Carleman estimates for a Schrödinger equation with dynamic boundary conditions and applications

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Joint work with

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Supported by Qualifica Grant QUAL21 005 USE.

August 27, 2024

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Outline

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Carleman estimates

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, $\partial\Omega = \Gamma_0 \cap \Gamma_1$ with Γ_0, Γ_1 closed subsets and $\Gamma_0 \cap \Gamma_1 = \emptyset$. We consider the following no conservative Schrödinger equation with dynamic boundary conditions

$$\begin{cases} i\partial_t y + d\Delta y + \vec{q}_1 \cdot \nabla y + q_0 y = f & \text{in } \Omega \times (0,T), \\ i\partial_t y_\Gamma - d\partial_\nu y + \delta\Delta_\Gamma y_\Gamma + \vec{q}_{\Gamma,1} \cdot \nabla_\Gamma y_\Gamma + q_{\Gamma,0} y_\Gamma = f_\Gamma & \text{on } \Gamma_1 \times (0,T), \\ y\big|_{\Gamma_1} = y_\Gamma & \text{on } \Gamma_1 \times (0,T), \\ y\big|_{\Gamma_0} = 0 & \text{on } \Gamma_0 \times (0,T), \\ (y(\cdot,0), y_\Gamma(\cdot,0)) = (y_0, y_{\Gamma,0}) & \text{in } \Omega \times \Gamma_1. \end{cases}$$

Here,

- $i := \sqrt{-1}$ and $d, \delta > 0$.
- ∂_ν the normal derivative operator.
- ∇_Γ the tangential derivative operator.
- $\Delta_\Gamma = \text{div}_\Gamma(\nabla_\Gamma)$ is the Laplace Beltrami operator.

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First Goal

To establish a Carleman estimate with boundary observations in an open set

$$\Gamma_* \subseteq \Gamma_0.$$

- The controllability properties of the Schrödinger equation have been studied mainly in the case of **Dirichlet boundary conditions**.[a]
- The Schrödinger equation with dynamic boundary conditions has been studied before by M. Cavalcanti et. al. in 2016. [b]
- One of the main problems in obtaining Carleman estimates is to choose appropriate **weight functions** adapted to the geometry of the problem.[c]

[a]Zuazua, E. (2002). Remarks on the controllability of the Schrödinger equation. In CRM Workshop (pp. 193-211).

[b]Cavalcanti, M. M., Corrêa, W. J., Lasiecka, I., & Lefler, C. (2016). Well-posedness and uniform stability for nonlinear Schrödinger equations with dynamic/Wentzell boundary conditions. Indiana University Mathematics Journal, 1445-1502.

[c]Baudouin, L., & Puel, J. P. (2002). Uniqueness and stability in an inverse problem for the Schrödinger equation. Inverse problems, 18(6), 1537.

Assumptions

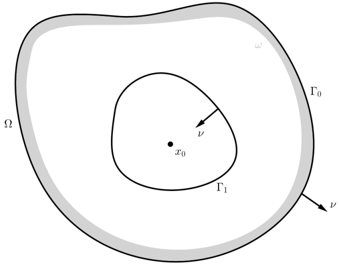
Geometric assumptions

We suppose that:

- Ω has smooth boundary and it can be written as

$$\Omega = \Omega_0 \setminus \overline{\Omega_1},$$

  where Ω_1 is strongly convex.
- $0 \in \Omega_1$.



Weight functions

For each $x \in \mathbb{R}^n$, we set

$$\mu(x) := \text{ínf}\{\lambda > 0 \,:\, x \in \lambda\Omega_1\}.$$

Now, we write $\psi(x) = \mu^2(x)$ and for $\lambda > 0$ we set

$$\theta(x,t) := \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \varphi(x,t) = \frac{\alpha - e^{\lambda\psi(x)}}{t(T-t)}, \quad \forall (x,t) \in \Omega \times (0,T),$$

where $\alpha > \|e^{\lambda\psi}\|_{L^\infty(\Omega)}$.

