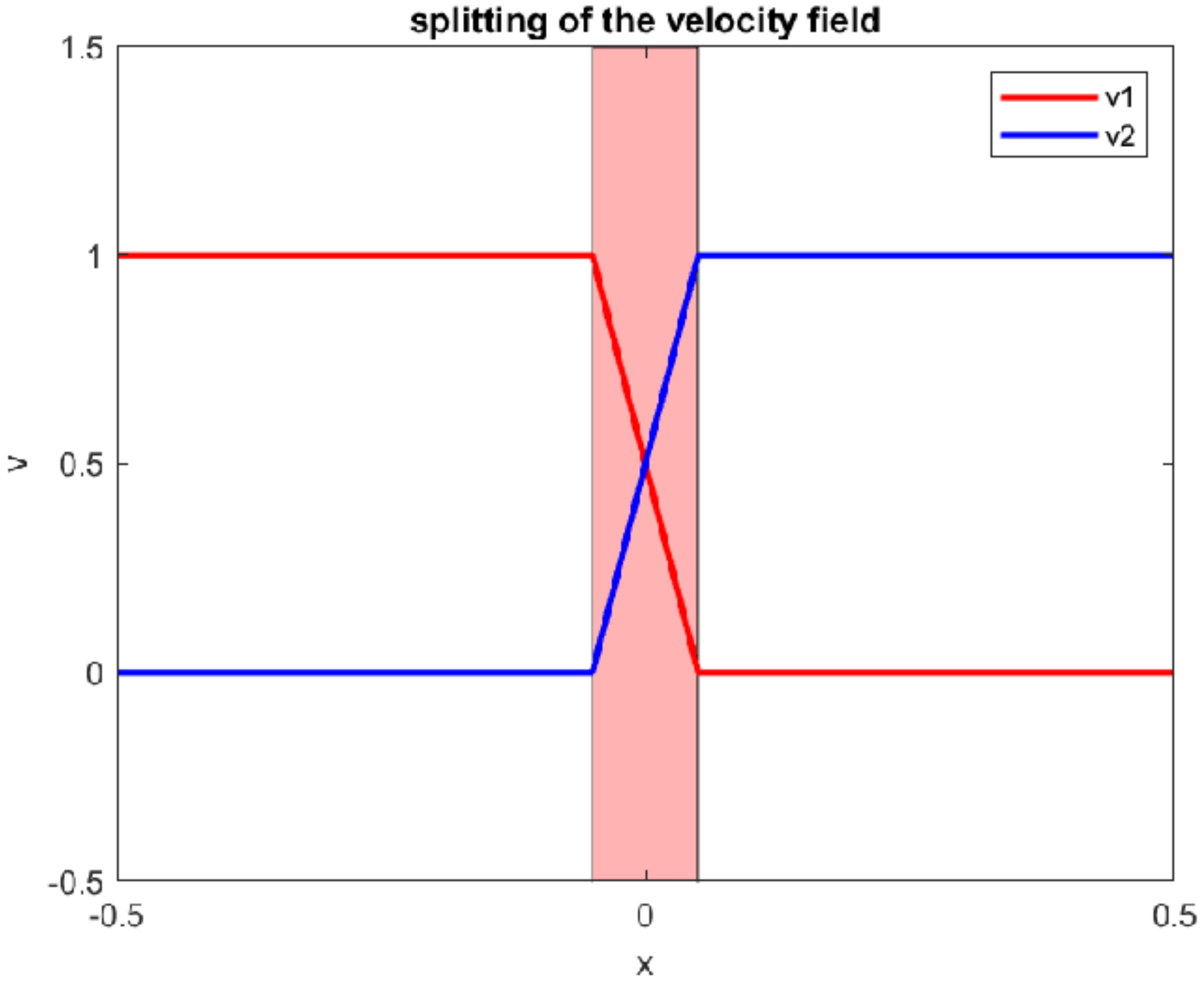
$$v_h(\boldsymbol{\omega}, x) = \frac{M}{P} \sum_{m \in B_k} v_m(x), \qquad t \in [t_{k-1}, t_k).$$

☑ Let $y_h(\omega, t, x)$ be the solution resulting from $v_h(\boldsymbol{\omega}, t, x)$, $h = max_{k \in \{1,\dots,K\}} t_k - t_{k-1}$, then there exists a constant C such that

$$\mathbb{E}[\,|y_h(t,x) - y(t,x)|^2] \leq C_{[P,M,T,Lip]} h\,.$$

Transport equation: Visualization

$$v(x) \equiv 1 = \color{red}{v_1(x)} + \color{blue}{v_2(x)}$$



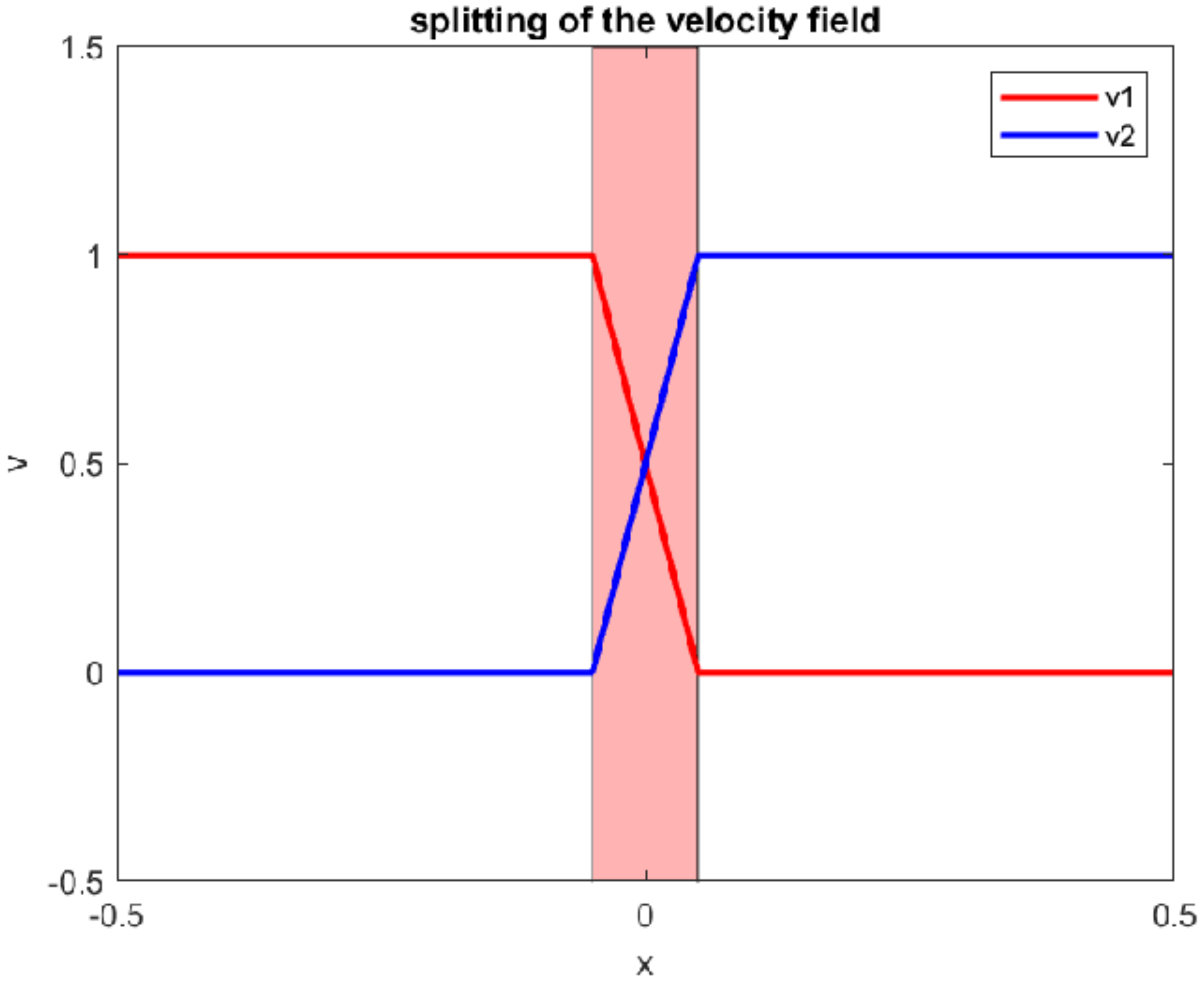
$$v_1(x) = v_0 \begin{cases} 1 & x < -\varepsilon, \\ \frac{1}{2} - \frac{x}{2\varepsilon} & x \in [-\varepsilon, \varepsilon], \\ 0 & x > \varepsilon, \end{cases}$$

$$v_2(x) = v_0 \begin{cases} 0 & x < -\varepsilon, \\ \frac{1}{2} + \frac{x}{2\varepsilon} & x \in [-\varepsilon, \varepsilon], \\ 1 & x > \varepsilon, \end{cases}$$

