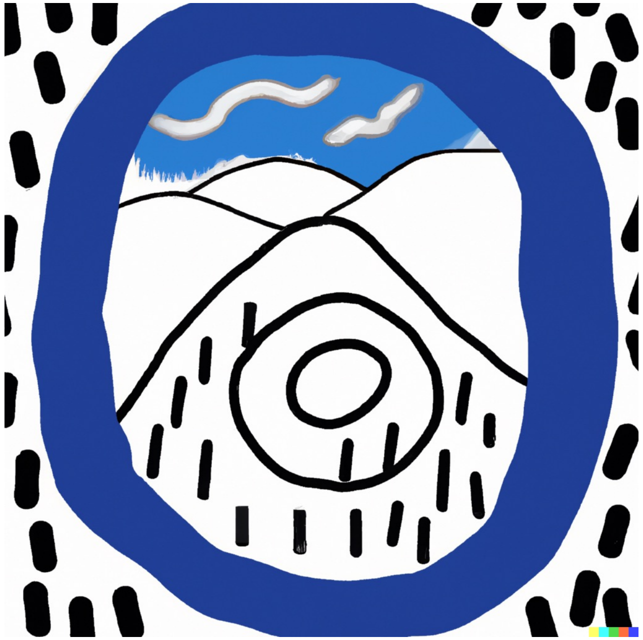
Spin Chains, Entanglement, and All That

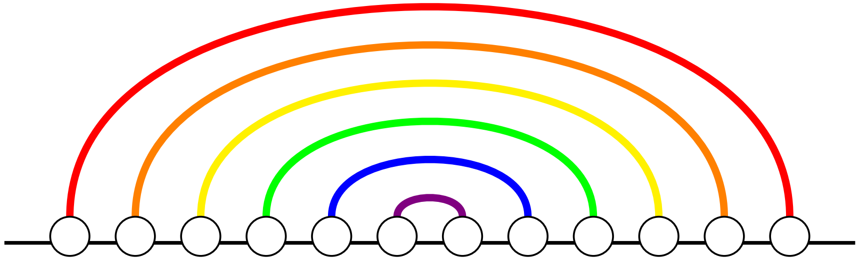


Germán Sierra
Instituto de Física Teórica UAM-CSIC, Madrid

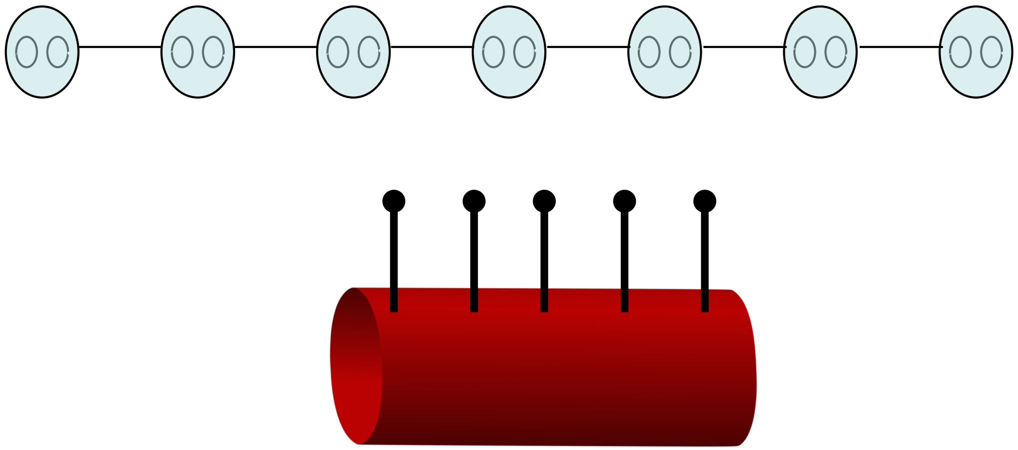
Workshop “Quantum Matter Simulations”

