Estimation of the controllable subspace for an induced earthquakes model

Lucas Davron

Université Paris-Dauphine, under the supervision of Olivier Glass, Pierre Lissy (ENPC, Ecole des Ponts), and Swann Marx (LS2N, Ecole Centrale Nantes)

23/08/2024

CENTRO DE CIENCIAS DE BENASQUE PEDRO PASCUAL

Presentation of the model

$$\begin{cases} p_t = p_{xx}, & 0 < x < 1, \\ p_x(t,0) = 0, & \\ p_x(t,1) = q(t), & \\ w_{tt} = w_{xx}, & -1 < x < 0, \\ w_x(t,0) = p(t,0), & \\ w(t,-1) = 0. & \end{cases} \tag{1}$$

- p is the pressure of the fluid, w the displacement of the earth´s crust
- Toy model, linear 1d coupled heat/wave equations
- Similar to a system studied by Zhang and Zuazua in 2003

Formulation of the control problem

We wish to compute

$$\left\{ \begin{pmatrix} p^0 \\ w^0 \\ w_t^0 \end{pmatrix} \text{s.t. } \exists q : [0,2] \to \mathbb{R}, \quad \begin{pmatrix} p(2,\cdot) \\ w(2,\cdot) \\ w_t(2,\cdot) \end{pmatrix} \equiv 0 \right\}$$

Litterature

In 2003 X.Zhang and E.Zuazua propose a general technique to guess a class of initial data that are null controllable in any time $T > 2$

Formulation of the control problem

We wish to compute

$$\left\{ \begin{pmatrix} p^0 \\ w^0 \\ w_t^0 \end{pmatrix} \text{s.t. } \exists q : [0,2] \to \mathbb{R}, \quad \begin{pmatrix} p(2,\cdot) \\ w(2,\cdot) \\ w_t(2,\cdot) \end{pmatrix} \equiv 0 \right\}$$

Litterature

In 2003 X.Zhang and E.Zuazua propose a general technique to guess a class of initial data that are null controllable in any time $T > 2$

A more realistic goal in time $T = 2$

Estimate the subspace

$$\mathcal{N} := \left\{ \begin{pmatrix} 0 \\ w^0 \\ w_t^0 \end{pmatrix} \text{s.t. } \exists q : [0,2] \to \mathbb{R}, \quad w(2,x) \equiv w_t(2,x) \equiv 0 \right\}$$

The functional setting

Proposition: Well-posedness

The system (1) has a "canonical" realization as a well-posed linear and time invariant. In particular:

- The state space variable

$$z := \begin{pmatrix} p \\ w \\ w_t \end{pmatrix}$$

  belongs to the state space

$$X := L^2(0,1) \times H^1_{(-1)}(-1,0) \times L^2(-1,0)$$

- For all $z^0 \in X$ and $q \in L^2(0,2)$, the system (1) has a unique solution.

1D hyperbolic uniqueness

Proposition:

For all $(w^0, w_t^0) \in H^1_{(-1)}(-1,0) \times L^2(-1,0)$, there is a unique control $c \in L^2(0,2)$ so that the solution of

$$\begin{cases} w_{tt} & = & w_{xx}, & -1 < x < 0, \quad 0 < t < 2, \\ w_x(t,0) & = & c(t), & \\ w(t,-1) & = & 0, & \\ w(0,x) & = & w^0(x), & \\ w_t(0,x) & = & w_t^0(x), & \end{cases}$$

vanishes at time $T = 2$.

1D hyperbolic uniqueness

Proposition:

For all $(w^0, w_t^0) \in H^1_{(-1)}(-1,0) \times L^2(-1,0)$, there is a unique control $c \in L^2(0,2)$ so that the solution of

$$\begin{cases} w_{tt} = w_{xx}, & -1 < x < 0, \quad 0 < t < 2, \\ w_x(t,0) = c(t), & \\ w(t,-1) = 0, & \\ w(0,x) = w^0(x), & \\ w_t(0,x) = w_t^0(x), & \end{cases}$$

vanishes at time $T = 2$.

Consequence: $(w^0, w_t^0) \in \mathcal{N}$ if and only if

$$(w^0, w_t^0) = \mathrm{FORMULA}(p(\cdot, 0)),$$

we are left to estimate

$$\mathcal{P} := \{p(\cdot,0) : q \in L^2(0,2), \quad p^0 = 0\}$$

Detour on the general time regularity for the heat equation

Suppose $y(t,x)$ is a local solution of the heat equation on $(0,T)\times(-L,L)$. Then

- $y(\cdot,0)$ is Gevrey of order 2 on all $(t_0,t_1)\subset\subset(0,T)$:

$$\forall (t_0,t_1)\subset\subset(0,T), \quad \exists \gamma, R>0, \quad \left|\frac{d^k}{dt^k}y(t,0)\right| \leq \gamma\frac{(2k)!}{R^{2k}}$$