h=0.01

Transport equation: Visualization

$$v(x) \equiv 1 = {\color{red} v_1(x)} + {\color{blue} v_2(x)}$$



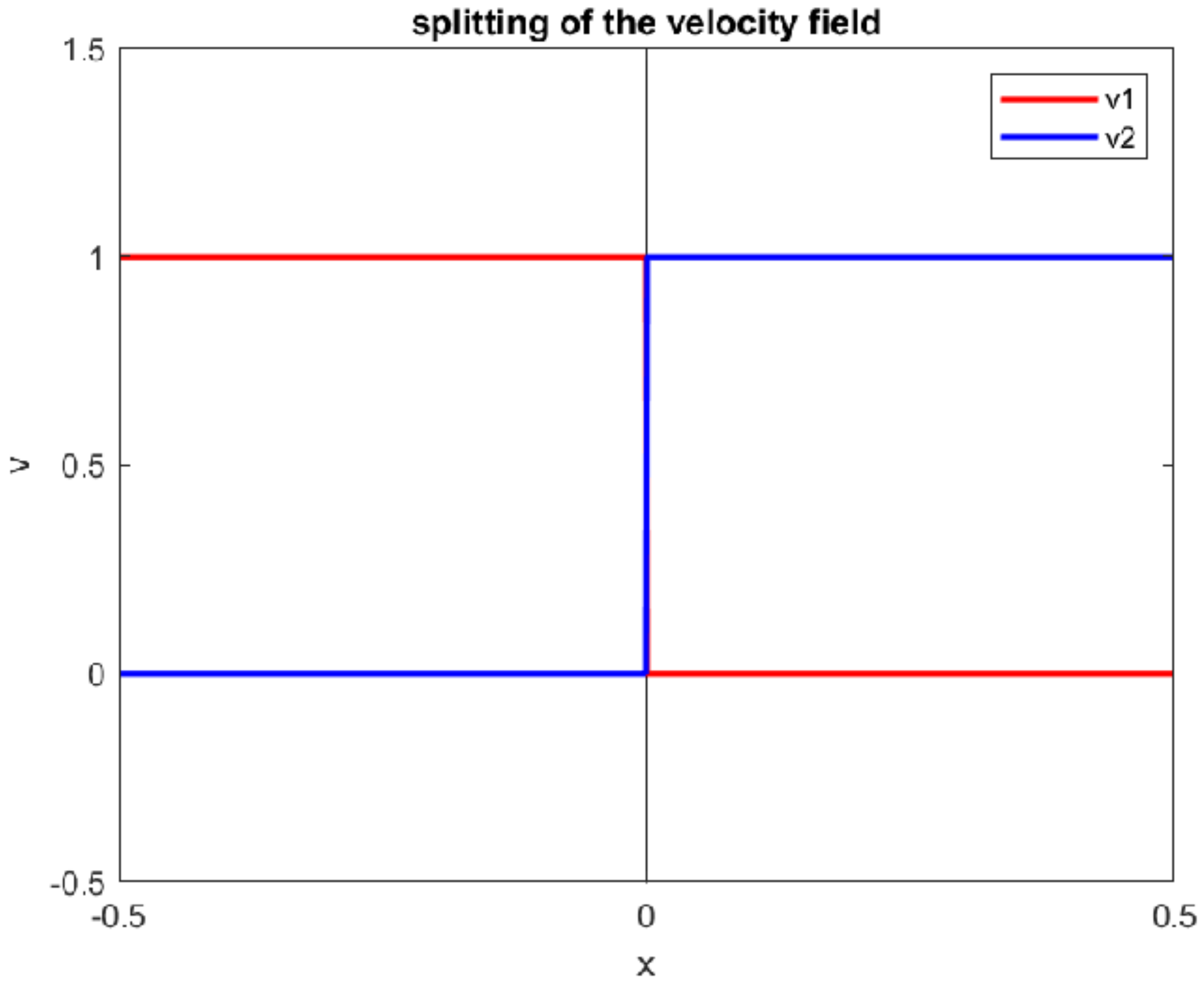
$$v_1(x) = v_0 \begin{cases} 1 & x < -\varepsilon, \\ \frac{1}{2} - \frac{x}{2\varepsilon} & x \in [-\varepsilon, \varepsilon], \\ 0 & x > \varepsilon, \end{cases}$$

$$v_2(x) = v_0 \begin{cases} 0 & x < -\varepsilon, \\ \frac{1}{2} + \frac{x}{2\varepsilon} & x \in [-\varepsilon, \varepsilon], \\ 1 & x > \varepsilon, \end{cases}$$

h=0.001

Transport equation: Visualization (splitting at the node)

$$v(x) \equiv 1 = {\color{red} v_1(x)} + {\color{blue} v_2(x)}$$



$\epsilon = 0$

Networked Linear Hyperbolic Systems

General hyperbolic system after rescaling on $(t, x) \in (0, T) \times (0, L)$

$$\frac{\partial \mathbf{y}}{\partial t}(t,x) + \mathbf{\Lambda}(t,x)\frac{\partial \mathbf{y}}{\partial x}(t,x) = \mathbf{G}(t,x)\mathbf{y}(t,x) + \mathbf{B}(t,x)\mathbf{u}_{int}(t,x) + \mathbf{f}(t,x),$$
$$\mathbf{y}_{\text{in}}(t) = \mathbf{K}(t)\mathbf{y}_{\text{out}}(t) + \mathbf{P}(t)\mathbf{u}_b(t) + \mathbf{g}(t), \qquad \mathbf{y}(0,x) = \mathbf{y}_0(x),$$

- $\mathbf{y}(t,x) := (\underbrace{y^1, ..., y^p}_{\mathbf{y}^+}, \underbrace{y^{p+1}, ..., y^n}_{\mathbf{y}^-}).$
- The matrix $\mathbf{\Lambda}(t,x)$ is diagonal and its entries $\lambda_i(t,x)$ are Lipschitz continuous in x, and real $\lambda_i(t,x) > 0 (i = 1, ..., p)$, $\lambda_i(t,x) < 0 (i = p+1, ..., n)$.
- $\mathbf{y}_{\text{in}}(t) = \begin{bmatrix} \mathbf{y}^+(t,0) \\ \mathbf{y}^-(t,L) \end{bmatrix}, \qquad \mathbf{y}_{\text{out}}(t) = \begin{bmatrix} \mathbf{y}^+(t,L) \\ \mathbf{y}^-(t,0) \end{bmatrix},$

Networked Linear Hyperbolic Systems

General hyperbolic system after rescaling on $(t, x) \in (0, T) \times (0, L)$

$$\frac{\partial \mathbf{y}}{\partial t}(t,x) + \mathbf{\Lambda}(t,x)\frac{\partial \mathbf{y}}{\partial x}(t,x) = \mathbf{G}(t,x)\mathbf{y}(t,x) + \mathbf{B}(t,x)\mathbf{u}_{int}(t,x) + \mathbf{f}(t,x),$$
$$\mathbf{y}_{\text{in}}(t) = \mathbf{K}(t)\mathbf{y}_{\text{out}}(t) + \mathbf{P}(t)\mathbf{u}_b(t) + \mathbf{g}(t), \qquad \mathbf{y}(0,x) = \mathbf{y}_0(x),$$

- Compatibility condition and regularity assumptions are required.
  - $\mathbf{y}_0 \in \mathrm{Lip}(0, L; \mathbb{R}^n)$,
  - $\mathbf{g} \in L^\infty(0, T; \mathrm{Lip}(0, L; \mathbb{R}^{n\times n}))$,
  - $\mathbf{B} \in L^\infty(0, T; \mathrm{Lip}(0, L; \mathbb{R}^{n\times m_{\text{int}}}))$,
  - $\mathbf{f} \in L^\infty(0, T; \mathrm{Lip}(0, L; \mathbb{R}^{n}))$,
  - $\mathbf{K} \in \mathrm{Lip}(0, T; \mathbb{R}^{n\times n})$, $\mathbf{P} \in \mathrm{Lip}(0, T; \mathbb{R}^{n\times m_{\text{b}}})$, $\mathbf{g} \in \mathrm{Lip}(0, T; \mathbb{R}^{n})$.

Randomized Splitting Scheme

Step 1 Split the matrices $\mathbf{\Lambda}(t,x)$, $\mathbf{G}(t,x)$, and $\mathbf{B}(t)$ as

$$\mathbf{\Lambda}(t,x)=\sum_{m=1}^{M}\mathbf{\Lambda}_m(t,x),\quad \mathbf{G}(t,x)=\sum_{m=1}^{M}\mathbf{G}_m(t,x),\quad \mathbf{B}(t)=\sum_{m=1}^{M}\mathbf{B}_m(t),$$

Step 2 Divide the time interval $(0,T)$ into K subintervals with maximal length h, i.e.

$$0=t_0<t_1<t_2<\ldots<t_{K-1}<t_K=T,\qquad h=\max_{k\in\{1,2,\ldots,K\}} t_k-t_{k-1}$$

Step 3 In each time step, randomly choose batch $B_k\subset\{1,\ldots,M\}$ of size P, and define the randomized matrices

$$\mathcal{L}_h(\omega,t,x)=\frac{M}{P}\sum_{m\in B_k}\mathbf{\Lambda}_m(t,x),$$

and $\mathcal{G}_h$, $\mathcal{B}_h$ similarly, where $\omega=(\omega_1,\omega_2,\ldots,\omega_K)$ recording the chosen batches.

$$\frac{\partial \mathbf{y}}{\partial t}(t,x) + \mathbf{\Lambda}(t,x)\frac{\partial \mathbf{y}}{\partial x}(t,x) = \mathbf{G}(t,x)\mathbf{y}(t,x) + \mathbf{B}(t,x)\mathbf{u}_{int}(t,x) + \mathbf{f}(t,x),$$
$$\mathbf{y}_{\text{in}}(t) = \mathbf{K}(t)\mathbf{y}_{\text{out}}(t) + \mathbf{P}(t)\mathbf{u}_b(t) + \mathbf{g}(t), \qquad \mathbf{y}(0,x) = \mathbf{y}_0(x),$$

⇓ RBM (STEPs 1,2,3) ⇓

Randomized-System

$$\frac{\partial \mathbf{y}_h}{\partial t}(\omega,t,x) + \mathcal{L}_h(\omega,t,x)\frac{\partial \mathbf{y}_h}{\partial x}(\omega,t,x) = \mathcal{G}_h(\omega,t,x)\mathbf{y}_h(\omega,t,x) + \mathcal{B}_h(\omega,t,x)\mathbf{u}_{int}(t,x) + \mathbf{f}(t,x),$$
$$\mathbf{y}_{h,\text{in}}(\omega,t) = \mathbf{K}(t)\mathbf{y}_{h,\text{out}}(\omega,t) + \mathbf{P}(t)\mathbf{u}_b(t) + \mathbf{g}(t), \qquad \mathbf{y}_h(\omega,0,x) = \mathbf{y}_0(x),$$

where $\mathbb{E}[\mathcal{L}_h(t,x)] = \Lambda(t,x), \quad \mathbb{E}[\mathcal{G}_h(t,x)] = \mathbf{G}(t,x), \quad \mathbb{E}[\mathcal{B}_h(t)] = \mathbf{B}(t).$

Question 1: Is $\mathbf{y}_h(\omega,t,x)$ a good approximation of $\mathbf{y}(t,x)$ for h sufficiently small?