Benasque Science Center, Feb 18 – Feb 23, 2024

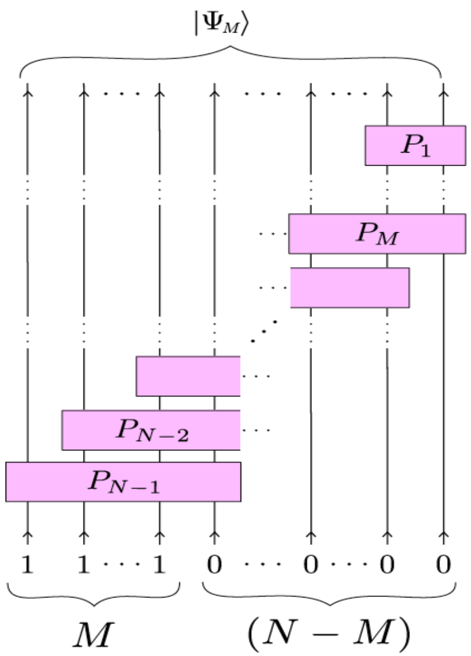
Rainbow spin chain



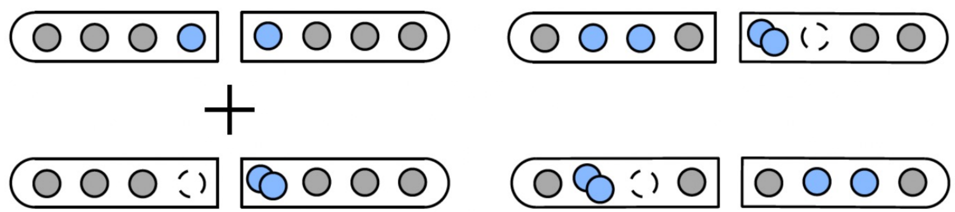
Infinite MPS – Haldane Shastry

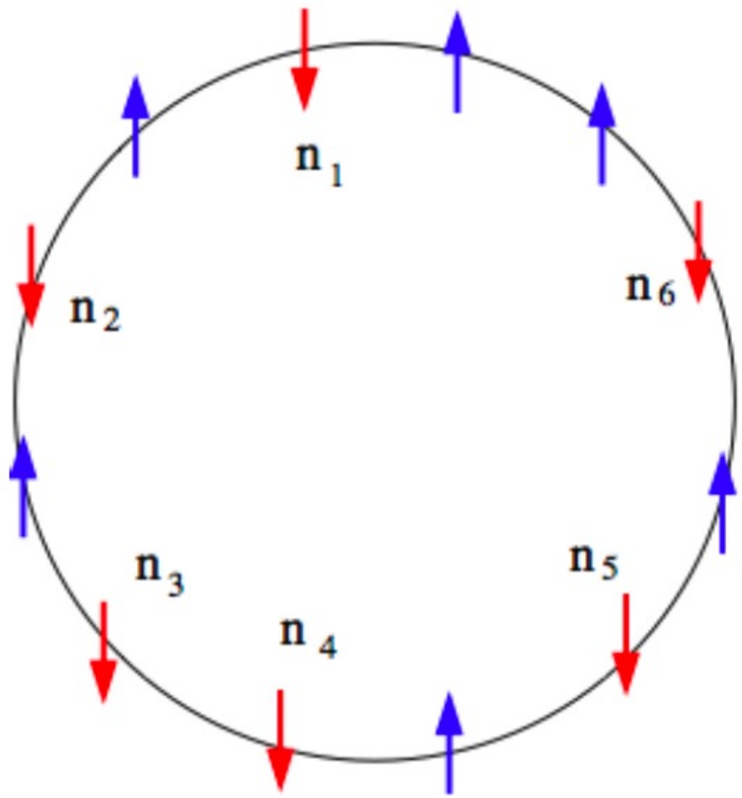


Algebraic Bethe Circuits

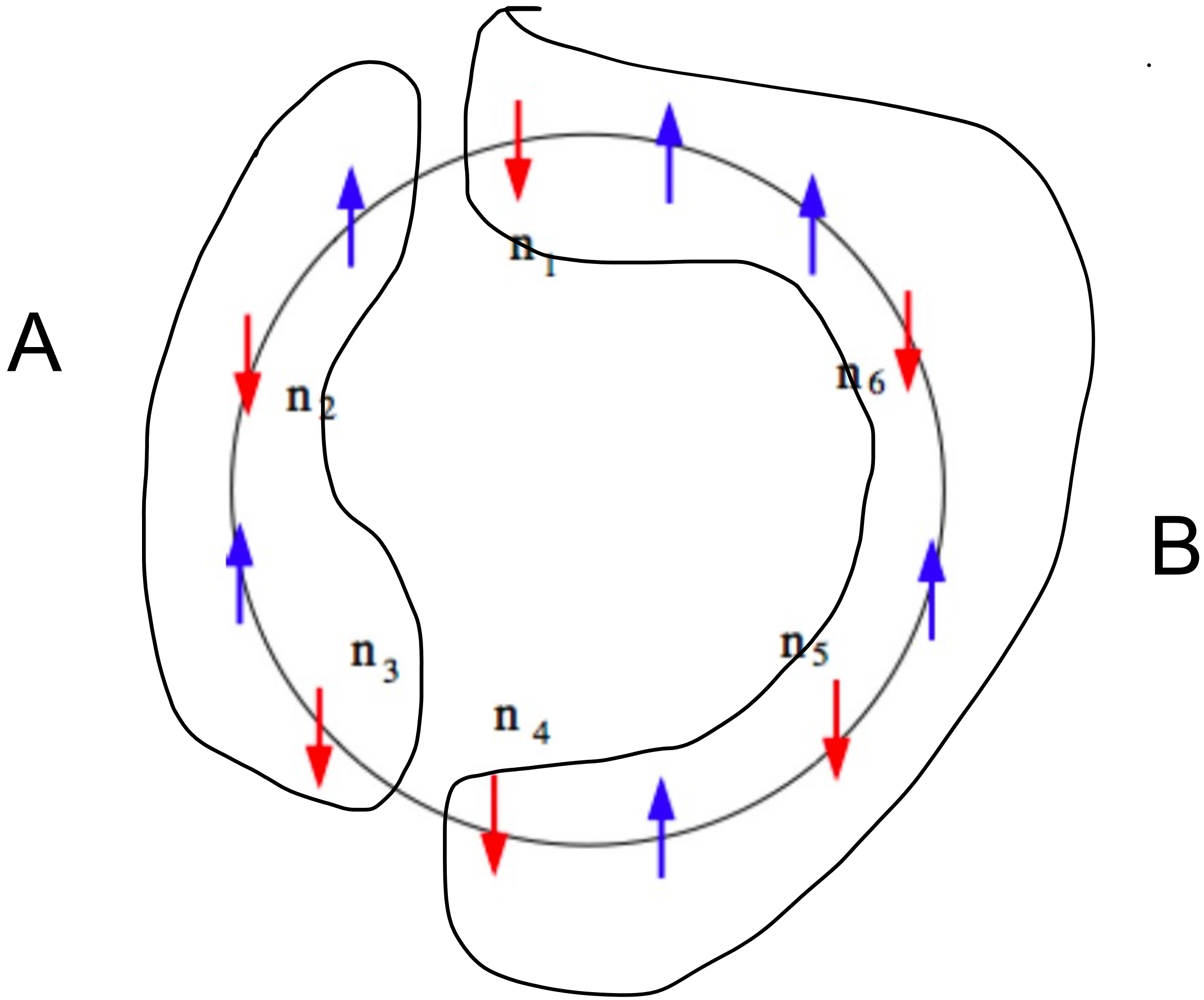


Symmetry Resolved Entanglement

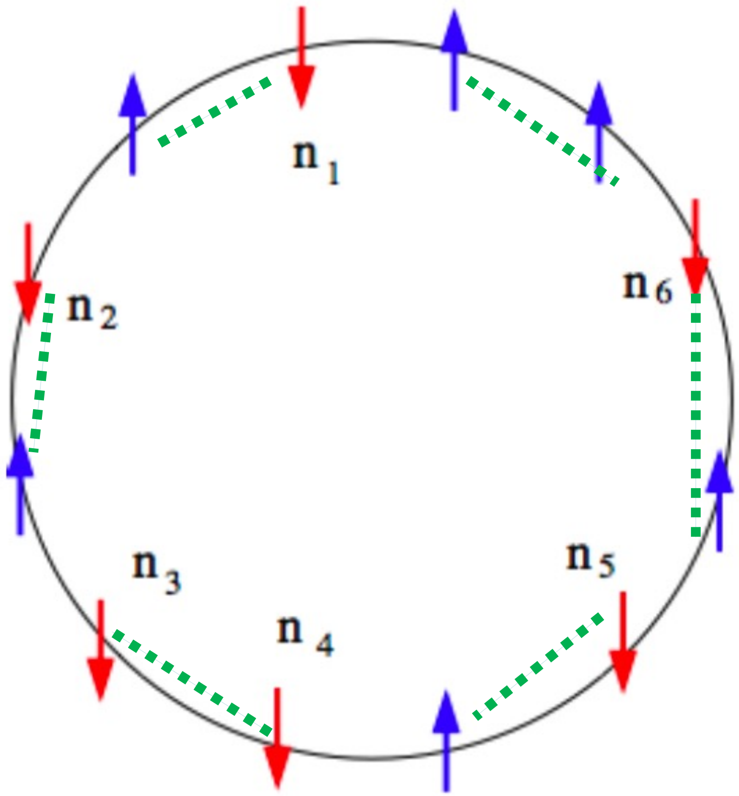



n_1
n_2
n_3
n_4
n_5
n_6

entanglement



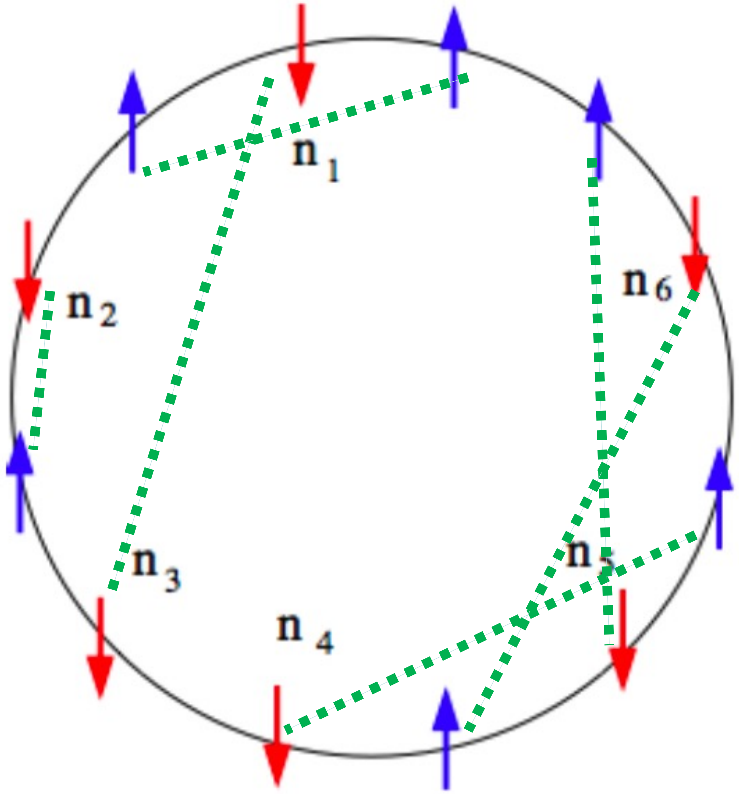
Short range entanglement



Hasting’s thm: ground states of local gapped Hamiltonians

entanglement “area” law in 1D

Long range entanglement



ground states of local gapless Hamiltonians

breaking of entanglement "area" law in 1D

Examples of long range entanglement

- Critical systems described by CFT

$$S_A = \frac{c}{3}\log|A| + c'$$

Vidal, Latorre, Rico, Kitaev, 2002
Calabrese, Cardy, 2004

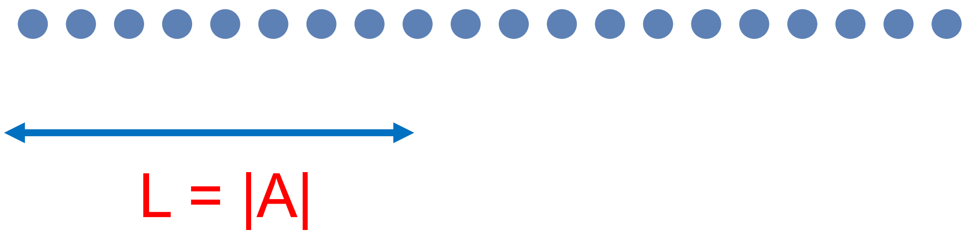
- Inhomogenous Hamiltonians (rainbow model)

$$S_A \propto |A|$$

Vitagliano, Riera, Latorre, 2010
Ramirez, Rodriguez-Laguna, GS, 2014

Uniform Hamiltonian $H = \sum_n h_{n,n+1}$

2L sites



Critical spin chain described by CFT

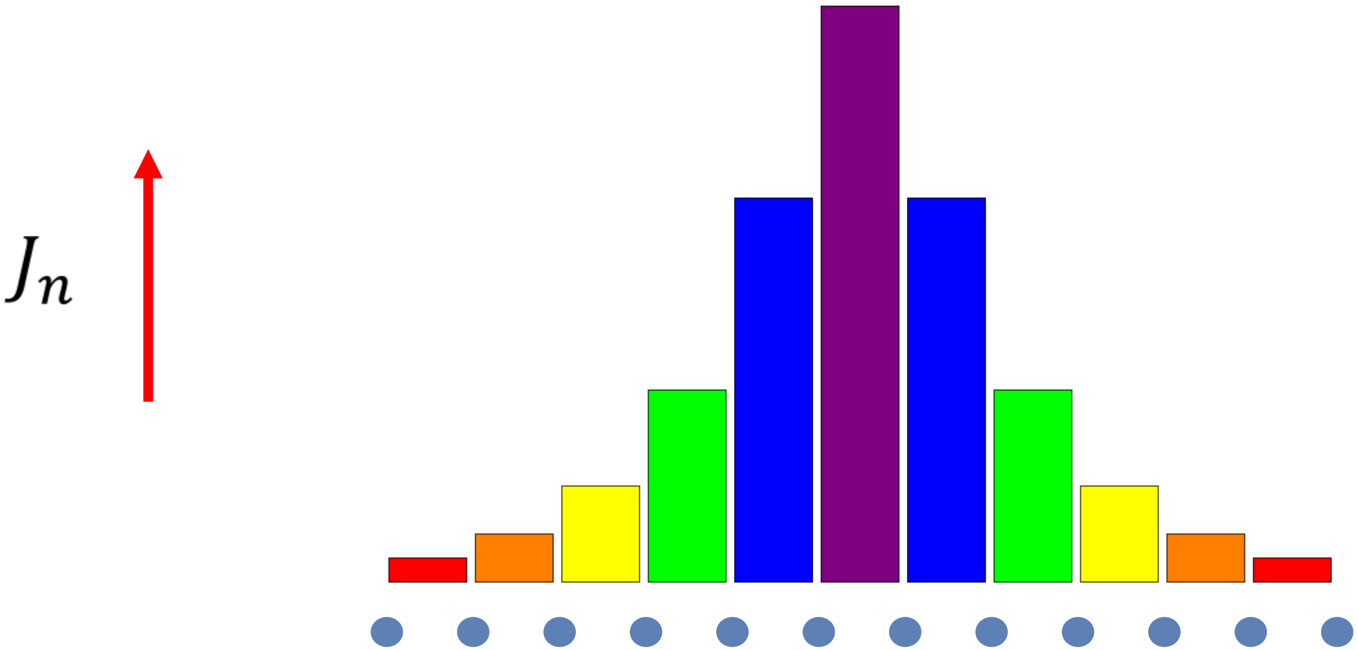
Entanglement entropy

$$S_A = \frac{c}{6} \ln L + \cdots$$

open chain

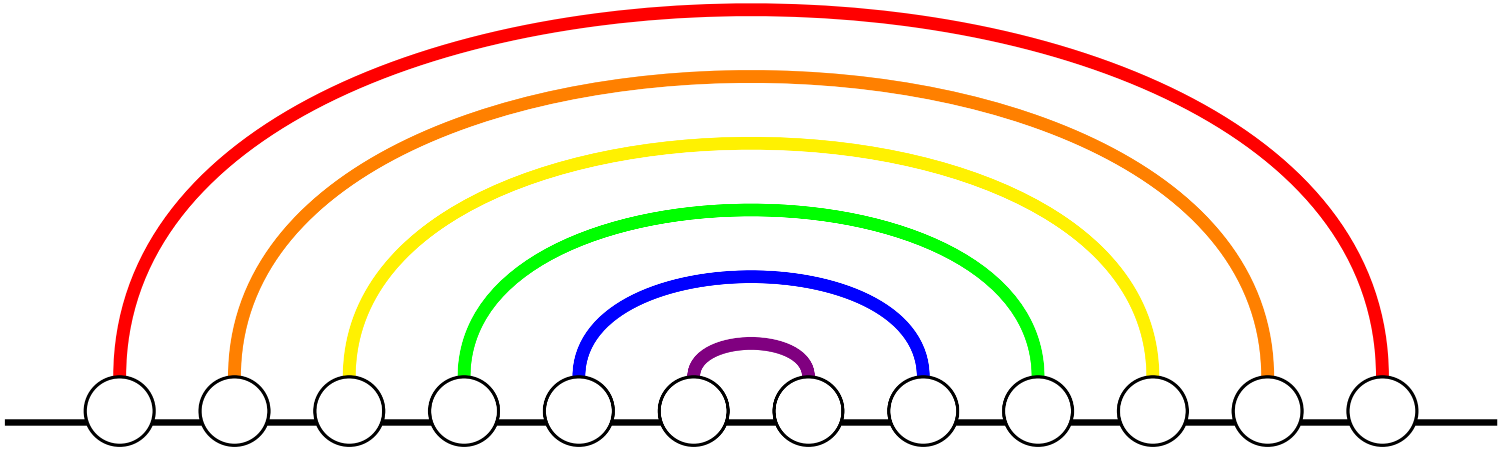
Non-uniform Hamiltonian $$H = \sum_n J_n \, h_{n,n+1}$$

$L = 12$



$$S_A \propto L$$

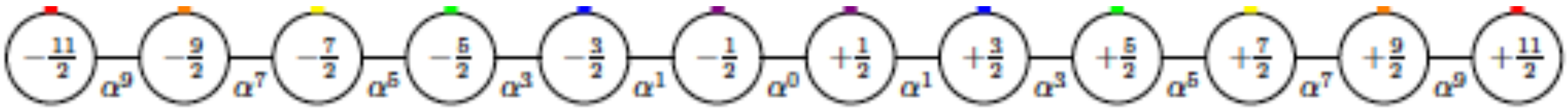
Ground State is a product of valence bonds states



- Concentric singlet phase (Vitagliano et al. 2010)
- Rainbow state (Ramirez et al. 2014)
- Matrioska state (Di Franco et al 2008)
- Nested Bell state (Alkurtass, et al 2014)