Theorem (A. Mercado, R.M, 2023)

Consider the Geometric Assumptions. Suppose that $(\vec{q}_1, \vec{q}_{\Gamma,1})$ and $(q_0, q_{\Gamma,0})$ are L^∞. Suppose that $\delta > d$. Then, there exist constants C, s_0 and λ_0 such that

$$\begin{aligned} &\int_0^T \int_\Omega e^{-2s\varphi}(s^3\lambda^4\theta^3|v|^2 + s\lambda\theta|\nabla v|^2 + s\lambda^2\theta|\nabla\psi\cdot\nabla v|^2)\,dxdt \\ &+ \int_0^T \int_{\Gamma_1} e^{-2s\varphi}(s^3\lambda^3\theta^3|v_\Gamma|^2 + s\lambda\theta|\partial_\nu v|^2 + s\lambda\theta|\nabla_\Gamma v_\Gamma|^2)\,d\sigma dt \\ \leq& C\int_0^T \int_\Omega e^{-2s\varphi}|L(v)|^2\,dxdt + C\int_0^T \int_{\Gamma_1} e^{-2s\varphi}|N(v,v_\Gamma)|^2 d\sigma dt \\ &+ Cs\lambda\int_0^T \int_{\Gamma_*} e^{-2s\varphi}\theta|\partial_\nu v|^2\,d\sigma dt, \end{aligned}$$

for all $\lambda \geq \lambda_0$ and $s \geq s_0$, $L(v) \in L^2(\Omega\times(0,T))$, $N(v,v_\Gamma) \in L^2(\Gamma_1\times(0,T))$, $\partial_\nu v \in L^2(\partial\Omega\times(0,T))$ with

$$\Gamma_* := \{x\in\partial\Omega \,:\, \partial_\nu\psi(x) \geq 0\} \subseteq \Gamma_0.$$

Sketch of the proof

- We follow the classical **conjugation** for suitable operators involving the Schrödinger equation with dynamic boundary conditions.
- **Several boundary terms arise** from the dynamic boundary conditions. We shall prove that these conditions can be managed properly.
- We point out that $\partial_\nu \psi < 0$ on Γ_1 and

$$\nabla_\Gamma \theta = \nabla_\Gamma \varphi = 0 \text{ on } \Gamma_1 \times (0,T).$$

  Thus, several terms vanish in the conjugation process.
- We also use the **Surface Divergence Theorem**

$$\int_{\Gamma_1} \Delta_\Gamma y z \, d\sigma = -\int_{\Gamma_1} \nabla_\Gamma y \cdot \nabla_\Gamma z \, d\sigma, \quad \forall y \in H^2(\Gamma_1), \quad \forall z \in H^1(\Gamma_1)$$

  to estimate some global boundary terms.

Application #1: Boundary Controllability

Consider the following (controlled) problem

$$\begin{cases} i\partial_t y + d\Delta y - \vec{q}_1 \cdot \nabla y + q_0 y = 0 & \text{in } \Omega \times (0,T), \\ i\partial_t y_\Gamma - d\partial_\nu y + \delta\Delta_\Gamma y_\Gamma - \vec{q}_{\Gamma,1} \cdot \nabla_\Gamma y_\Gamma + q_{\Gamma,0} y_\Gamma = 0 & \text{on } \Gamma_1 \times (0,T), \\ y\big|_{\Gamma_1} = y_\Gamma & \text{on } \Gamma_1 \times (0,T), \\ y\big|_{\Gamma_0} = \mathbb{1}_{\Gamma_*} h & \text{on } \Gamma_0 \times (0,T), \\ (y(\cdot,0), y_\Gamma(\cdot,0)) = (y_0, y_{\Gamma,0}) & \text{in } \Omega \times \Gamma_1. \end{cases}$$