Randomized Optimal Control Problem

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$$\min_{\mathbf{u}_{int},\mathbf{u}_b} J(\mathbf{u}_{int},\mathbf{u}_b) = \frac{1}{2}\|\mathbf{y}-\mathbf{y}_d\|^2_{L^2(Q)} + \frac{s_0}{2}|\mathbf{u}_{int}|^2_{W^{1,2}(Q)} + \frac{s_1}{2}|\mathbf{u}_b|^2_{W^{1,2}(0,T)}$$

where $\mathbf{y}(t,x)$ is the solution of

$$\frac{\partial \mathbf{y}}{\partial t}(t,x) + \mathbf{\Lambda}(t,x)\frac{\partial \mathbf{y}}{\partial x}(t,x) = \mathbf{G}(t,x)\mathbf{y}(t,x) + \mathbf{B}(t,x)\mathbf{u}_{int}(t,x) + \mathbf{f}(t,x),$$
$$\mathbf{y}_{\text{in}}(t) = \mathbf{K}(t)\mathbf{y}_{\text{out}}(t) + \mathbf{P}(t)\mathbf{u}_b(t) + \mathbf{g}(t), \qquad \mathbf{y}(0,x) = \mathbf{y}_0(x),$$

The minimizer is denoted by $(\mathbf{u}^*_{int}(t,x), \mathbf{u}^*_b(t))$.

$$\min_{\mathbf{u}_{int},\mathbf{u}_b} J_h(\omega,\mathbf{u}_{int},\mathbf{u}_b) = \frac{1}{2}\|\mathbf{y}_h(\omega)-\mathbf{y}_d\|^2_{L^2(Q)} + \frac{s_0}{2}|\mathbf{u}_{int}|^2_{W^{1,2}(Q)} + \frac{s_1}{2}|\mathbf{u}_b|^2_{W^{1,2}(0,T)}$$

where $\mathbf{y}_h(\omega,t,x)$ is the solution of

$$\frac{\partial \mathbf{y}_h}{\partial t}(\omega,t,x) + \mathcal{L}_h(\omega,t,x)\frac{\partial \mathbf{y}_h}{\partial x}(\omega,t,x) = \mathcal{G}_h(\omega,t,x)\mathbf{y}_h(\omega,t,x) + \mathcal{B}_h(\omega,t,x)\mathbf{u}_{int}(t,x) + \mathbf{f}(t,x),$$
$$\mathbf{y}_{h,\text{in}}(\omega,t) = \mathbf{K}(t)\mathbf{y}_{h,\text{out}}(\omega,t) + \mathbf{P}(t)\mathbf{u}_b(t) + \mathbf{g}(t), \qquad \mathbf{y}_h(\omega,0,x) = \mathbf{y}_0(x).$$

The minimizer (for a fixed ω) is denoted by $(\mathbf{u}^*_{h,int}(\omega,t,x), \mathbf{u}^*_{h,b}(\omega,t))$.

Question 2: Does $(\mathbf{u}^*_{h,int}(\omega,t,x), \mathbf{u}^*_{h,b}(\omega,t))$ approximate $(\mathbf{u}^*_{int}(t,x), \mathbf{u}^*_b(t))$for h small?

Convergence Results

D.W.M. VELDMAN, Y. WANG, E. ZUAZUA 2024. Efficient Simulation and Optimal Control of Networked Linear Hyperbolic Systems by the Random Batch Method (proceeding).

Theorem 1

For $\mathbf{u}_{int}(t,x) \in L^\infty(0,T;\mathrm{Lip}(0,L;\mathbb{R}^{m_{\mathrm{int}}}))$ and $\mathbf{u}_b(t) \in \mathrm{Lip}(0,T;\mathbb{R}^{m_{\mathrm{b}}})$, there exists a constant C independent of the considered time grid such that for all $t \in [0,T]$

$$\mathbb{E}[\,|\,\mathbf{y}_h(t) - \mathbf{y}(t)\,|^2_{L^\infty(0,L;\mathbb{R}^n)}] \leq Ch\log(h^{-1})\,.$$

Remark: $Ch\log(h^{-1}) \leq \dfrac{C}{\epsilon e}h^{1-\epsilon},\ \forall \epsilon > 0.$

Theorem 2

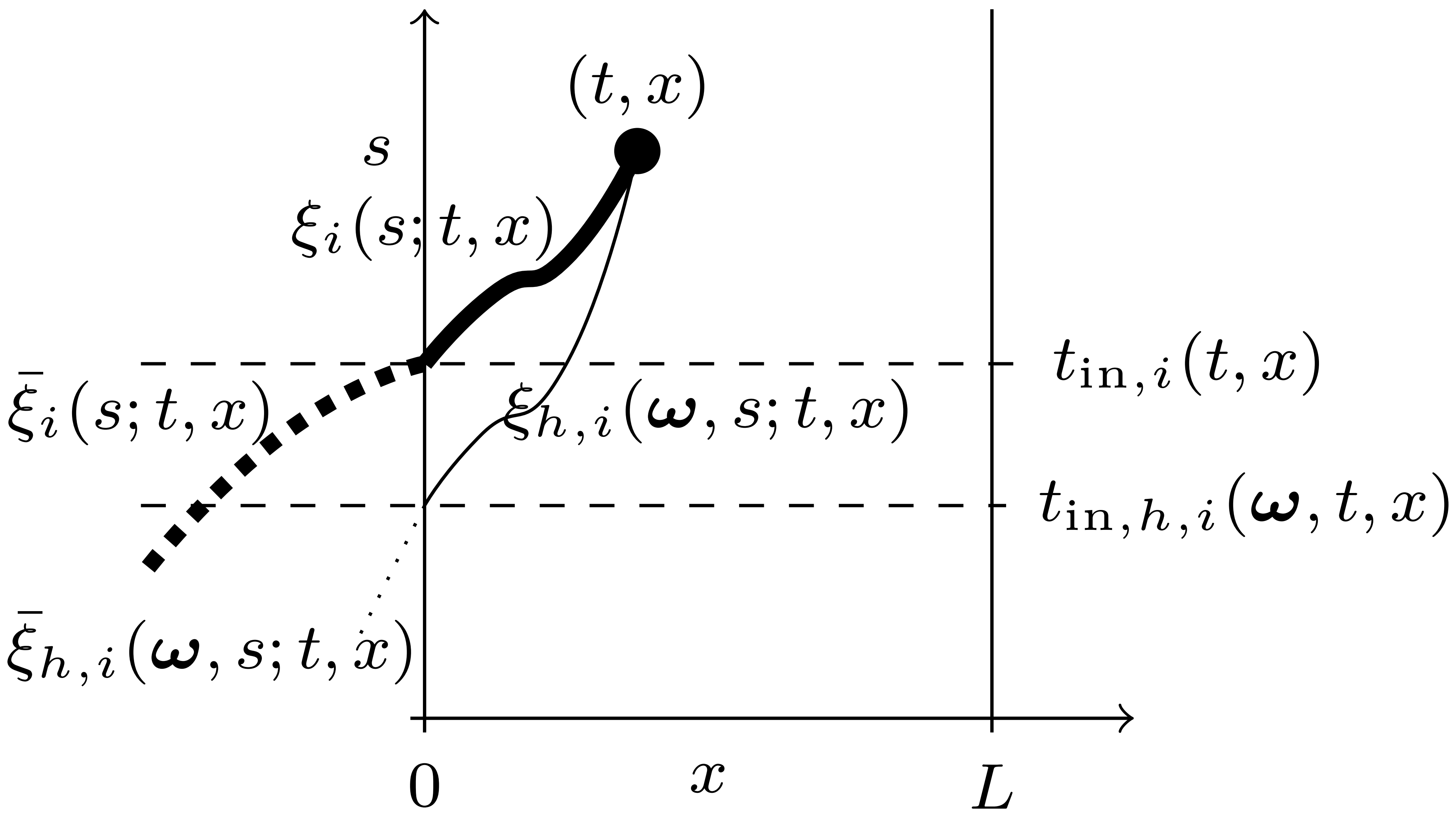
Convergence in controls:

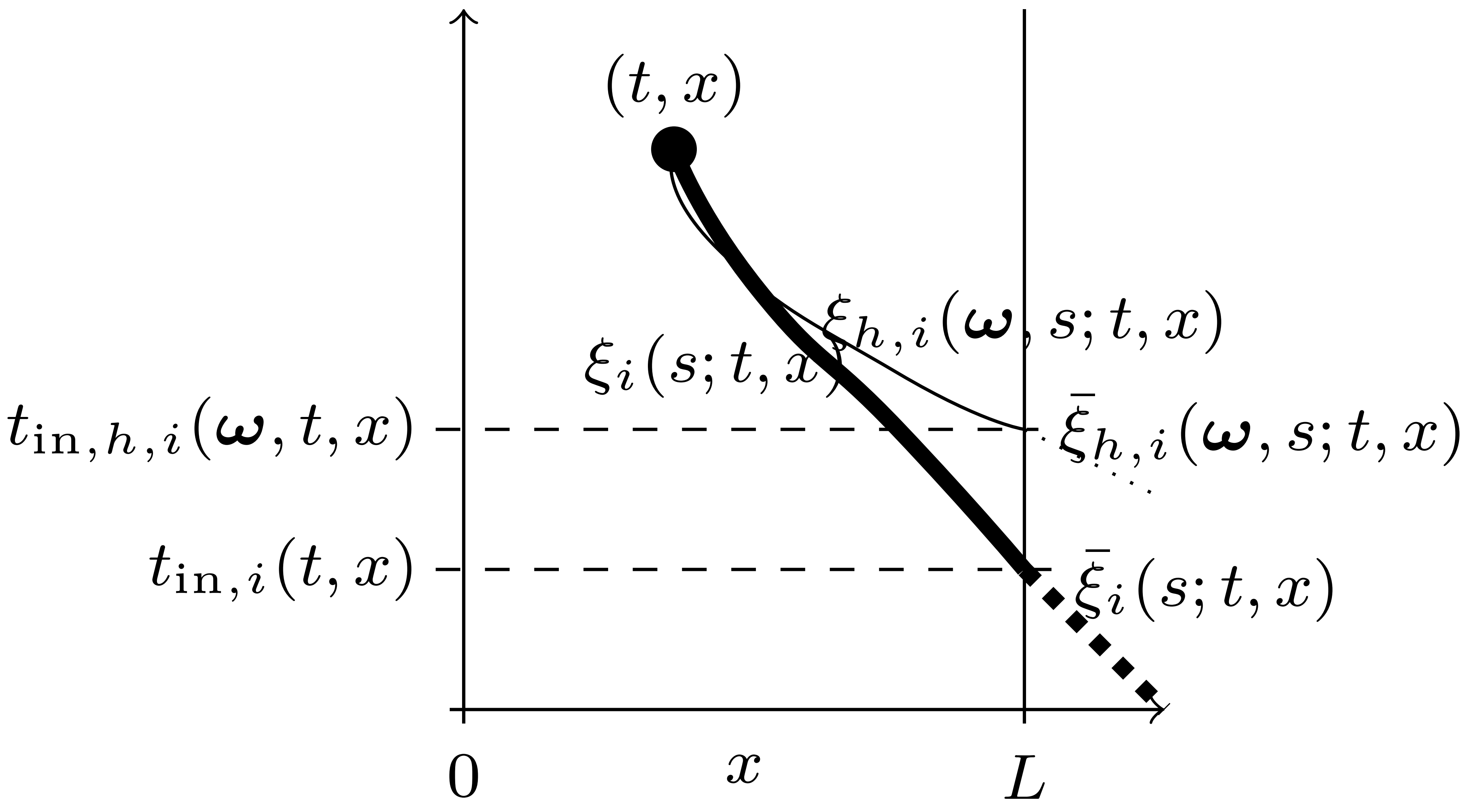
$$lim_{h\to 0}\mathbb{E}[\,|\,\mathbf{u}^*_{h,int} - \mathbf{u}^*_{int}\,|^2_{L^2((0,L)\times(0,T);\mathbb{R}^{m_{int}})} + |\,\mathbf{u}^*_{h,b} - \mathbf{u}^*_b\,|^2_{L^2(0,T;\mathbb{R}^{m_b})}] = 0.$$