Rainbow XX model

Vitagliano, Riera and Latorre (2010)
Ramirez, Rodriguez-Laguna, GS (2014)



$$H = \sum_n J_n \left(S_n^X S_{n+1}^X + S_n^Y S_{n+1}^Y \right)$$

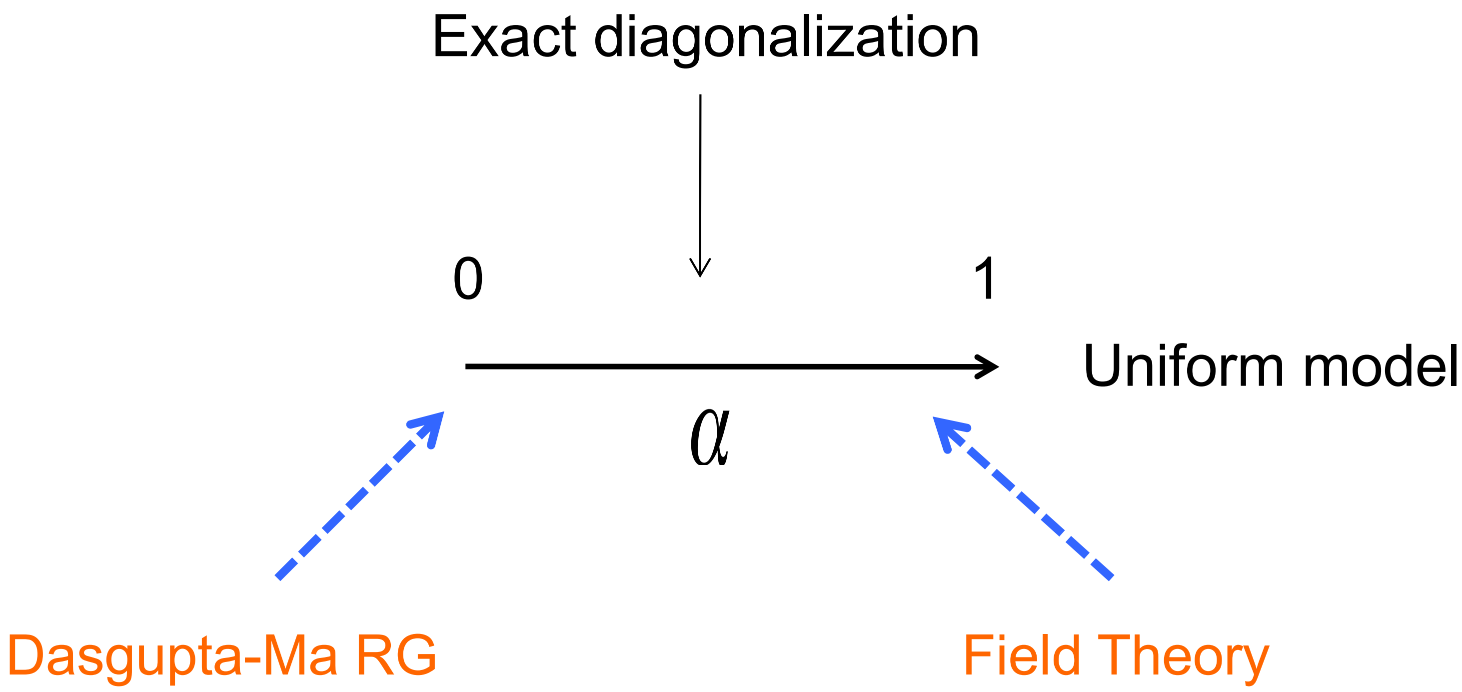
Jordan
Wigner

$$H \equiv -\frac{J_0}{2} c^\dagger_{\frac{1}{2}} c_{-\frac{1}{2}} - \sum_{n=\frac{1}{2}}^{L-\frac{3}{2}} \frac{J_n}{2} \left[c^\dagger_n c_{n+1} + c^\dagger_{-n} c_{-(n+1)} \right] + \text{h.c.} \, ,$$

$$\begin{cases} J_0(\alpha) & = & 1, \\ J_n(\alpha) & = & \alpha^{2n}, \; n = \frac{1}{2}, \ldots, L - \frac{3}{2}. \end{cases}$$

$$0 < \alpha \leq 1$$

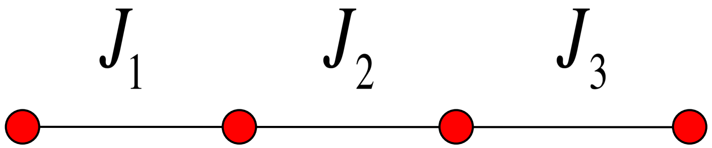
Methods:



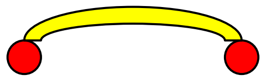
Exact diagonalization
0
1
Uniform model
α
Dasgupta-Ma RG
Field Theory

Dasgupta-Ma RG

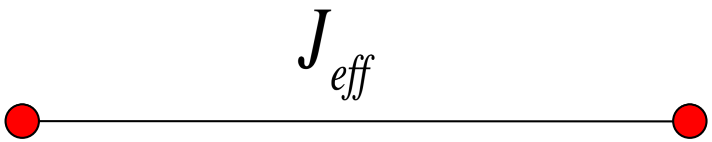
XX Hamiltonian



$$J_2 >> J_1, J_3$$



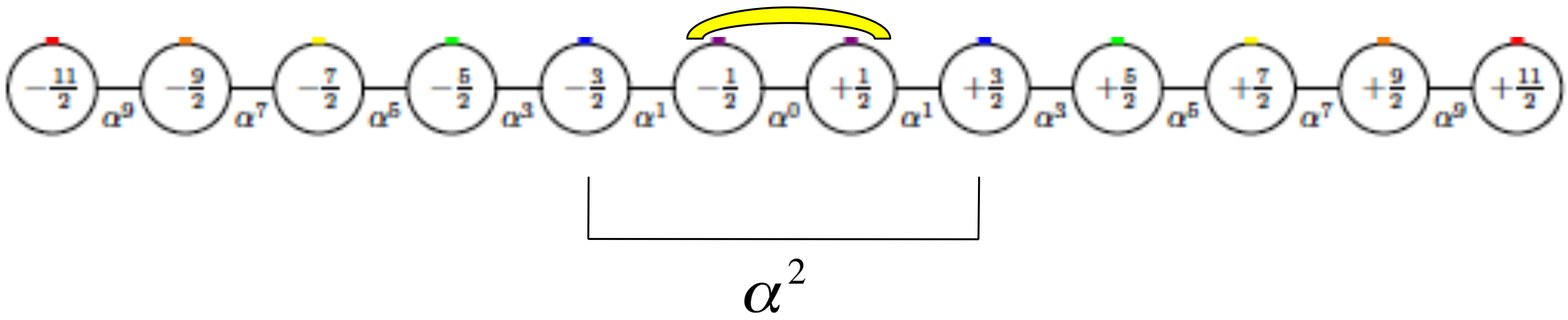
$$|bond\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$



$$J_{eff} = \frac{J_1 J_3}{J_2}$$

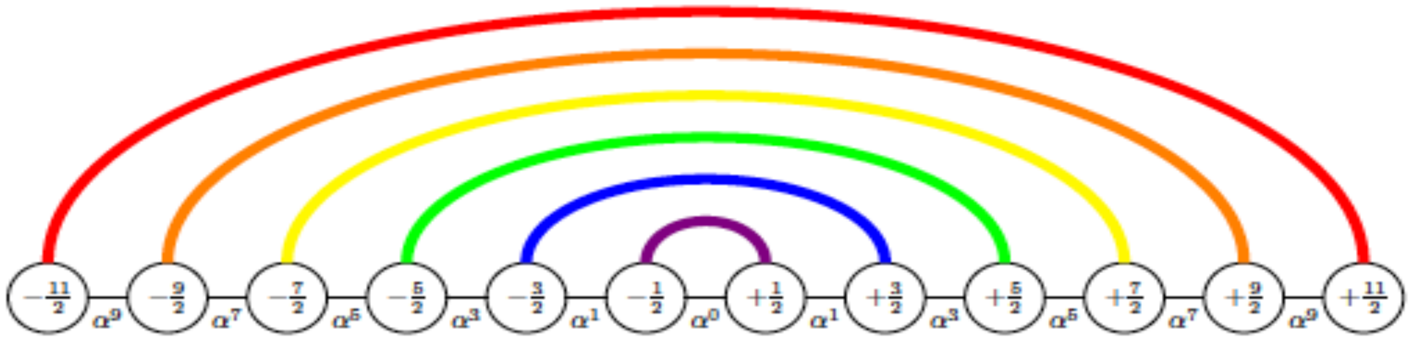
In the limit $\alpha \to 0^+$

Effective coupling: $$J_{eff} = \frac{J_{1/2} J_{1/2}}{J_0} = \alpha^2$$



This new bond is again the strongest one because $\alpha^2 > \alpha^3$

Iterate the RG ⇨ GS: valence bond state

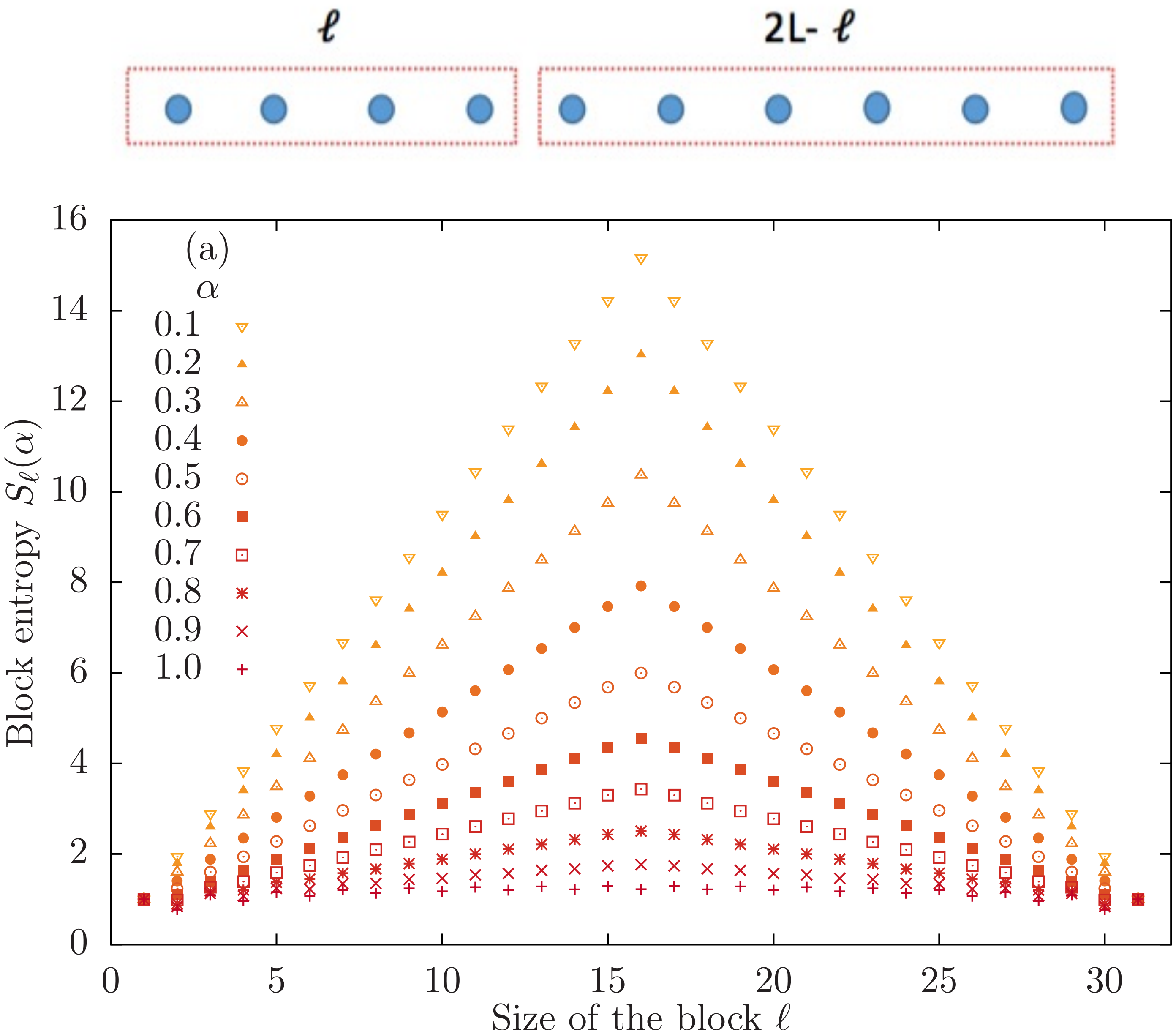


Exact in the limit $\alpha \to 0^+$

Half-block entanglement entropy

$$S_A = L \ln 2$$

All inhomogeneities : Exact Diagonalization



Weak inhomogeneity : Field theory

$$\alpha = e^{-h/2} \rightarrow 1$$

Fast-slow separation of degrees of freedom

$$\frac{c_m}{\sqrt{a}} \simeq e^{ik_F x}\psi_L(x) + e^{-ik_F x}\psi_R(x)$$

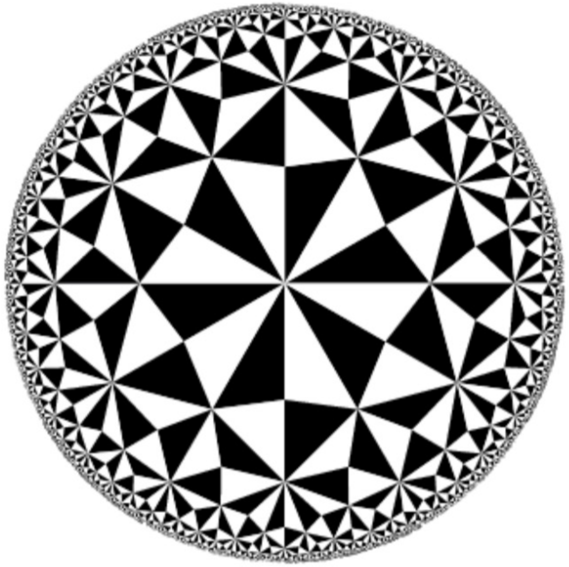
$$H \simeq iJa \int_{-aL}^{aL} dx\, e^{-\frac{h|x|}{a}} \left[\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L - \frac{h}{2a} \mathrm{sign}(x) (\psi_R^\dagger \psi_R - \psi_L^\dagger \psi_L) \right]$$