We shall suppose that

$$\vec{q}_1 := d\nabla\pi - i\vec{r}, \quad \vec{q}_{\Gamma,1} := \delta\nabla_\Gamma \pi_\Gamma - i\vec{r}_\Gamma,$$

are smooth enough, $\pi = \pi_\Gamma$ on $\Gamma_1 \times (0,T)$, $\vec{r} \cdot \nu \le 0$ on $\partial\Omega \times (0,T)$.[a] Without loss of generality, we can put $\pi = 0$ in $\Omega \times (0,T)$ and $\pi_\Gamma = 0$ on $\Gamma_1 \times (0,T)$.

[a] Lasiecka, I., Triggiani, R., & Zhang, X. (2004). Global uniqueness, observability and stabilization of nonconservative Schrodinger equations via pointwise Carleman estimates. Part I: $H^1(\Omega)$-estimates. Journal of inverse and Ill Posed Problems, 12(1), 43-123.

Functional spaces

We consider the closed subspace of $H^1(\Omega)$:

$$H^1_{\Gamma_0}(\Omega) := \{u \in H^1(\Omega) \,:\, u = 0 \text{ on } \Gamma_0\}.$$

We also define

$$\mathcal{V} := \{(u, u_\Gamma) \in H^1_{\Gamma_0}(\Omega) \times H^1(\Gamma_1) \,:\, u\big|_\Gamma = u_\Gamma \text{ on } \Gamma_1\}.$$

This is a Hilbert space in $\mathbb{C}$ with

$$\langle (u, u_\Gamma), (v, v_\Gamma) \rangle_{\mathcal{V}} = \int_\Omega \nabla u \cdot \nabla \vec{v} \, dx + \int_{\Gamma_1} \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma \overline{v}_\Gamma \, d\sigma.$$

Exact Boundary Controllability

Theorem (A. Mercado, R. M., 2023)

Assume the same assumptions on Carleman estimates and the regularity hypotheses on the coefficients $(\vec{q}_1, \vec{q}_{\Gamma,1})$ and $(q_0, q_{\Gamma,0})$. Then, for all states $(y_0, y_{\Gamma,0}), (y_T, y_{\Gamma,T}) \in \mathcal{V}'$, there exists a control $h \in L^2(\Gamma_* \times (0,T))$ such that the solution (y, y_Γ) (defined in the sense of tranposition) satisfies

$$(y(\cdot,T), y_\Gamma(\cdot,T)) = (y_T, y_{\Gamma,T}) \text{ in } \Omega \times \Gamma_1.$$

A sketch of the proof

We introduce the adjoint system

$$\begin{cases} i\partial_t z + d\Delta z + i\vec{r}\cdot\nabla z + qz = 0 & \text{in } \Omega\times(0,T), \\ i\partial_t z_\Gamma - d\partial_\nu z + \delta\Delta_\Gamma z_\Gamma + i\vec{r}_\Gamma\cdot\nabla_\Gamma z_\Gamma + q_\Gamma z_\Gamma = 0 & \text{on } \Gamma_1\times(0,T), \\ z\big|_{\Gamma_1} = z_\Gamma & \text{on } \Gamma_1\times(0,T), \\ z\big|_{\Gamma_0} = 0 & \text{on } \Gamma_0\times(0,T), \\ (z(\cdot,T), z_\Gamma(\cdot,T)) = (z_T, z_{\Gamma,T}) & \text{in } \Omega\times\Gamma_1, \end{cases}$$

where (q, q_Γ) are given by

$$q := i\mathrm{div}(\vec{r}) + \overline{q}_0, \quad q_\Gamma := i\mathrm{div}_\Gamma(\vec{r}_\Gamma) - i\vec{r}\cdot\nu + \overline{q}_{\Gamma,0}.$$