Sketch of proof for Theorem 1

- Consider the characteristics terminating at (t,x) for $s \in [0,t]$

$$\dot{\xi}_{h,i}(\omega,s;t,x) = \ell_{h,i}(\omega,s,\xi_{h,i}(\omega,s;t,x)), \qquad \dot{\xi}_i(s;t,x) = \lambda_i(s,\xi_i(s;t,x)),$$
$$\xi_{h,i}(\omega,t;t,x) = \xi_i(t;t,x) = x.$$





$$y_i(t,x) = \int_{t_{\mathrm{in},i}(t,x)}^{t} [\mathbf{G}\mathbf{y} + \mathbf{B}\mathbf{u}_{int} + \mathbf{f}]_i(s,\xi_i(s;t,x))\ \mathrm{d}s + \begin{cases} y_{0,i}(\xi_i(0;t,x)) & \text{if } t_{\mathrm{in},i}(t,x) = 0, \\ [\mathbf{K}\mathbf{y}_{\mathrm{out}} + \mathbf{P}\mathbf{u}_b + \mathbf{g}]_i(t_{\mathrm{in},i}(t,x)) & \text{if } t_{\mathrm{in},i}(t,x) > 0. \end{cases}$$

Sketch of proof for Theorem 1

- Consider the characteristics terminating at (t, x) for $s \in [0, t]$

$$\dot{\xi}_{h,i}(\omega, s; t, x) = \ell_{h,i}(\omega, s, \xi_{h,i}(\omega, s; t, x)), \qquad \dot{\xi}_i(s; t, x) = \lambda_i(s, \xi_i(s; t, x)),$$
$$\xi_{h,i}(\omega, t; t, x) = \xi_i(t; t, x) = x.$$

- We show that there is a constant C such that for $s \in [0, t]$

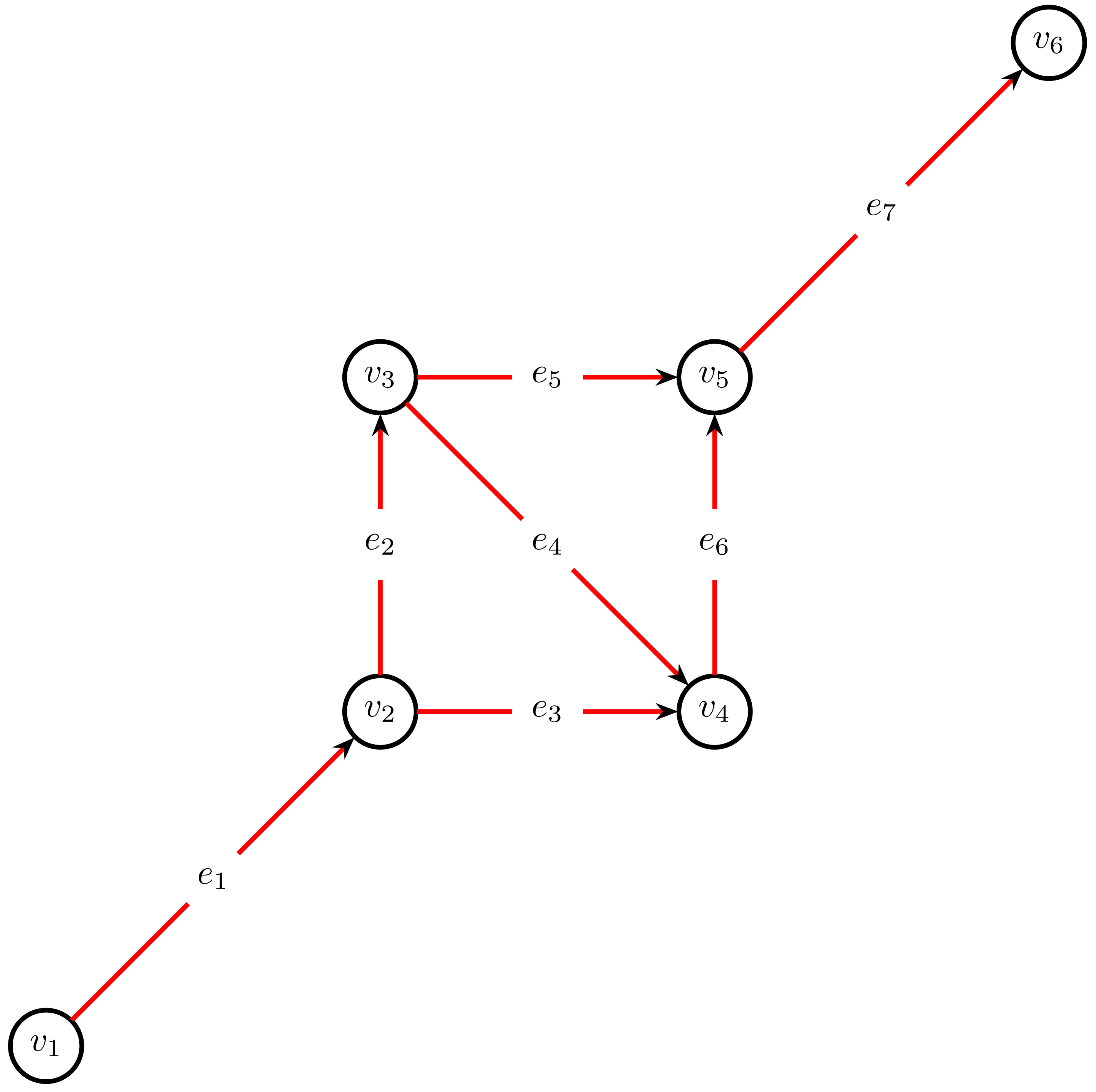
$$\mathbb{E}[|\xi_{h,i}(s; t, x) - \xi_i(s; t, x)|^2] \leq Ch, \qquad \mathbb{E}[|\xi_{h,i}(s; t) - \xi_i(s; t)|^2_{L^\infty(0,L;\mathbb{R}^n)}] \leq Ch \log(h^{-1}).$$

- Let $t_{h,in,i}(\omega, t, x)$ and $t_{in,i}(t, x)$ denote the values of s for which $(s, \xi_{h,i}(\omega, s; t, x))$ and $(s, \xi_i(s; t, x))$ leave $(0, T) \times (0, L)$. We then show there is a constant C s.t.

$$\mathbb{E}[|t_{h,in,i}(t, x) - t_{in,i}(t, x)|^2] \leq Ch, \qquad \mathbb{E}[|t_{h,in,i}(t) - t_{in,i}(t)|^2_{L^\infty(0,L;\mathbb{R}^n)}] \leq Ch \log(h^{-1}).$$

- Integrating $y_{h,i}(\omega, s, \xi_{h,i}(\omega, s; t, x))$ and $y_i(s, \xi_i(s; , t, x))$ along characteristics and some cumbersome calculations, we arrive at Theorem 1.