- The radius of convergence R vanishes when $t\to 0$ with rate at worst

$$R(t)\gtrsim\sqrt{t}$$

In our situation we may hope to have a more precise estimate on R

The main result

Set

$$\mathcal{G}^{2,1}_{(0)}[0,T] := \left\{ \varphi \in C^{\infty}_{(0)}[0,T] : \sum_{k=1}^{\infty} \frac{\|\varphi^{(k)}\|_{L^2(0,T)}}{(2k-1)!} < \infty \right\}$$

and

$$\mathcal{G}^{2,1/\sqrt{2}}_{(0)}[0,T] := \left\{ \varphi \in C^{\infty}_{(0)}[0,T] : |\varphi^{(k)}(t)| \lesssim k^{-1/4} \frac{(2k)!}{(1/\sqrt{2})^{2k}} \right\}$$

Theorem

The set $\mathcal{P}$ is sandwiched as

$$\mathcal{G}^{2,1}_{(0)}[0,2] \subset \mathcal{P} \subset \mathcal{G}^{2,1/\sqrt{2}}_{(0)}[0,2].$$

The consequence for earthquakes

Consider the compatibility conditions for the hyperbolic initial conditions

$$\frac{d^{2k}}{dx^{2k}} w_t^0(-1) = 0, \quad \frac{d^{2k}}{dx^{2k}} w^0(-1) = 0, \quad \frac{d^k}{dx^k} w_t^0(0) = \frac{d^{k+1}}{dx^{k+1}} w(0) \quad (2)$$

Corollary

The set $\mathcal{N}$ is sandwiched as

$$\{(w^0, w_t^0) : w_x^0, w_t^0 \in \mathcal{G}^{2,1}[-1,0], \quad (2)\} \subset \mathcal{N}$$

and

$$\mathcal{N} \subset \{(w^0, w_t^0) : w_x^0, w_t^0 \in \mathcal{G}^{2,1/\sqrt{2}}[-1,0], \quad (2)\}$$

Tracking for the heat equation

The inclusion $\mathcal{G}^{2,1}_{(0)}[0,2] \subset \mathcal{P}$ is a tracking result for the output $p(t,0)$ of

$$\begin{cases} p_t &= p_{xx}, \quad 0 < x < 1, \\ p_x(t,0) &= 0, \\ p_x(t,1) &= q(t), \\ p(0,x) &= 0. \end{cases}$$

This strengthens the standard result of [Laroche, Martin, Rouchon, 2000], which assumes φ to have radius of convergence > 2.

Sketch of the proof of the tracking result

The key point is to show that when $\varphi \in \mathcal{G}^{2,1}_{(0)}[0,2]$, the ensatz

$$p(t,x) := \sum_{k=0}^{\infty} \varphi^{(k)}(t) \frac{x^{2k}}{(2k)!}$$

solves the above heat system, with control

$$q(t) := \sum_{k=1}^{\infty} \varphi^{(k)}(t) \frac{1}{(2k-1)!}.$$

Sketch of the proof of the tracking result

The key point is to show that when $\varphi \in \mathcal{G}^{2,1}_{(0)}[0,2]$, the ensatz

$$p(t,x) := \sum_{k=0}^{\infty} \varphi^{(k)}(t) \frac{x^{2k}}{(2k)!}$$

solves the above heat system, with control

$$q(t) := \sum_{k=1}^{\infty} \varphi^{(k)}(t) \frac{1}{(2k-1)!}.$$

Gevrey 2 with radius R is exactly what you want for the first series to converge when $|x| < R$

Sketch of the proof of the tracking result

1. If we assume further that

$$|\varphi^{(k)}(t)| \lesssim \frac{(2k)!}{(1+\epsilon)^{2k}}$$

   then it is easy (the series converges normally in $C^1([0,T];C^2[0,1])$)

2. In the general case, use the regularization

$$\varphi_\epsilon(t) := \varphi\left(\frac{t}{1+\epsilon}\right)$$

   which is as in the previous point

3. Carefully pass to the limit $\epsilon \to 0$, weakly in the formulas but yet strongly thanks to *a priori* estimates

4. Take advantage of the series that makes p converges near $x = 0$ so that automatically

$$p(t,0) = \varphi(t).$$

A link with the reachable set for the 1D heat equation

For the heat equation on $(0,1)$ controlled by the Neumann action we know that

$$\mathcal{H}(\overline{\diamond}) \subset \mathcal{R} \subset \mathcal{H}(\diamond).$$

However, in this work we essentially have

$$\mathcal{H}(\overline{D}(0,1)) \subset \mathcal{R} \subset \mathcal{H}(D(0,1/\sqrt{2})).$$

Thank you!