Remark

If $(z_T, z_{\Gamma,T}) \in \mathcal{V}$, this problem has a unique solution $(z, z_\Gamma) \in C^0([0,T];\mathcal{V})$. Moreover, there is a constant $C > 0$ such that

$$\|(z, z_\Gamma)\|_{C^0([0,T];\mathcal{V}} + \|\partial_\nu z\|_{L^2(\partial\Omega\times(0,T))} \leq C\|(z_T, z_{\Gamma,T})\|_{\mathcal{V}}$$

Then, the **exact controllability** is equivalent to proving the **observability inequality**

$$\|(z_T, z_{\Gamma,T})\|_{\mathcal{V}}^2 \leq C \int_0^T \int_{\Gamma_*} |\partial_\nu z|^2 \, d\sigma dt, \quad \forall (z_T, z_{\Gamma,T}) \in \mathcal{V}.$$

This is done by using Carleman estimates + energy estimates for the Schrödinger operator with dynamic boundary conditions. □

Application #2: An Inverse source problem

We consider the problem

$$
\begin{cases}
i\partial_t y + d\Delta y + p(x)y = g & \text{in } \Omega \times (0,T),\\
i\partial_t y_\Gamma - d\partial_\nu y + \delta\Delta_\Gamma y_\Gamma + p_\Gamma(x) y_\Gamma = g_\Gamma & \text{on } \Gamma_1 \times (0,T),\\
y\big|_{\Gamma_1} = y_\Gamma & \text{on } \Gamma_1 \times (0,T),\\
y\big|_{\Gamma_0} = g_{\Gamma,0} & \text{on } \Gamma_0 \times (0,T),\\
(y(\cdot,0), y_\Gamma(\cdot,0)) = (y_0, y_{\Gamma,0}) & \text{in } \Omega \times \Gamma_1.
\end{cases}
$$

Coefficient Inverse problem (CIP)

Is it possible to retrieve the complex potentials (p, p_Γ) from measurements of the normal derivative $\partial_\nu y$ on $\Gamma_* \times (0,T)$? $(\Gamma_* \subseteq \Gamma_0)$.

Classical questions on Inverse Problems

- **Uniqueness:** Does the inequality $\partial_\nu y[p, p_\Gamma] = \partial_\nu y[q, q_\Gamma]$ on $\Gamma_* \times (0, T)$ imply $p = q$ in Ω and $p_\Gamma = q_\Gamma$ on Γ_1?
- **Stability:** Is it possible to estimate $\|q - p\|_{L^2(\Omega)}$ and $\|q_\Gamma - p_\Gamma\|_{L^2(\Gamma_1)}$ by a suitable norm of $(\partial_\nu y[q, q_\Gamma] - \partial_\nu y[p, p_\Gamma])$ on $\Gamma_* \times (0, T)$?
- **Reconstruction formula:** Can we find an algorithm to compute the potentials p and p_Γ by partial knowledge of $\partial_\nu y[p, p_\Gamma]$?

An stability result

Theorem (H. Carrillo, A. Mercado, R. M., 2024)

For $m > 0$, choose $(p, p_\Gamma) \in \mathbb{L}^\infty_{\leq m}$. Under geometric assumptions and regularity hypotheses, the following inequalities hold

$$\frac{1}{C}\|(q, q_\Gamma) - (p, p_\Gamma)\|_{\mathbb{L}^2} \leq \|\partial_\nu y[q, q_\Gamma] - \partial_\nu y[p, p_\Gamma]\|_{H^1(0,T;L^2(\Gamma_*))}$$

and

$$\|\partial_\nu y[q, q_\Gamma] - \partial_\nu y[p, p_\Gamma]\|_{H^1(0,T;L^2(\Gamma_*))} \leq C\|(q, q_\Gamma) - (p, p_\Gamma)\|_{\mathbb{L}^2},$$

for all $(q, q_\Gamma) \in \mathbb{L}^\infty_{\leq m}$

Notation

$$\mathbb{L}^\infty_{\leq m} := \{(p, p_\Gamma) \in L^\infty(\Omega) \times L^\infty(\Gamma_1) \;:\; \|(p, p_\Gamma)\|_{L^\infty(\Omega)\times L^\infty(\Gamma_1)} \leq m\}$$