Coupled Wave Equations on Diamond Networks



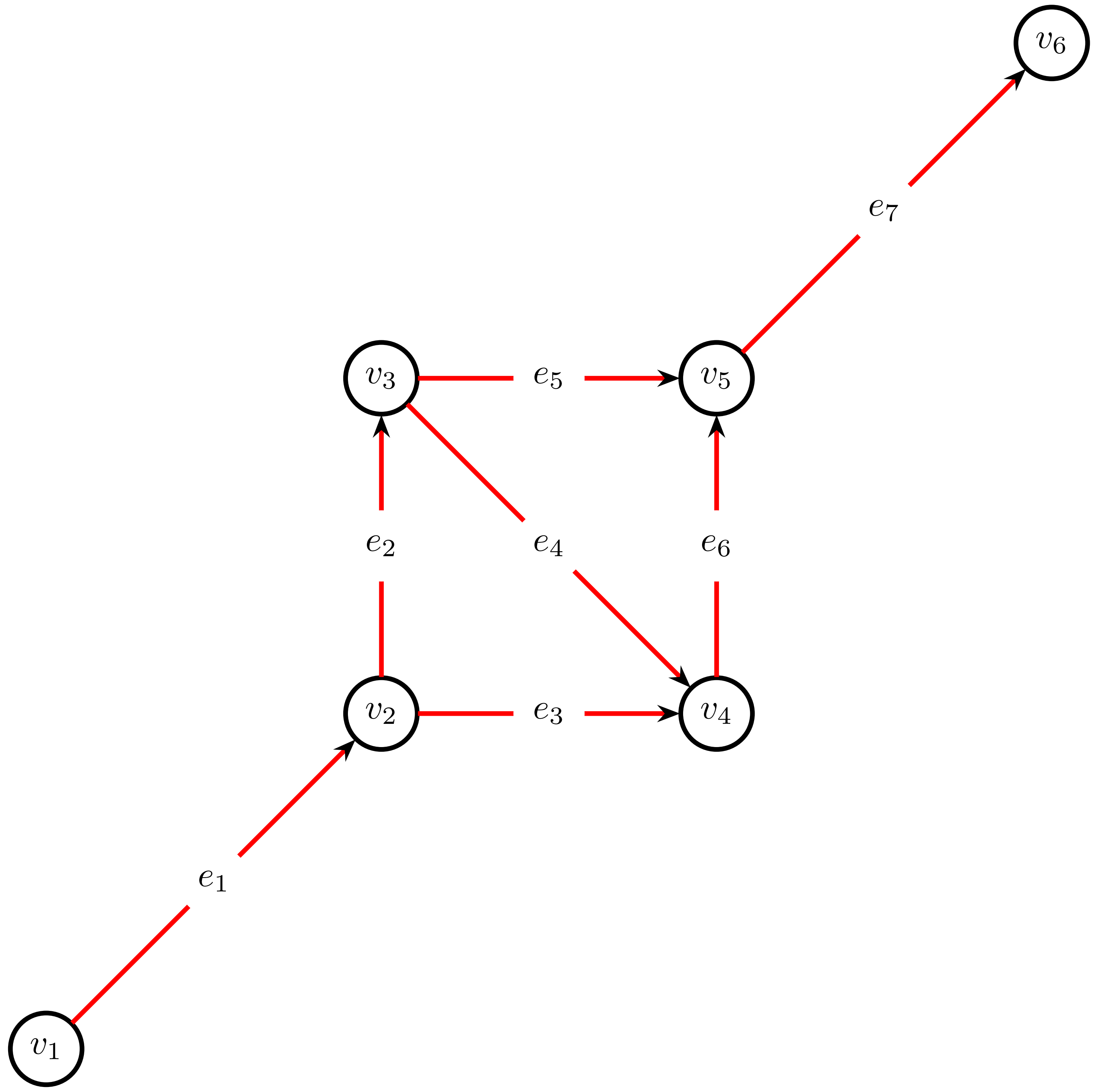
Diamond Directed Graph

$$
\begin{cases}
y^{e_i}_{tt}(t,x) - c^2_{e_i} y^{e_i}_{xx}(t,x) = 0 & e_i \in E, \\
\displaystyle\sum_{e_i \in E(v_j)} D_{ji} c_{e_i} y^{e_i}_x(t, v_j) = \bar{u}^{v_j}(t) & v_j \in V, \\
y^{e_i}(t, v_j) = y^{e_k}(t, v_j), & \forall e_i, e_k \in E(v_j), v_j \in V, \\
y^{e_i}(0,x) = y^{e_i}_0(x), \quad y^{e_i}_t(0,x) = y^{e_i}_1(x), & e_i \in E,
\end{cases}
$$

Vertices: $V = \{v_1, v_2, \ldots, v_6\}$
Edges: $E = \{e_1, \ldots, e_7\}$
Incidence Matrix:

$$
D = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Coupled Wave Equations on Diamond Networks



Diamond Directed Graph

Riemann Variables for Each Wave Equation

$$\mathbf{w}^{e_i}(t,x) = \begin{pmatrix} w_-^{e_i}(t,x) \\ w_+^{e_i}(t,x) \end{pmatrix} = \begin{pmatrix} y_t^{e_i}(t,x) + c_{e_i} y_x^{e_i}(t,x) \\ y_t^{e_i}(t,x) - c_{e_i} y_x^{e_i}(t,x) \end{pmatrix}$$

$$\Rightarrow \begin{cases} w_{-,t}^{e_i}(t,x) - c_{e_i} w_{-,x}^{e_i}(t,x) = 0, \\ w_{+,t}^{e_i}(t,x) + c_{e_i} w_{+,x}^{e_i}(t,x) = 0. \end{cases}$$

$$W_t - \Lambda W_x = 0, \quad \text{where } \Lambda = \begin{pmatrix} -c_1 & 0 & \dots & \dots \\ 0 & c_1 & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

+ B.C. in the form of

$$w_{\text{in}}^{e_i}(t,v_j) = -w_{\text{out}}^{e_i}(t,v_j) + \frac{2}{|E(v_j)|}\left(\sum_{e_k \in E(v_j)} w_{\text{out},j}^{e_k}(t,v_j) - \bar{u}^{v_j}(t)\right)$$

RBM Scheme for Coupled Wave Equations on Diamond Networks

① Enumerate the subsets of E as $E_1, E_2, \ldots, E_{2^{|E|}}$. Assign to each subset $S_\omega(\omega \in \{1,2,\ldots,2^{|E|}\}$ a probability $p_\omega \geq 0$.

② Divide$[0,T]$ into K subintervals $(t_{k-1}, t_k]$ with $h = \max_{k\in\{1,2,\ldots,K\}} t_k - t_{k-1}$. For each$(t_{k-1}, t_k]$, randomly choose an index $\omega_k \in \{1,2,\ldots,2^{|E|}\}$ according to the probabilities p_ω. Set vector $\omega = (\omega_1, \omega_2, \ldots, \omega_K)$.

③ We introduce $\chi_{e_i}(\omega) = \begin{cases} 1, & e_i \in E_\omega, \\ 0, & e_i \notin E_\omega\,. \end{cases}$, $\qquad \pi_{e_i} := \mathbb{E}[\chi_{e_i}] = \sum_{\omega\in\{\omega|e_i\in E_\omega\}} p_\omega,$

where $\pi_{e_i} \in [0,1]$ represents the probability that an edge e_i is an element of the selected subset. Define the new propagation speed as

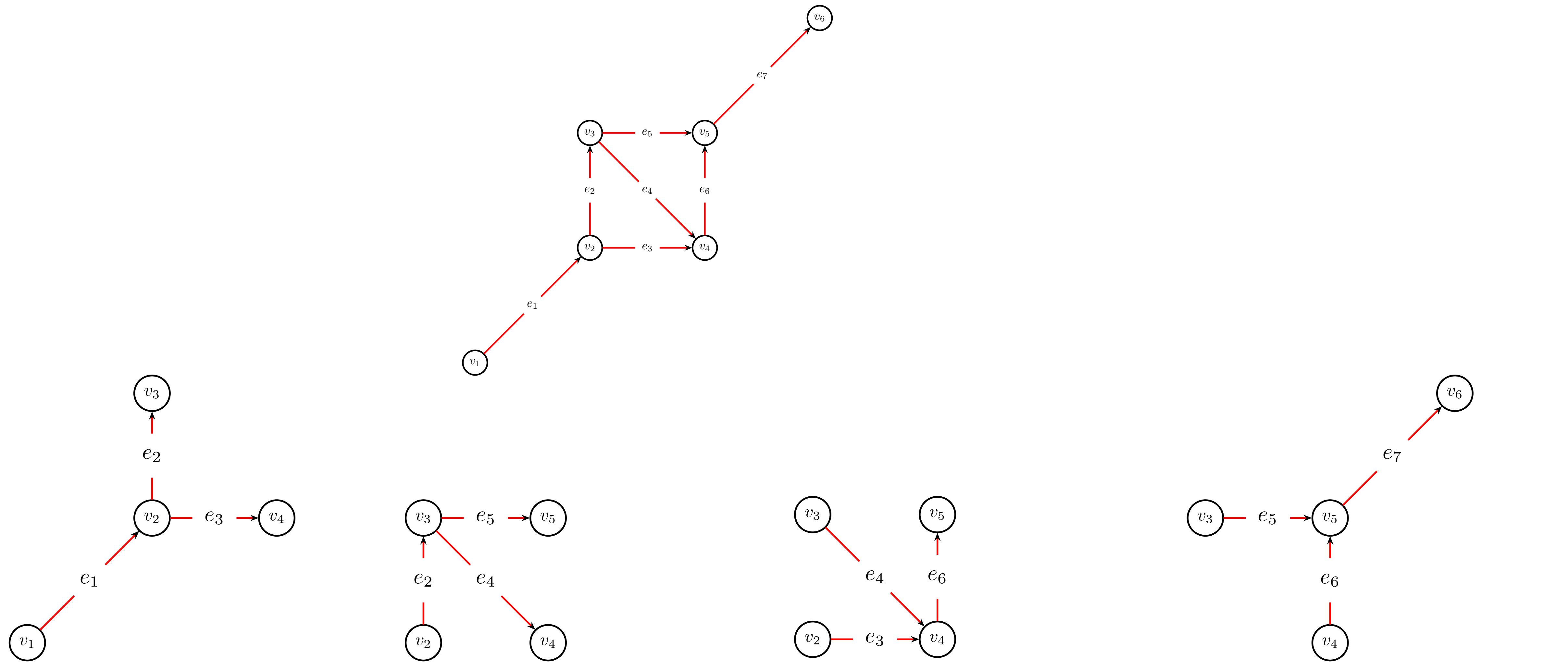
$$c_{h,e_i}(\omega, t) := \frac{c_{e_i}}{\pi_{e_i}}\chi_{e_i}(\omega_k), \qquad t \in (t_{k-1}, t_k]\,.$$