$$\psi_R(\pm L) = \mp i\, \psi_L(\pm L)$$

Massless Dirac fermion in curved spacetime

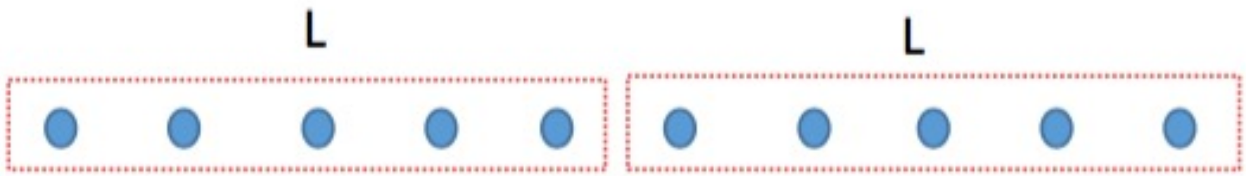
Curvature $R(x) = -h^2 - 4h\,\delta(x)$

AdS_2



??? → Holography

Half-block entropy



$$S_L \approx \frac{1}{6} \ln \frac{e^{hL} - 1}{h}$$

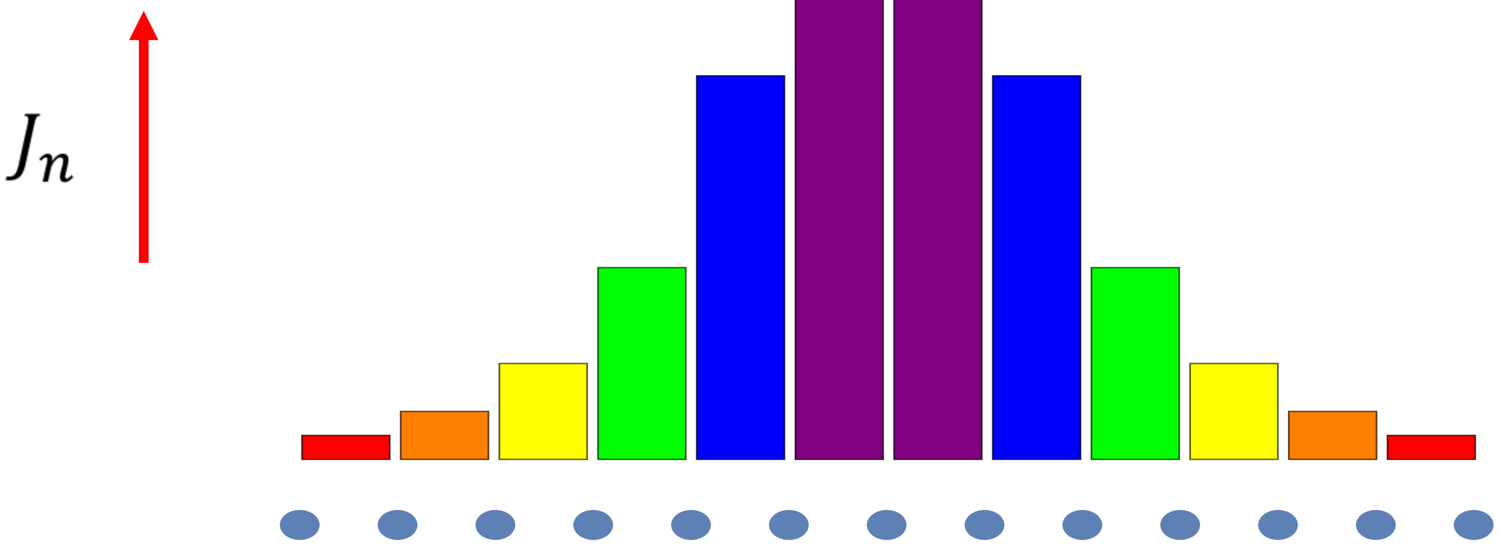
Uniform chain $h \to 0$ $\qquad S_L \approx \frac{1}{6} \ln L$ $\qquad$ CFT law

$h\,L \gg 1 \;\to\; \qquad S_L \approx \frac{h\,L}{6}$ $\qquad$ Thermodynamic entropy

Topological rainbow

$$H = \sum_n J_n \, h_{n,n+1}$$

Open odd chain

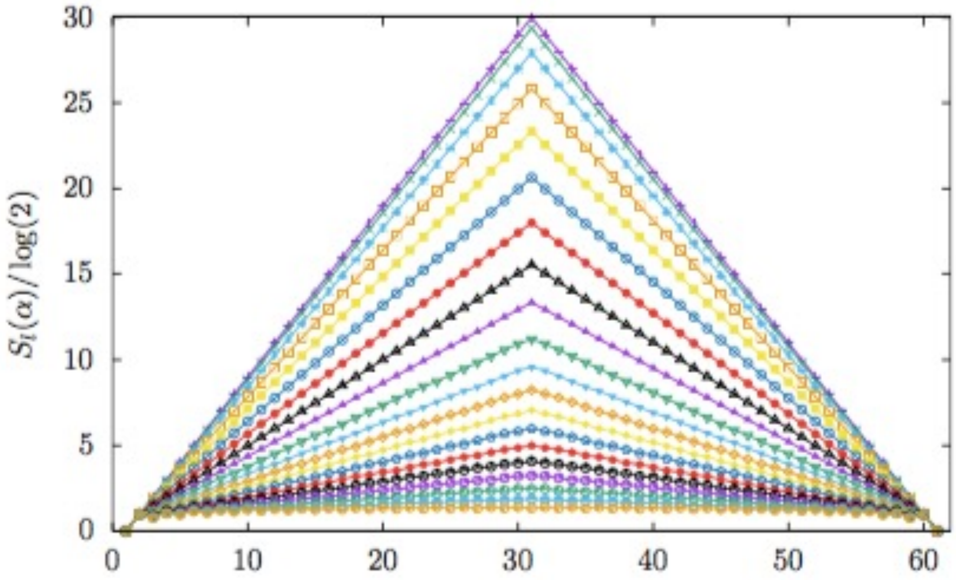


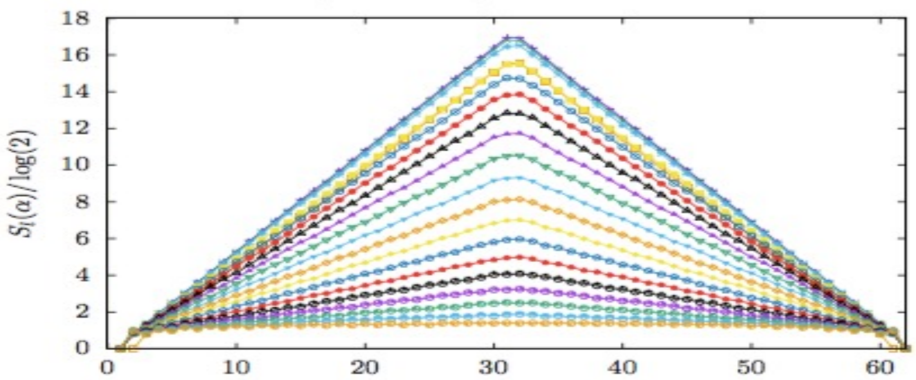
Topological rainbow

XX chain

2L sites

2L+1 sites





block entanglement entropies

$$S_A = \ell \ \ln 2$$

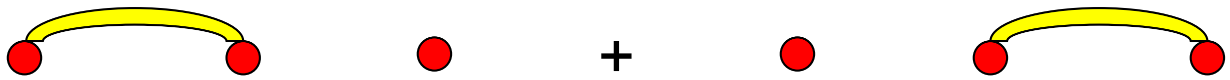
$$S_A = (\ell + \frac{1}{2})\,(2\ \ln 2 - 1)$$

Strong inhomogeneity

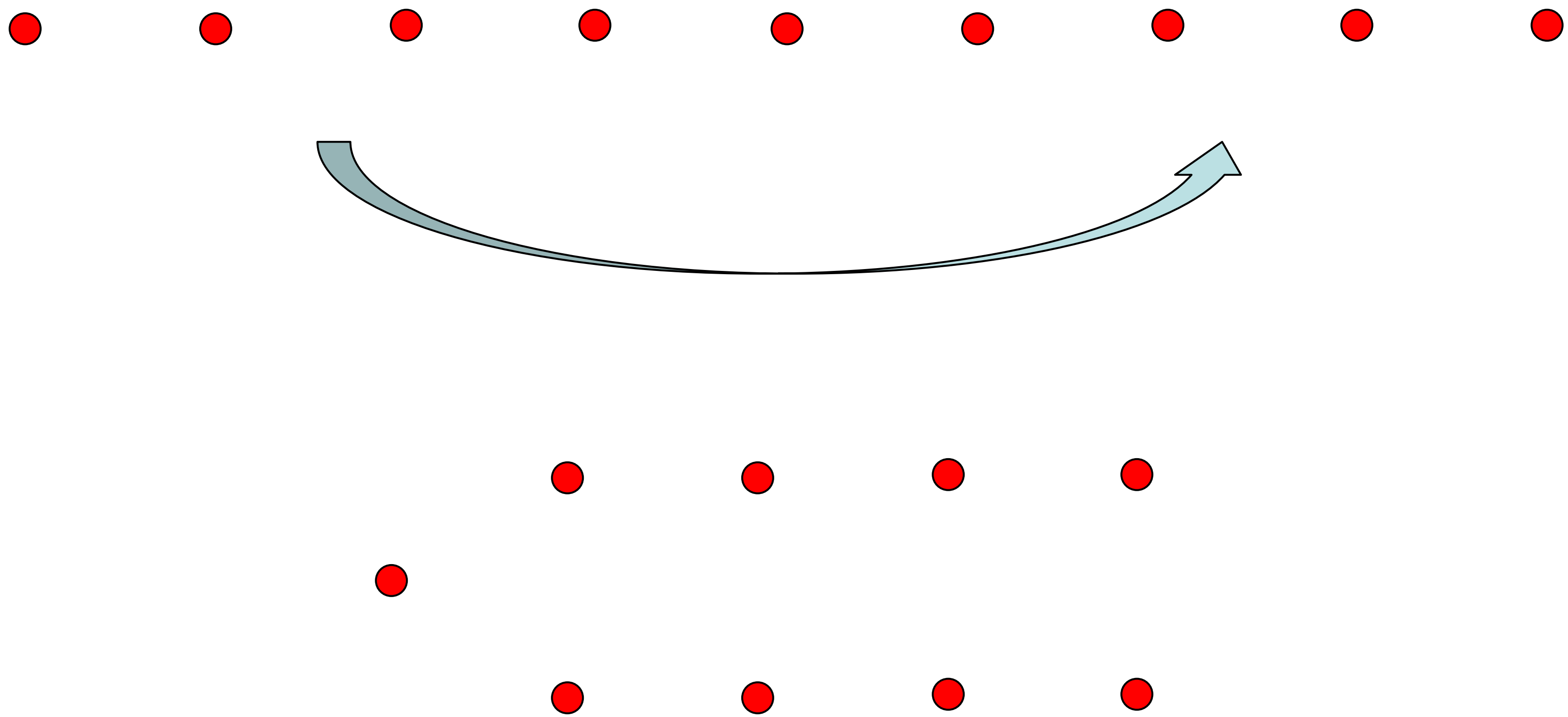
$$H_{\text{center}} = J_0 \left(\vec{\mathbf{S}}_0 \cdot \vec{\mathbf{S}}_1 + \vec{\mathbf{S}}_0 \cdot \vec{\mathbf{S}}_{-1} \right)$$