A reconstruction formula

Fix $(\zeta,\zeta_\Gamma)\in\mathbb{L}^2$, $h\in L^2(0,T;L^2(\Gamma_*))$ and choose $s_0>0$ as in Carleman estimates. Then, for all $s\geq s_0$, we introduce the functional $J:\mathcal{W}\to\mathbb{R}$ defined by

$$
\begin{aligned}
&J[\zeta,\zeta_\Gamma,h](u,u_\Gamma)\\
:=&\frac{1}{2}s\int_0^T\int_\Omega e^{-2s\varphi}|L(u)-\zeta|^2\,dx\,dt+\frac{1}{2s}\int_0^T\int_{\Gamma_1} e^{-2s\varphi}|N(u,u_\Gamma)-\zeta_\Gamma|^2\,d\sigma dt\\
&+\frac{1}{2}\int_0^T\int_{\Gamma_*} e^{-2s\varphi}|\partial_\nu u-h|^2 d\sigma dt,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{W}:=\{(y,y_\Gamma)\in L^2(0,T;\mathcal{V})\ :&(L(y),N(y,y_\Gamma))\in L^2(0,T;\mathbb{L}^2),\\
&\partial_\nu y\in L^2(0,T;L^2(\Gamma_*))\}.
\end{aligned}
$$

On the functional J

- The problem

$$\begin{cases} \text{Minimize} & J[\zeta, \zeta_\Gamma, h](u, u_\Gamma), \\ \text{Subject to} & (u, u_\Gamma) \in \mathcal{W} \end{cases}$$

has a unique minimizer $(u^*, u_\Gamma^*) \in \mathcal{W}$. Moreover, $\exists C > 0$ such that

$$\|(u, u_\Gamma)\|_{\mathcal{W}}^2 \leq \frac{C}{s} \int_0^T \int_\Omega e^{-2s\varphi} |\zeta|^2 \, dx dt + \frac{C}{s} \int_0^T \int_{\Gamma_1} e^{-2s\varphi} |\zeta_\Gamma|^2 \, d\sigma dt$$
$$+ C \int_0^T \int_{\Gamma_*} e^{-2s\varphi} |h|^2 \, d\sigma dt$$

- Consider $(\zeta^j, \zeta_\Gamma^j) \in L^2(0,T;\mathbb{L}^2)$, $h \in L^2(0,T;L^2(\Gamma_*))$ and $(u^{*,j}, u_\Gamma^{*,j}) \in \mathcal{W}$ is the corresponding minimizer of $J[\zeta^j, \zeta_\Gamma^j, h]$ for $j \in \{a,b\}$. Then, we have

$$
\begin{aligned}
& s^{3/2} \int_\Omega e^{-2s\varphi(\cdot,0)} |(u^{*,a} - u^{*,b})(\cdot,0)|^2 \, dx \\
& + s^{3/2} \int_{\Gamma_1} e^{-2s\varphi(\cdot,0)} |(u_\Gamma^{*,a} - u_\Gamma^{*,b})(\cdot,0)|^2 \, d\sigma \\
\leq & C \int_0^T \int_\Omega e^{-2s\varphi} |\zeta^a - \zeta^b|^2 \, dx \, dt + C \int_0^T \int_{\Gamma_1} e^{-2s\varphi} |\zeta_\Gamma^a - \zeta_\Gamma^b|^2 \, d\sigma dt,
\end{aligned}
$$

This property is the key point to reconstruct q and q_Γ using the Carleman-based Reconstruction algorithm (CbRec in short).[a] [b]

[a] Baudouin, L., De Buhan, M., & Ervedoza, S. (2013). Global Carleman estimates for waves and applications. Communications in Partial Differential Equations, 38(5), 823-859.