④ Compute the solution to (RD)

$$\begin{aligned}
&w^{e_i}_{h-,t}(\omega,t,x) - c_{h,e_i}(\omega,t)w^{e_i}_{h-,x}(\omega,t,x) = 0,\\
&w^{e_i}_{h+,t}(\omega,t,x) + c_{h,e_i}(\omega,t)w^{e_i}_{h+,x}(\omega,t,x) = 0,\\
&w^{e_i}_{h,\mathrm{in}}(\omega,t,v_j) = -\,w^{e_i}_{h,\mathrm{out}}(\omega,t,v_j) + \frac{2}{|E(v_j)|}\left(\sum_{e_k\in E(v_j)} w^{e_k}_{h,\mathrm{out},j}(\omega,t,v_j) - \bar{u}^{v_j}(t)\right),\\
&w^{e_i}_{h-}(\omega,0,x) = y^{e_i}_1(x) + c_{e_i}y^{e_i}_{0,x}(x),\\
&w^{e_i}_{h+}(\omega,0,x) = y^{e_i}_1(x) - c_{e_i}y^{e_i}_{0,x}(x)
\end{aligned}$$

and its optimal control problems

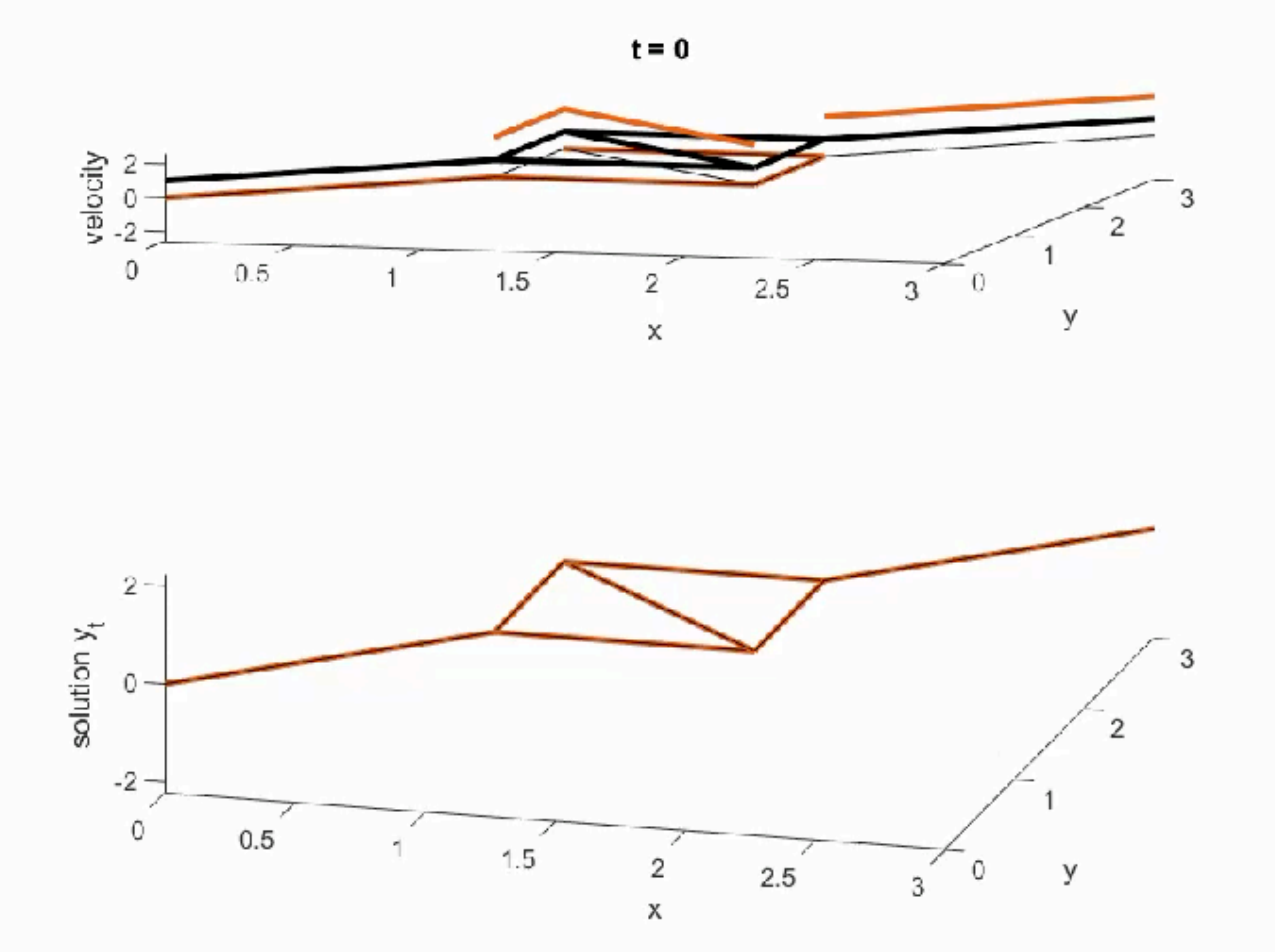
Numerical Illustration



(a) Subgraph (V_1, E_1, L_1) (b) Subgraph (V_2, E_2, L_2) (c) Subgraph (V_3, E_3, L_3) (d) Subgraph (V_4, E_4, L_4)

- Split the velocity field per edge.
- P=3 of M= 7 edges are active simultaneously.
- On each time interval, we randomly choose one of the subgraphs with the same possibility $p_\omega = \dfrac{1}{4}(\omega = 1,\ldots,4)$, and compute the solution.

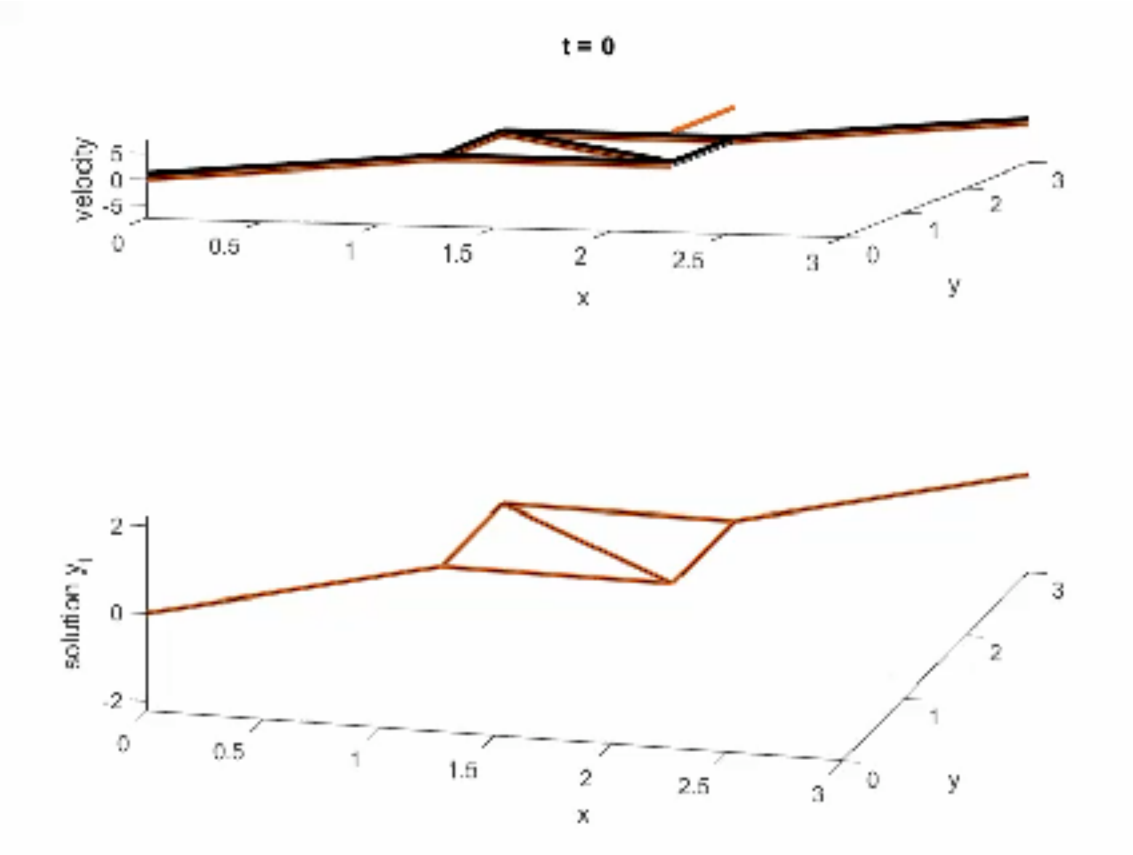
$$c_{h,i} = \begin{cases} 4c_i, & \text{for} \quad i \in \{1,7\} \cap S_{\omega_k} \\ 2c_i, & \text{for} \quad i \in \{2,3,4,5,6\} \cap S_{\omega_k} \\ 0, & i \notin S_{\omega_k}. \end{cases}$$



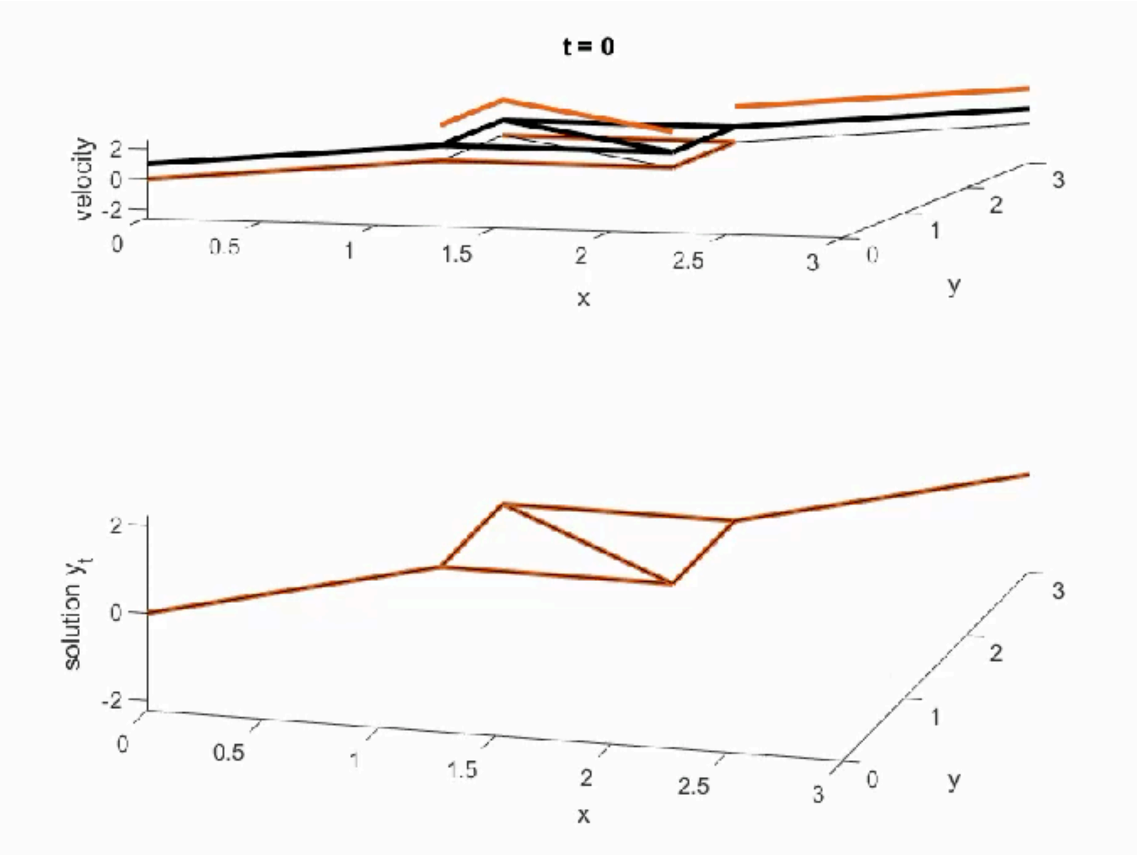
p=3

- $h = 0.005,\ dx = 0.05$
- Full model (black): 1.7s
- RBM-Sim. Time (orange): 1.1s
- Reduction: 37%
- Error: 33%