1st order perturbation theory

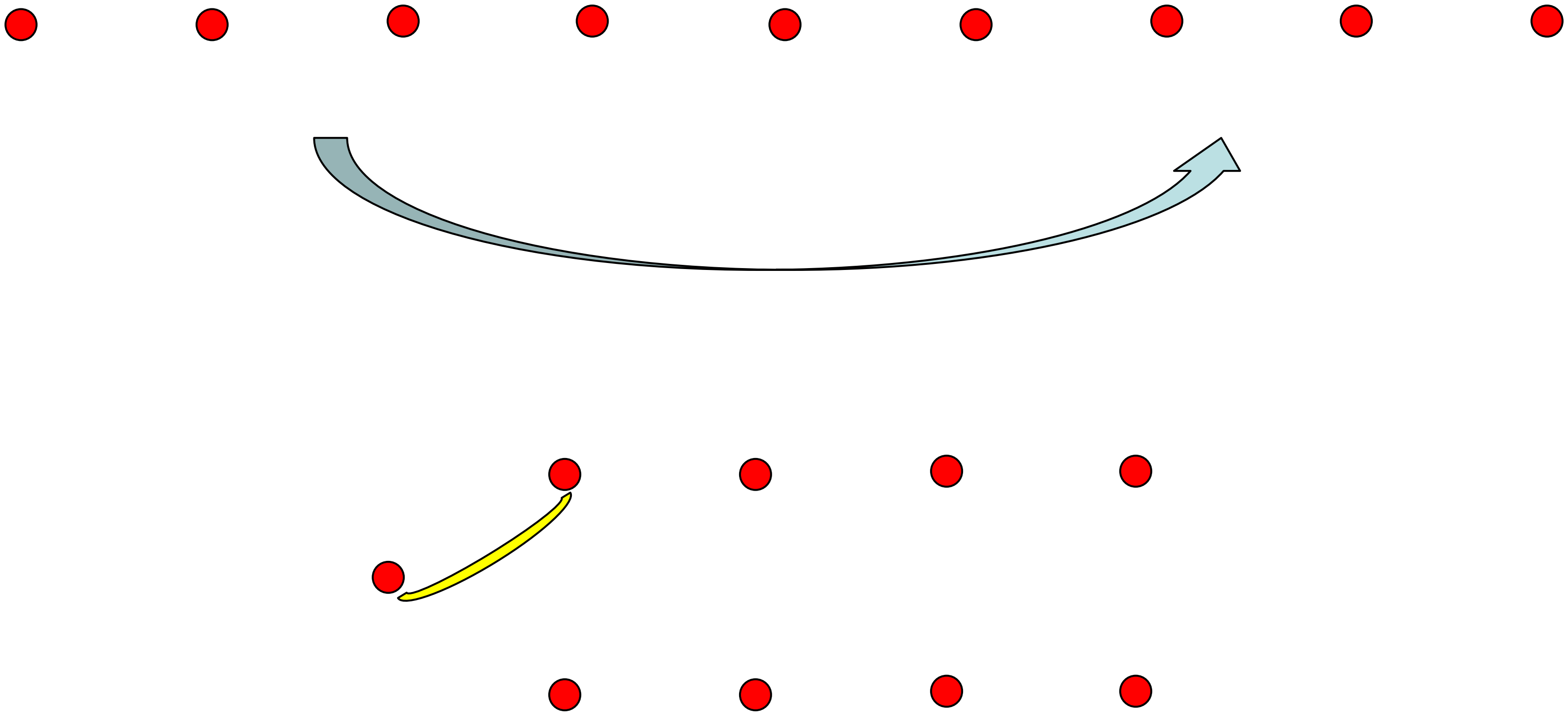
3 spins → 1 spin



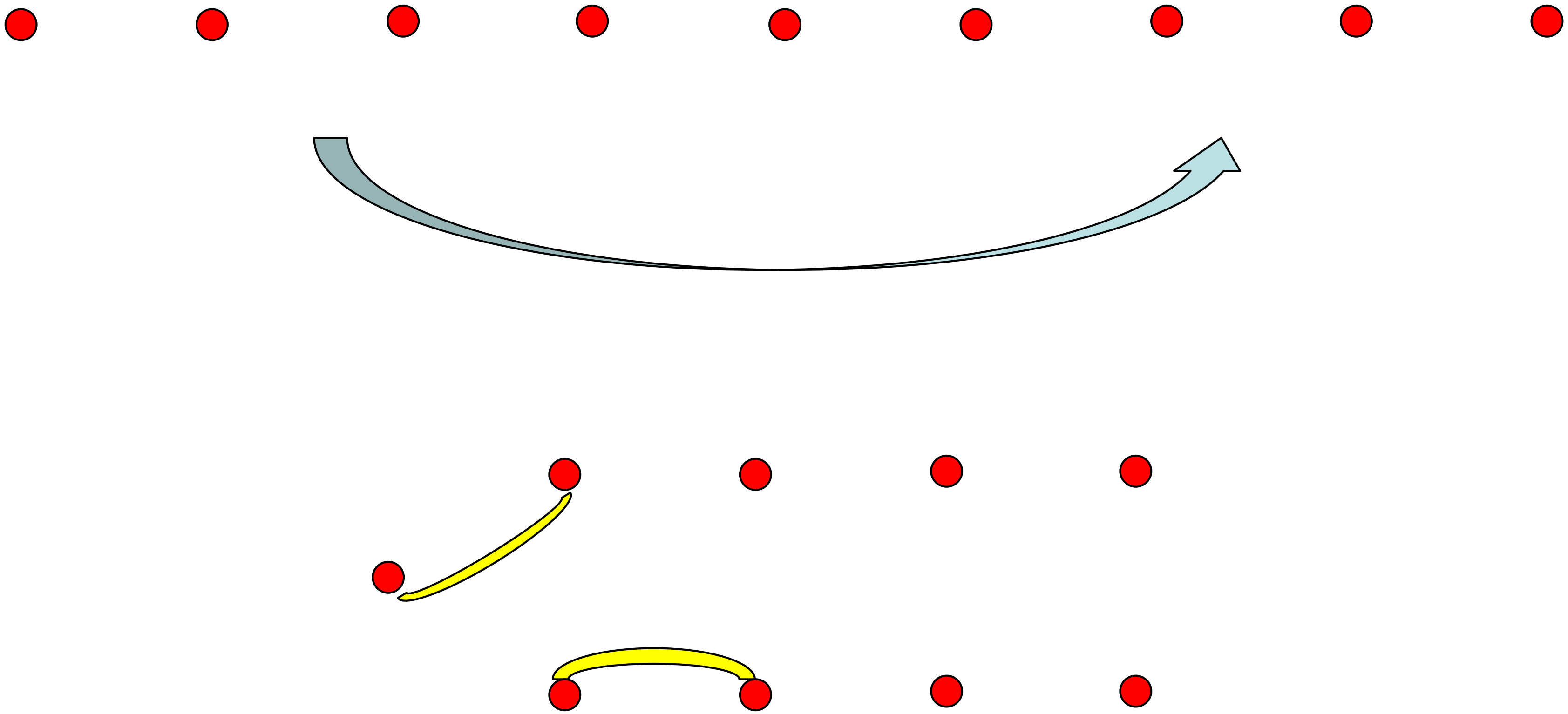
Folding the chain



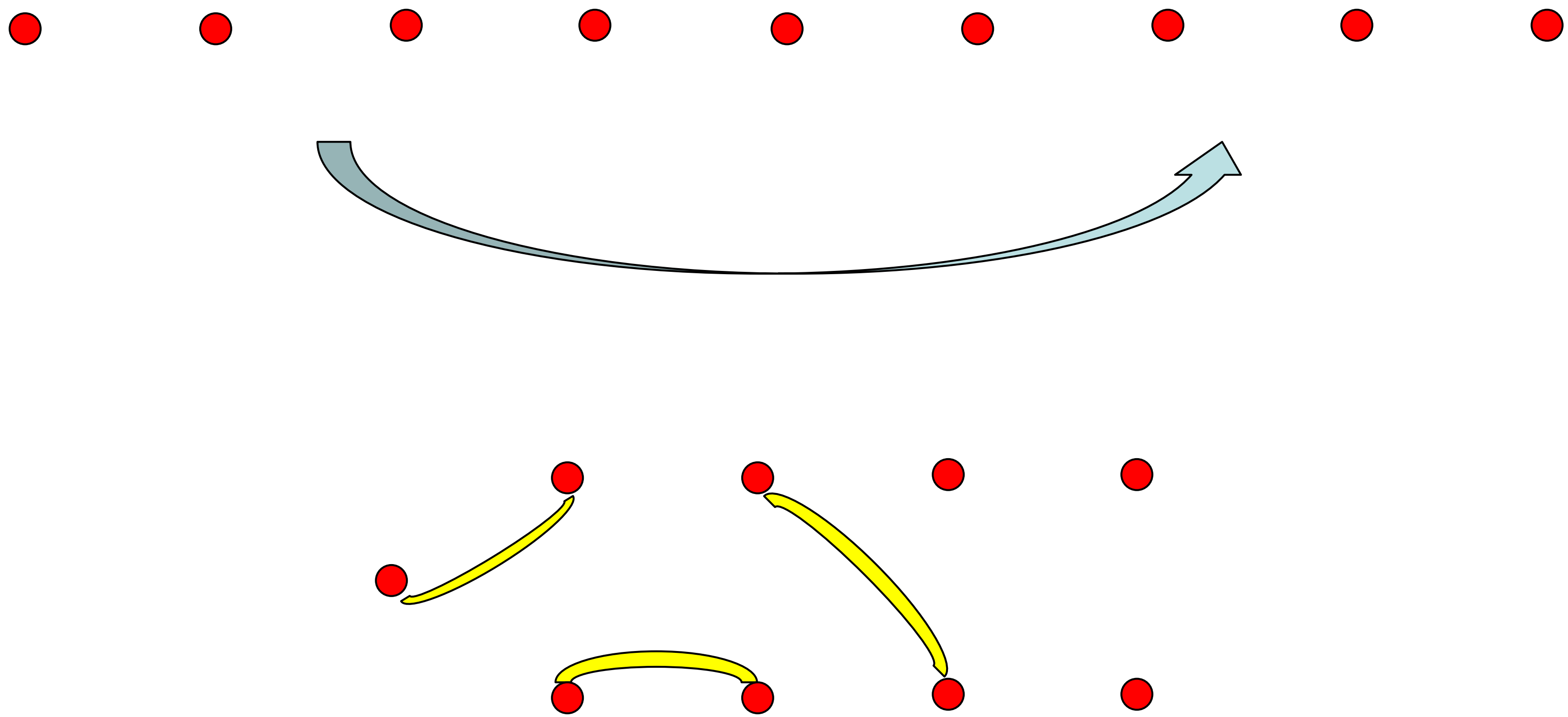
Folding the chain



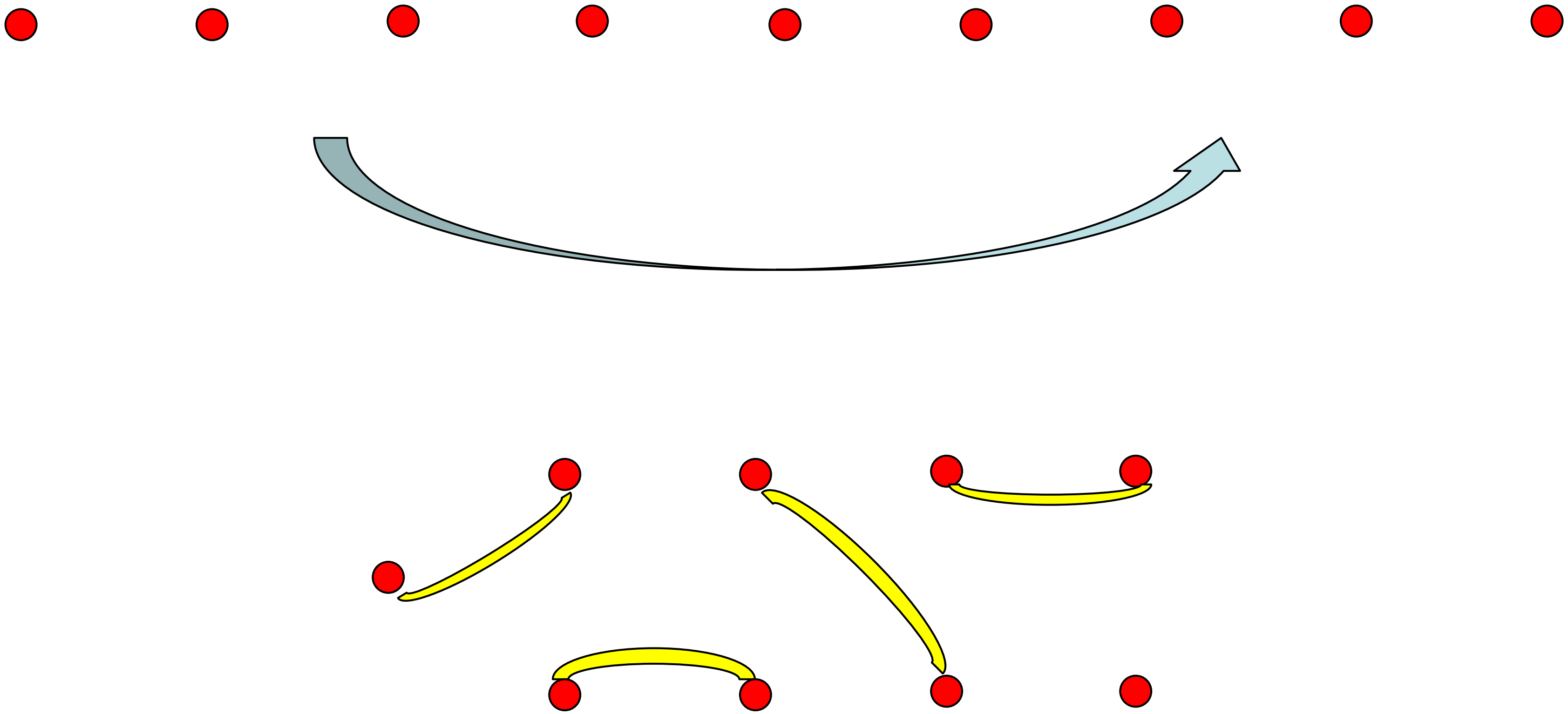
Folding the chain



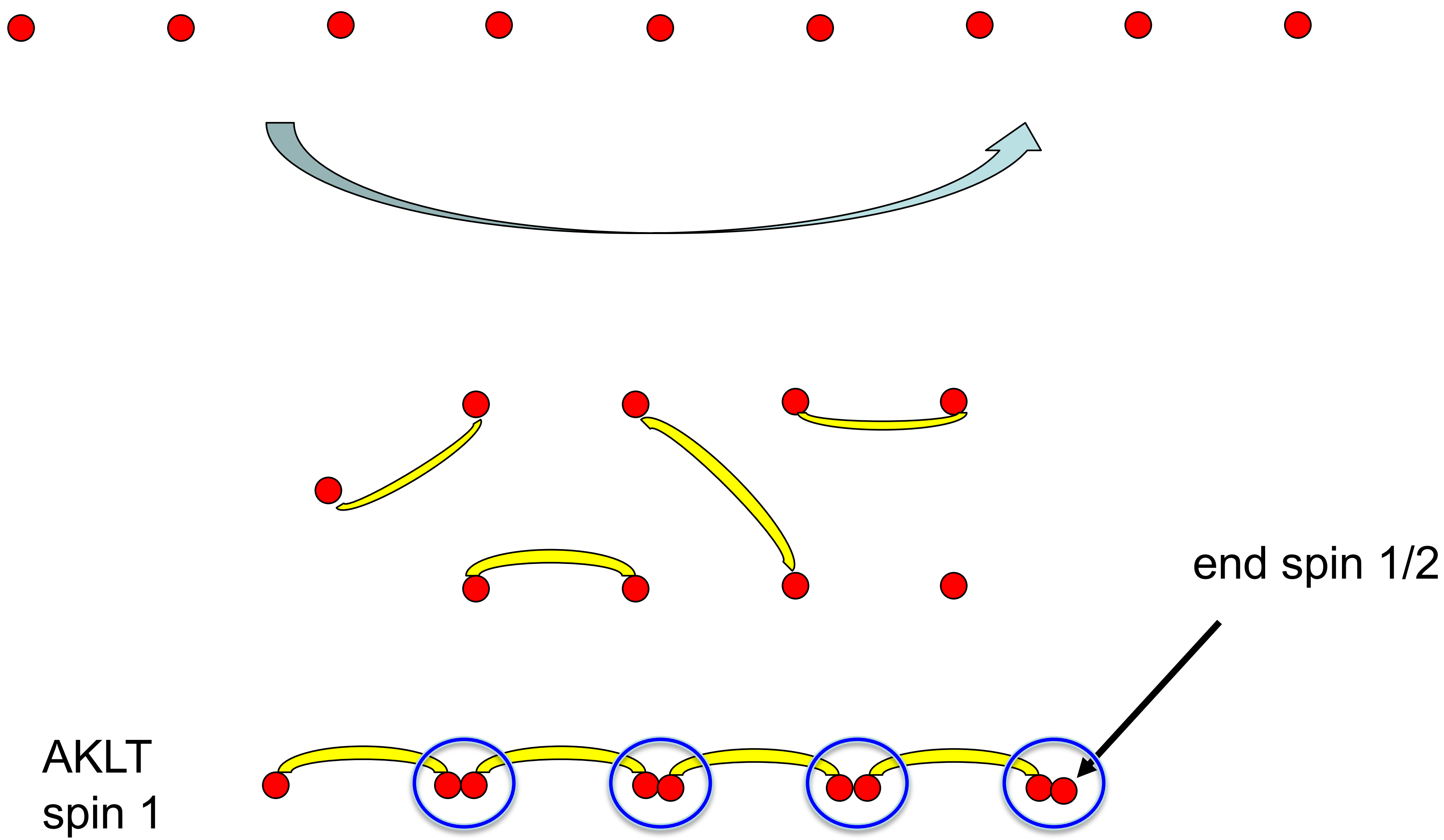
Folding the chain



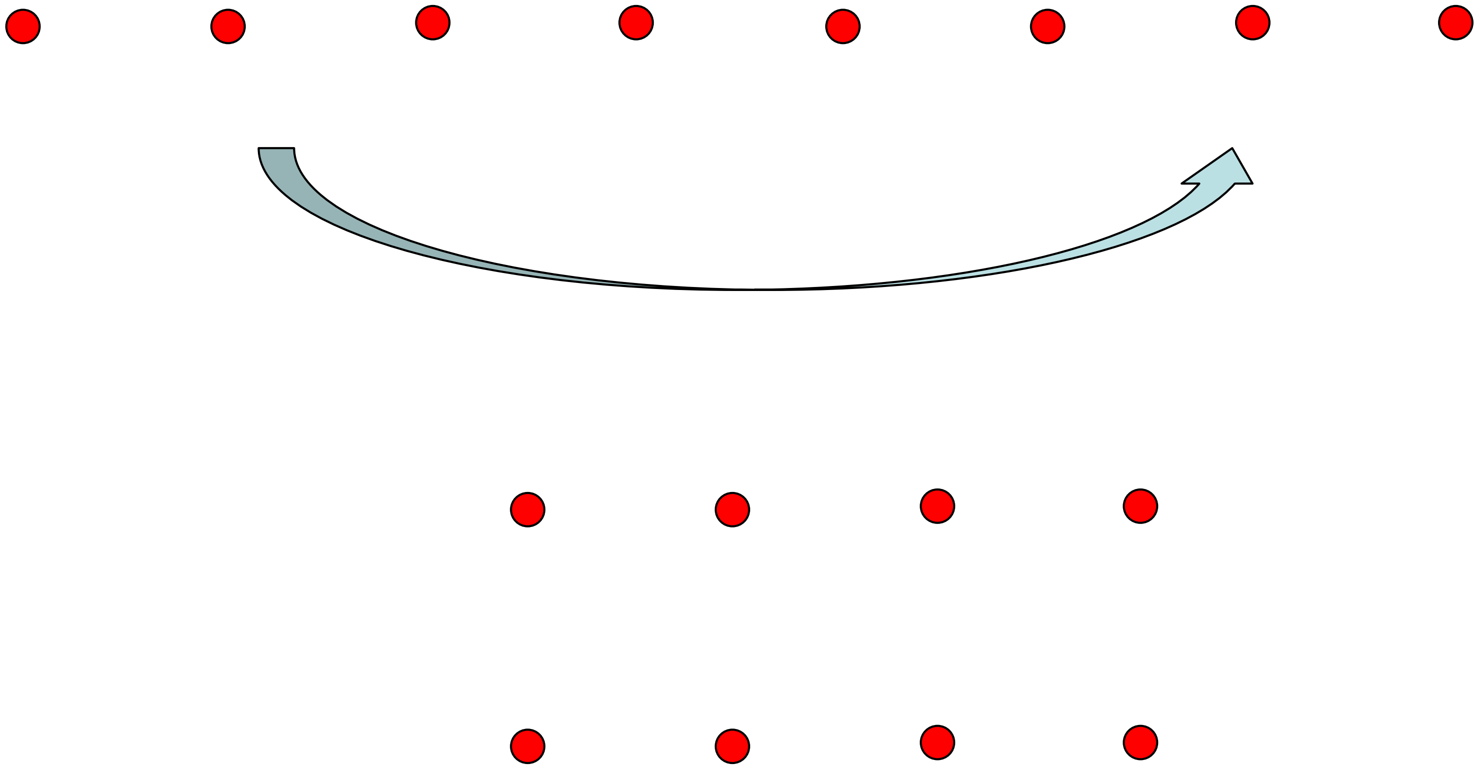
Folding the chain



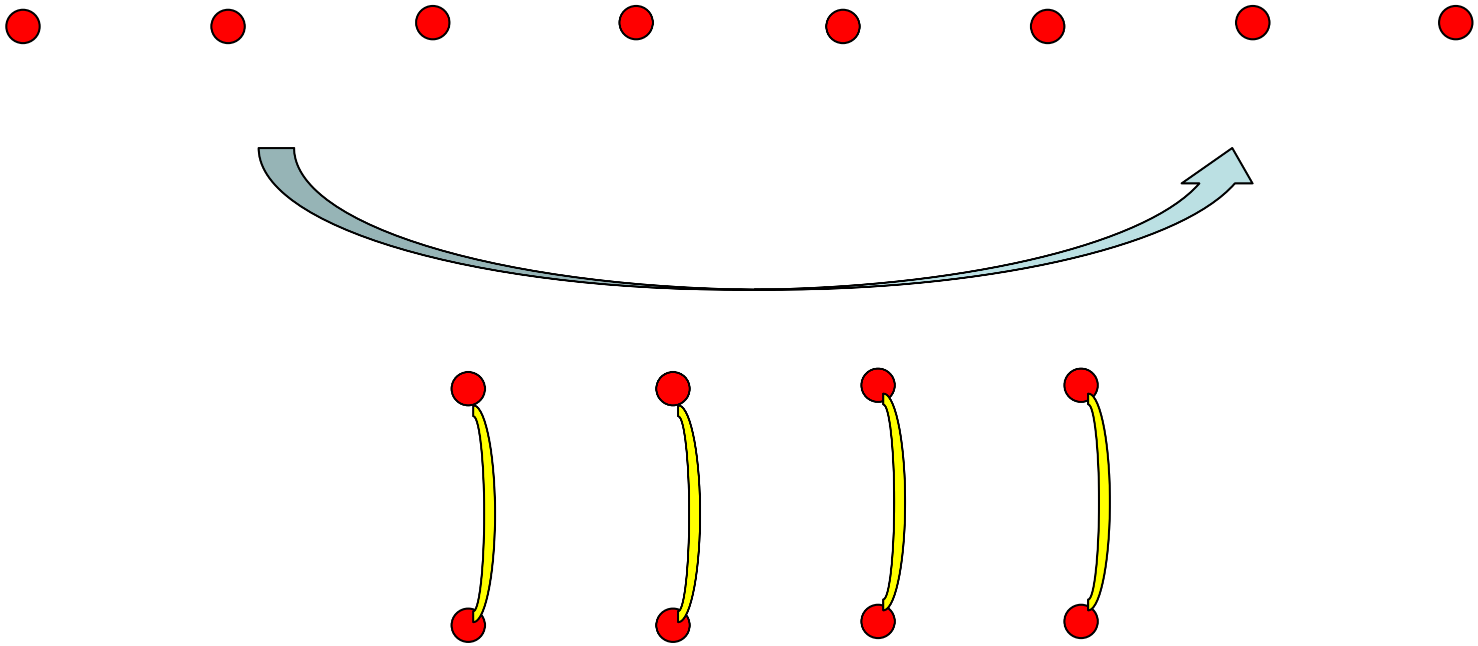
Folding the chain