[b] Baudouin, L., De Buhan, M., Ervedoza, S., & Osses, A. (2021). Carleman-based reconstruction algorithm for waves. SIAM Journal on Numerical Analysis, 59(2), 998-1039.

CbRec algorithm

Initialization:

- $(p^0, p^0_\Gamma) = (0,0)$ or any guess $(p^0, p^0_\Gamma) \in \mathbb{L}^\infty_{\leq m}$.

Iteration: From k to $k+1$.

- **Step 1:** Given (p^k, p^k_Γ), we set $h^k := \partial_t(\partial_\nu y[p^k, p^k_\Gamma] - \partial_\nu y[p, p_\Gamma])$.
- **Step 2:** Find the minimizer $(u^{*,k}, u^{*,k}_\Gamma)$ of the unconstrained problem

$$\begin{cases} \text{Minimize} & J[0,0,h^k](u,u_\Gamma) \\ \text{Subject to} & (u,u_\Gamma) \in \mathcal{W}. \end{cases}$$

- **Step 3:** Set

$$\tilde{p}^{k+1} = p^k + \frac{\partial_t u^{*,k}(0)}{y_0}, \quad \tilde{p}^{k+1}_\Gamma = p^k_\Gamma + \frac{\partial_t u^{*,k}_\Gamma(\cdot,0)}{y_{\Gamma,0}}.$$

- **Step 4:** Finally, define

$$p^{k+1} = \mathcal{T}(\tilde{p}^{k+1}) \text{ and } p_\Gamma^{k+1} = \mathcal{T}_\Gamma(\tilde{p}_\Gamma^{k+1})$$

where

$$\mathcal{T}(p) := \begin{cases} p & \text{if } \|p\|_{L^\infty(\Omega)} \leq m, \\ sgn(p)m & \text{if } \|p\|_{L^\infty(\Omega)} > m, \end{cases}$$

and

$$\mathcal{T}_\Gamma(p_\Gamma) := \begin{cases} p_\Gamma & \text{if } \|p_\Gamma\|_{L^\infty(\Gamma_1)} \leq m, \\ sgn(p_\Gamma)m & \text{if } \|p_\Gamma\|_{L^\infty(\Gamma_1)} > m \end{cases}$$

Theorem (H. Carrillo, A. Mercado, R.M., 2024)

Let $m > 0$, $(p, p_\Gamma) \in \mathbb{L}^\infty_{\leq m}$ and for each $k \in \mathbb{N}$, let $(p^k, p^k_\Gamma)_{k\in\mathbb{N}}$ be the sequence generated by **(CbRec)**. Then, under geometric and regularity assumptions, there exist a constant $C_0 > 0$ and $s_0 > 0$ such that for all $s \geq s_0$ and $k \in \mathbb{N}$, we have

$$\int_\Omega e^{-2s\varphi(\cdot,0)}|p^{k+1} - p|^2\,dx + \int_{\Gamma_1} e^{-2s\varphi(\cdot,0)}|p^{k+1}_\Gamma - p_\Gamma|^2\,d\sigma$$
$$\leq \frac{C_0}{s^{3/2}}\int_\Omega e^{-2s\varphi(\cdot,0)}|p^k - p|^2\,dx + \frac{C_0}{s^{3/2}}\int_{\Gamma_1} e^{-2s\varphi(\cdot,0)}|p^k_\Gamma - p_\Gamma|^2\,d\sigma$$

References

- Mercado, A., & Morales, R. (2023). Exact Controllability for a Schrödinger equation with dynamic boundary conditions. SIAM Journal on Control and Optimization, 61(6), 3501-3525.
- Carrillo, H., Mercado, A., & Morales, R. (2024) Simultaneous reconstruction of two complex potentials for a non conservative Schrödinger equation with dynamic boundary conditions. (In preparation).

Thank you for your attention!