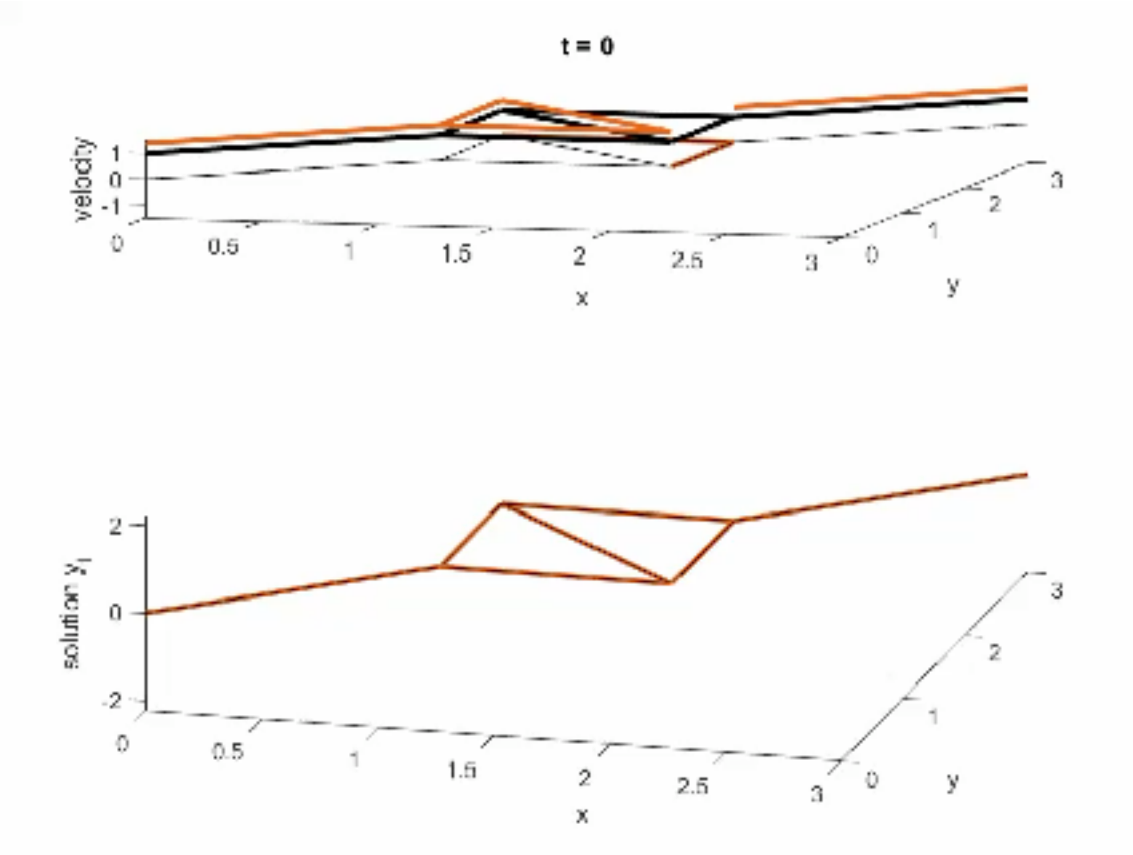
Numerical Illustration



p=1



p=3



P=5

dx	h	P	Full sim time*	RBM sim time*	Reduction in time* [%]	Error* [%]
0.05	0.005	1	1.7	1.1	31	67
0.05	0.005	2	1.7	1.0	34	43
0.05	0.005	3	1.7	1.1	37	37
0.05	0.005	4	1.7	1.2	29	29
0.05	0.005	5	1.7	1.4	14	19

*Reported are the averages values over 20 simulations

Convergence Results

D.W.M. VELDMAN, Y. WANG, E. ZUAZUA 2024. A Stochastic Algorithm for the Efficient Simulation and Optimal Control of Networked 1-D Wave Equations (pro

Theorem 3

If the initial conditions $(\mathbf{y}_0, \mathbf{y}_1)$ are such that $y_0^{e_i} \in C^2(0,\ell_{e_i})$ and $y_1^{e_i} \in C^1(0,\ell_{e_i})$ with compatibility conditions, and the control $\mathbf{u} \in C^1(0,T;\mathbb{R}^{|V_C|})$, there exists constants $C \geq 1$ and $\mu > 0$ independent of h such that

$$\mathbb{E}[\,|y_h^{e_i}(t) - y^{e_i}(t)|^2_{C^1(0,\ell_{e_i})}] \leq Cht^2e^{\mu t}.$$

y: original solution

y_h: solution to randomized system

- Remark: Markovs inequality $\quad \mathbb{P}[X \geq a] \leq \dfrac{\mathbb{E}[X]}{a}$

Theorem 4

$$\min_{\mathbf{u}} J(\mathbf{u}) = \frac{1}{2}\|\mathbf{y} - \mathbf{y}_d\|^2_{L^2(Q)} + \frac{s_0}{2}|\mathbf{u}|^2_{L^2(0,T)} + \frac{s_1}{2}|\mathbf{u}_t|^2_{L^2(0,T)}$$

If the initial conditions $(\mathbf{y}_0, \mathbf{y}_1)$ are such that $y_0^{e_i} \in H^2(0,\ell_{e_i})$ and $y_1^{e_i} \in H^1(0,\ell_{e_i})$ with compatibility conditions, then

If $s_1 > 0$, then $\quad \lim_{h\to 0} \mathbb{E}[\,|\mathbf{u}_h^* - \mathbf{u}^*|_{L^2(0,T);\mathbb{R}^{|V_C|}}] = 0.$

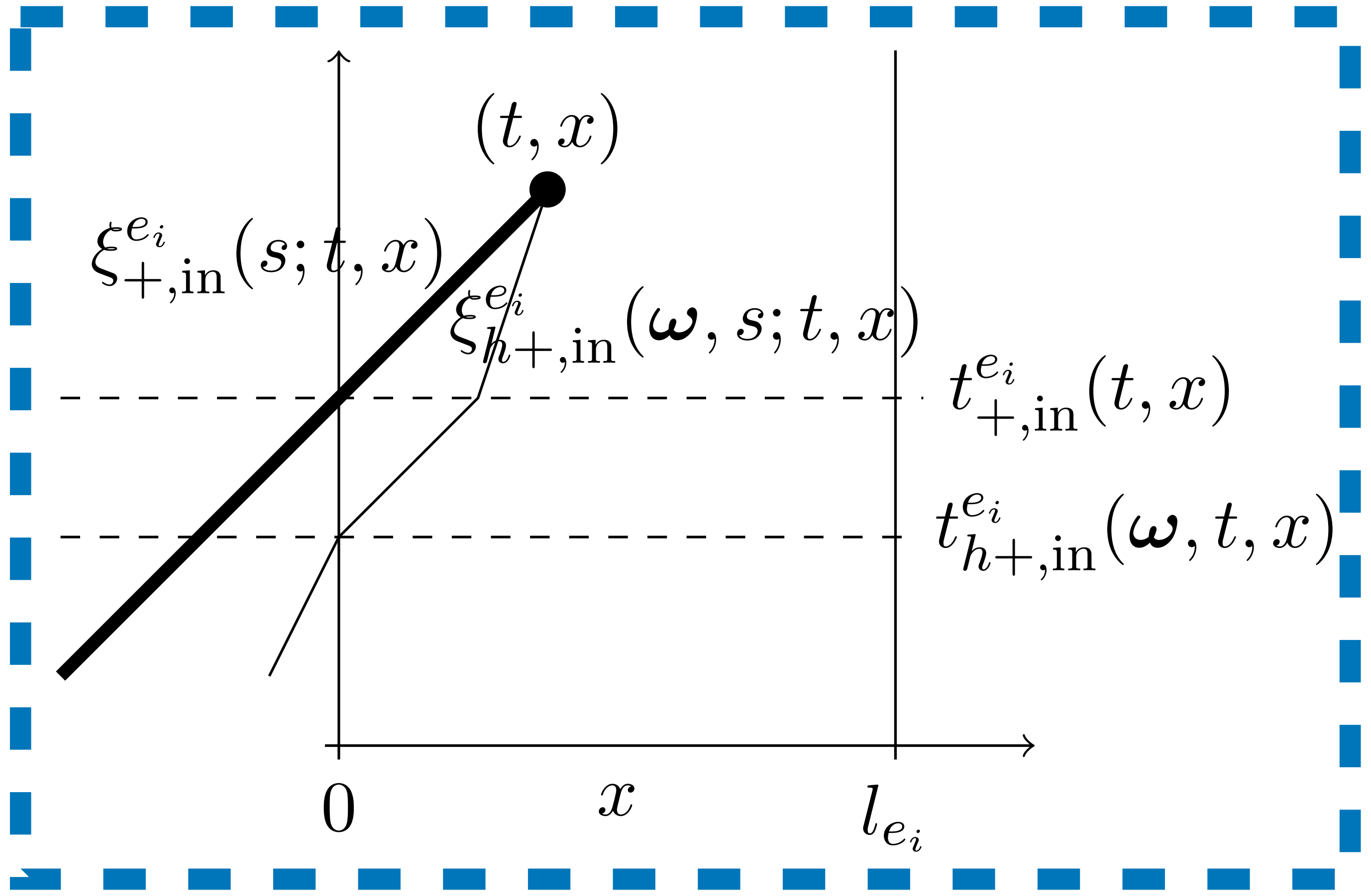
$\mathbf{u}^*$: optimal control to original system