Folding the even chain



Folding the even chain



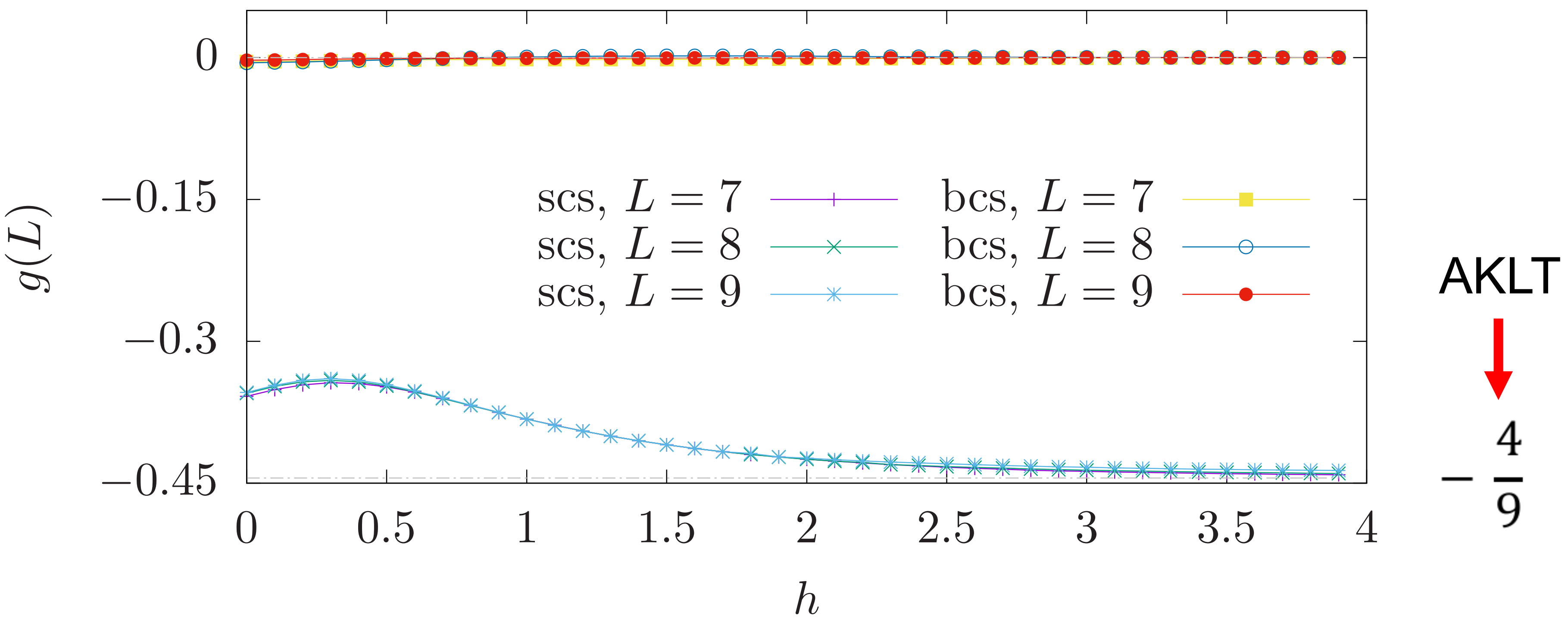
Even rainbow = product state in the folded basis

Even rainbow = trivial SPT

Odd rainbow = non trivial SPT

String order parameter (Heisenberg)

$$g(L) = \langle S_1^z \, e^{i\pi \sum_{j=2}^{L-1} S_j^z} \, S_L^z \rangle$$

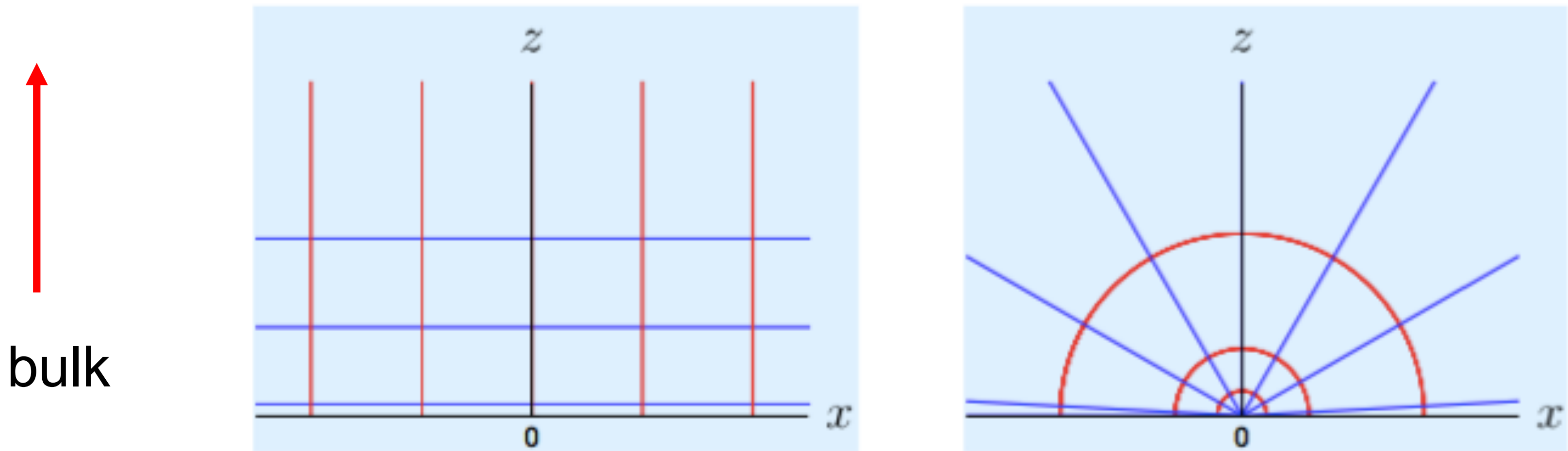


Holography and the rainbow

Cormack, Liu, Nozaki, Ryu (2018)

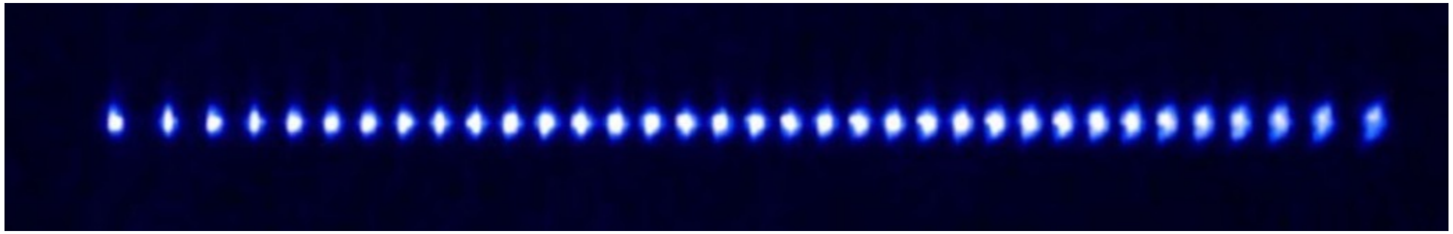