$\mathbf{u}_h^*$: optimal control to randomized system

Sketch of proof for Theorem 3

① Consider the characteristics terminating at (t, x) for $s \in [0,T]$

$$\frac{\mathrm{d}\xi_{\pm}^{e_i}}{\mathrm{d}s}(s;t,x) = \pm\, c_{e_i}, \quad \frac{\mathrm{d}\xi_{h\pm}^{e_i}}{\mathrm{d}s}(\omega,s;t,x) = \pm\, c_{h,e_i}(\omega,s), \quad \xi_{\pm}^{e_i}(t;t,x) = \xi_{h\pm}^{e_i}(\omega,t,t,x) = x\,.$$



② Let $\mathrm{Var}[c_{h,e_i}] = c_{e_i}^2 \sum_{\omega=1}^{2^M} \left(\frac{\chi_{e_i}(\omega)}{\pi_{e_i}} - 1 \right)^2 p_\omega$. We show that

$$\mathbb{E}\left[\, |\, \xi_{h\pm}^{e_i}(s;t) - \xi_{\pm}^{e_i}(s;t)\,|^2_{L^\infty(0,\ell_{e_i})} \right] \le h(t-s)\mathrm{Var}[c_{h,e_i}], \quad \forall 0 \le s \le t\,.$$

③ Let $t_{h\pm,\mathrm{in}}^{e_i}(\omega,t,x)$ and $t_{\pm,\mathrm{in}}^{e_i}(t,x)$ denote the values of s for which the characteristics $\xi_{h\pm}^{e_i}(\omega,s;t,x)$ and $\xi_{\pm}^{e_i}(s;t,x)$ leaving the domain $(0,T)\times(0,l_{e_i})$. We then show there exists a constant C independent of h s.t.

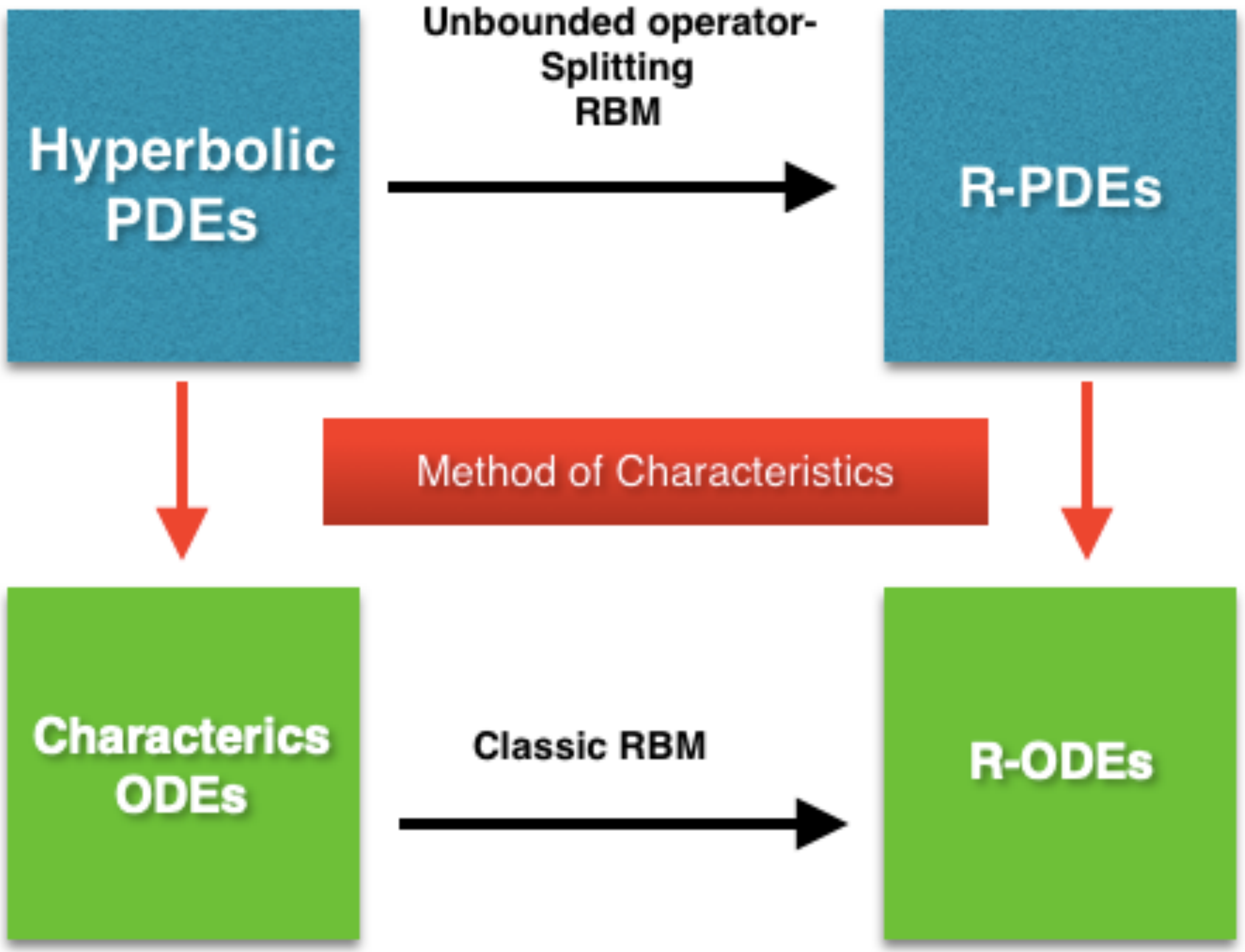
$$\mathbb{E}[\,|\max\{t_{h\pm,\mathrm{in}}^{e_i}(t),s\} - \max\{t_{\pm,\mathrm{in}}^{e_i}(t),s\}\,|^2_{L^\infty(0,\ell_{e_i})}] \le C_2 h(t-s), \quad \forall 0 \le s \le t \le T\,.$$

④ Integrating $w_{h\pm}^{e_i}(\omega,s,\xi_{h\pm}^{e_i}(\omega,s;t,x))$ and $w_{\pm}^{e_i}(s,\xi_{\pm}^{e_i}(s;t,x))$ along characteristics and some cumbersome calculations, we arrive at

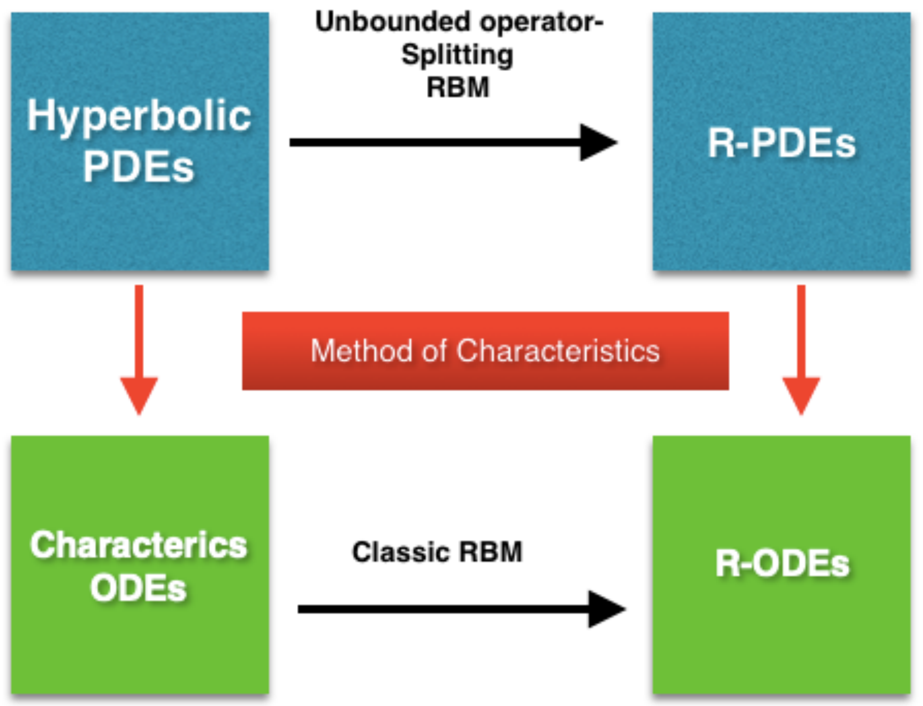
$$\mathbb{E}[\,|\,\mathbf{w}_{h+}(t) - \mathbf{w}_{+}(t)\,|^2_{L^\infty(\Omega)} + |\,\mathbf{w}_{h-}(t) - \mathbf{w}_{-}(t)\,|^2_{L^\infty(\Omega)}] \le Ch,$$ which implies to Theorem 3.

Summary and Perspectives

- The application of the RBM to (networked) hyperbolic PDEs combines
  (1) operator splitting for PDEs
  (2) stochastic methods for large-scale optimization
  (3) characteristic method for 1d Hyperbolic type PDEs.



Summary and Perspectives



- The application of the RBM to (networked) hyperbolic PDEs combines
  (1) operator splitting for PDEs
  (2) stochastic methods for large-scale optimization
  (3) characteristic method for 1d Hyperbolic type PDEs.

- We **efficiently approximate** the solution to networked linear hyperbolic equations and associated optimal control problems, and obtain the **convergence results**
  (1) $\mathbf{y}_h(\omega, t)$ converges to $\mathbf{y}(t)$ for $h \to 0$ (in expectation).
  (2) Convergence in the optimal controls can be proven along the lines of [E.Zuazua, D.Veldman 2022], but some regularity properties need to be verified.

- Extensions to nonlinear setting:
  Semi-linear case is straight forward, e.g. $y_t + \Lambda y_x = f(t, x)$ with f Lipschitz in x.
  Quasi-linear case for 1d hyperbolic systems in the framework of semi-global classical solution (Discussion with Tatsien Li, Shanghai).

- For networked case, extension to non-overlapping domain decomposition on complex spatial structures (Discussion with Günter Leugering) and XPINNs.

- What is the best splitting strategy/pattern for cutting sub-nets from a network with circles inside?

Thank you!

22 August, 2024
@ Benasque

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