Rainbow : AdS_2 with $R(x) = -h^2$

AdS_2 is a foliation of AdS_3



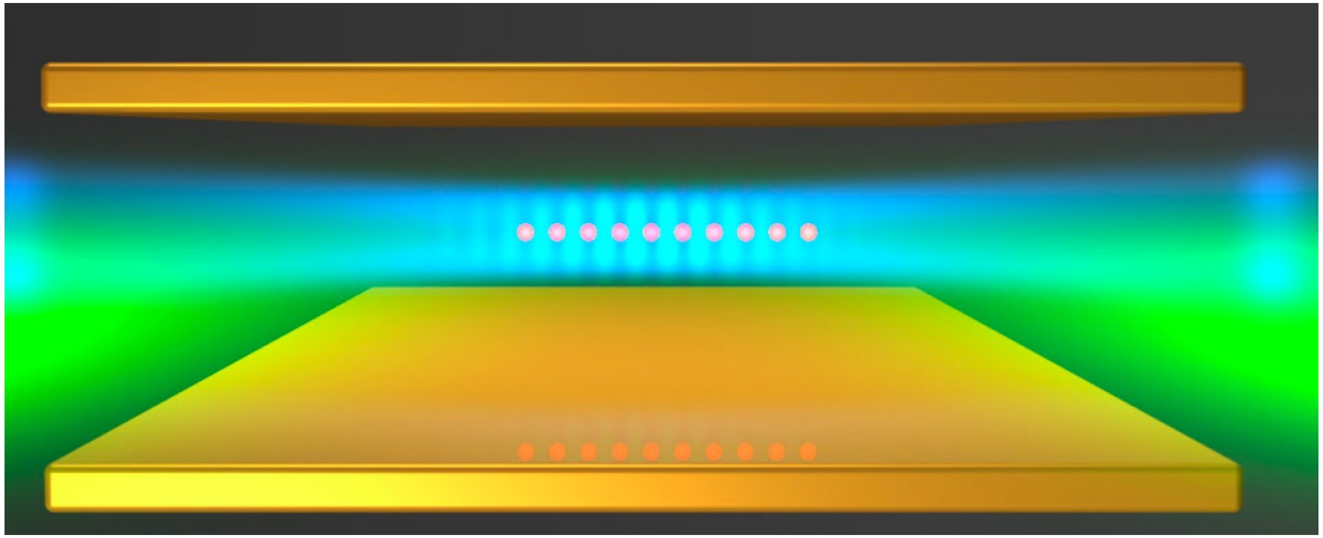
Can one simulate the rainbow chain?

with trapped ions ?



IonQ photo 2022

with Rydberg atoms ?

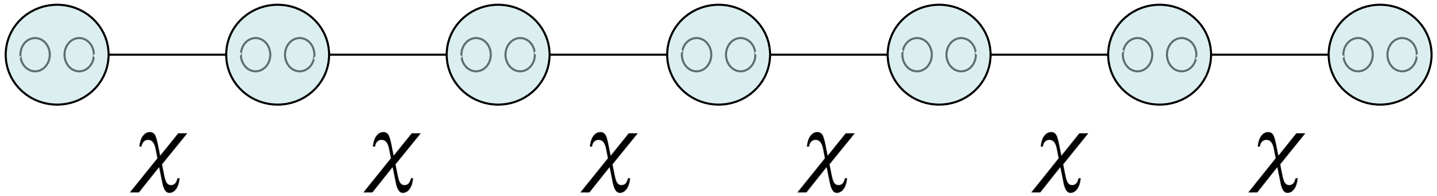


Nguyen et al 2018

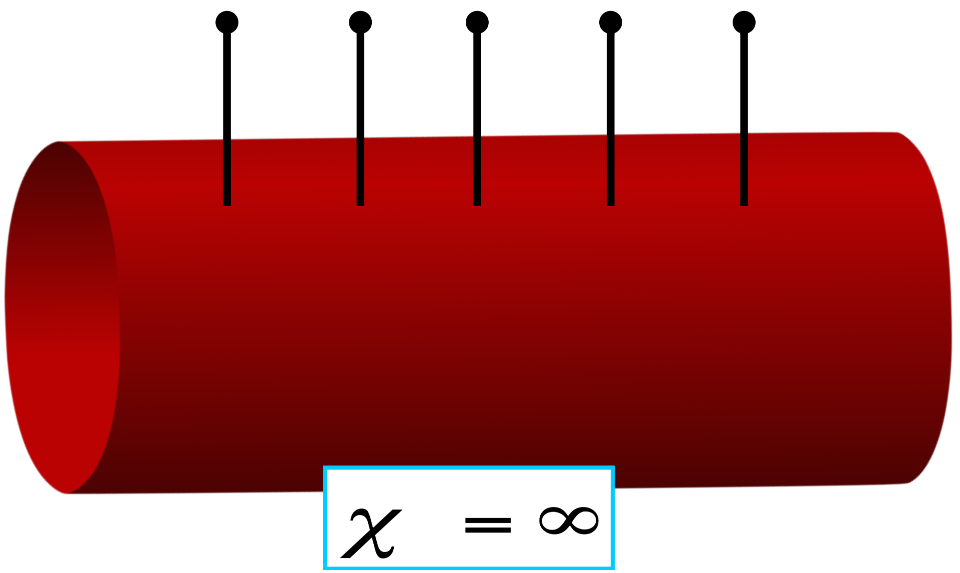
Infinite MPS
and
Haldane-Shastry

Infinite Matrix Product States and CFT

MPS state



iMPS state



Idea: use primary fields of a CFT as MPS "matrices"

MPS: $s_i \to A_i(s_i) : \chi \times \chi \; matrix$

iMPS: $s_i \to A_{z_i}(s_i) : primary \;\; field, \quad \chi = \infty$

$$\psi(s_1, s_2, \ldots, s_N) = \langle 0 | A_{z_1}(s_1) A_{z_2}(s_2) \cdots A_{z_N}(s_N) | 0 \rangle$$

iMPS and CFT

Cirac, GS 2009

Consider a chiral massless boson $\varphi(z)$

$$A_z(s) = \chi_s :e^{i s \sqrt{\alpha}\, \varphi(z)}: \quad \chi_s = \pm 1$$

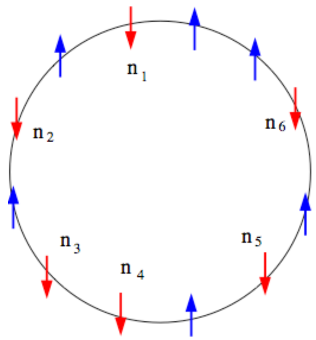
$$\psi(s_1, s_2, \ldots, s_N) = \prod_i \chi_{s_i} \prod_{i<j} (z_i - z_j)^{\alpha s_i s_j} \times \delta\left(\sum_i s_i\right)$$

$$S^z_{tot} = \frac{1}{2}\sum_{i=1}^{N} s_i = 0, \quad N : even$$

$\alpha, \; z_n, \chi_{s_n}$ are variational parameters obtained by minimization of the GS energy and imposing the symmetries of a Hamiltonian

XXZ model of a spin 1/2 chain

$$H_{XXZ} = \sum_{1=1}^{N} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$



Phases of the model

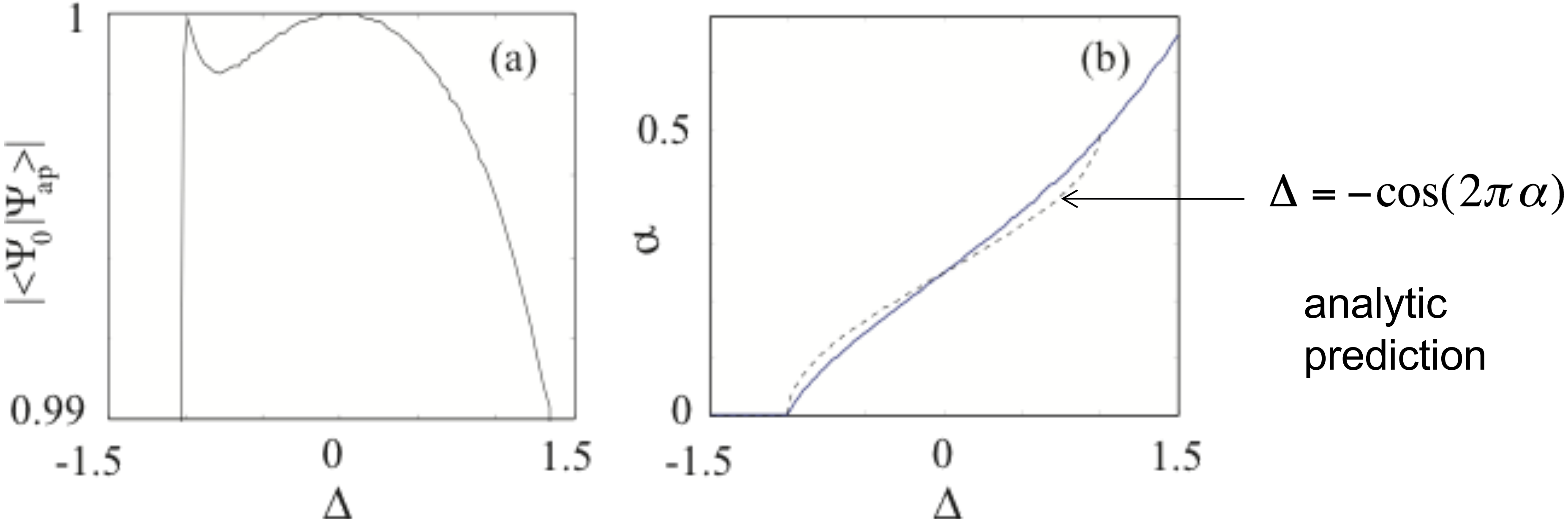
$$\Delta > 1 \quad \textit{gapped antiferromagnet}$$
$$-1 < \Delta \leq 1 \quad \textit{gapless} \ \ (c = 1\, CFT)$$
$$\Delta \leq -1 \quad \textit{Ferromagnetic}$$

Translational invariance -> $z_n = e^{2\pi i n/N}, \quad n = 1, \ldots, N$

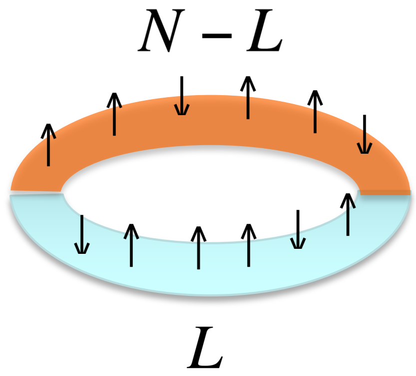
Marshall sign rule -> $\prod_{i} \chi_{s_i} = \prod_{i:odd} s_i$

Minimize the energy -> $\alpha = f(\Delta)$

Overlap of the exact and the CFT wave functions (N=20)



Entanglement properties



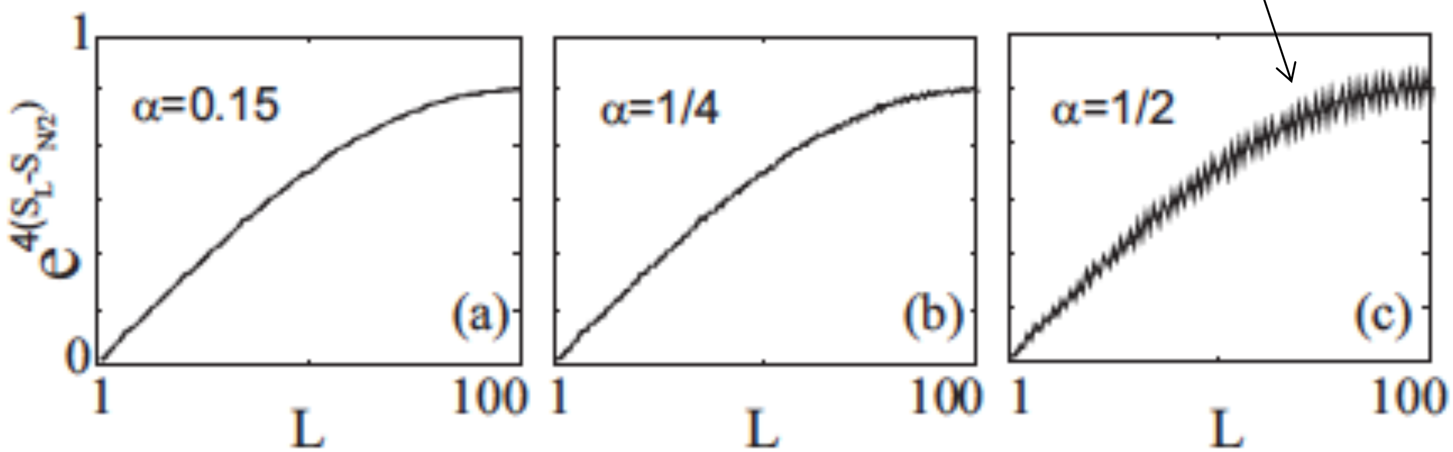
Renyi entropy $S_L = -\log Tr\,\rho_L^2$

CFT prediction $$S_L = \frac{c}{4}\log\left(\frac{N}{\pi}\sin\frac{\pi L}{N}\right) + c'$$

One finds $c = 1 \quad for \quad 0 < \alpha \le \frac{1}{2}$

Fluctuations depend on α

$N = 200$



Luttinger liquid of XXZ $-1 < \Delta \leq 1$

$$H_{XXZ} = \sum_{1=1}^{N} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

↓ bosonization

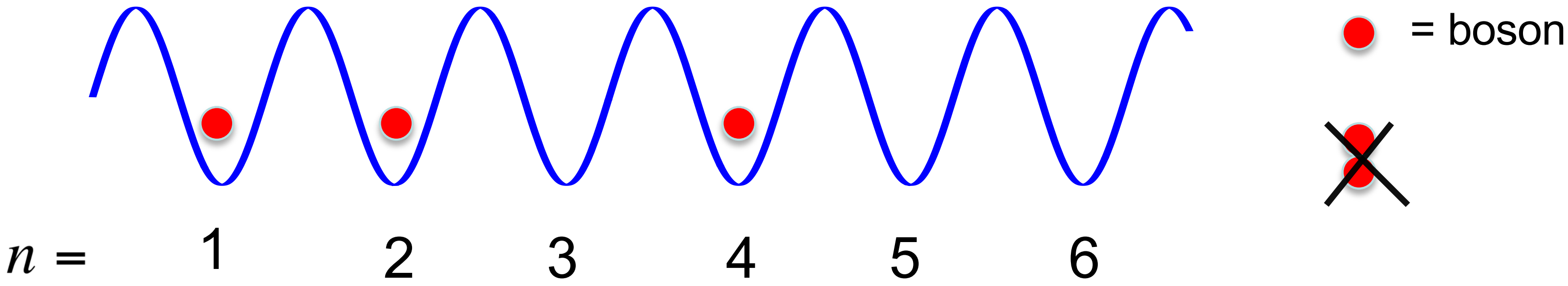
$$H_{XXZ}^{cont} = \frac{v}{2} \int dx \left[K \left(\partial_x \theta(x) \right)^2 + K^{-1} \left(\partial_x \phi(x) \right)^2 \right]$$

$$K = \frac{1}{2} \frac{1}{1 - \frac{1}{\pi} arcos\, \Delta}$$ (from Bethe ansatz)

$$\Delta = -\cos(2\pi\alpha) \rightarrow K = \frac{1}{4\alpha},$$

The Haldane-Shastry model (1988)

1D lattice of hard core bosons



If the site n is occupied $\longrightarrow$ $z_n = e^{2\pi i n/N} \quad , n = 1, \ldots, N$

many body state $|\psi\rangle = \sum_{n_1 < n_2 < \ldots < n_{N/2}} \psi(n_1, n_2, \ldots, n_{N/2}) \, |n_1, n_2, \ldots, n_{N/2}\rangle$

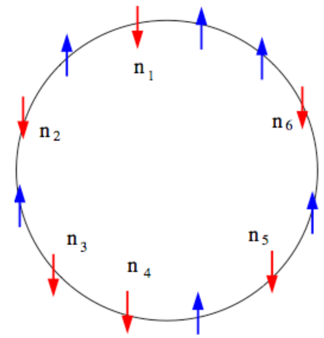
$$\psi_{HS}(n_1, \ldots, n_{N/2}) = \prod_i z_{n_i} \prod_{i<j} (z_{n_i} - z_{n_j})^2$$

1D bosonic Laughlin state with filling fraction $\nu = \frac{\# \, bosons}{\# \, sites} = \frac{1}{2}$

Map: hard core boson to spin 1/2

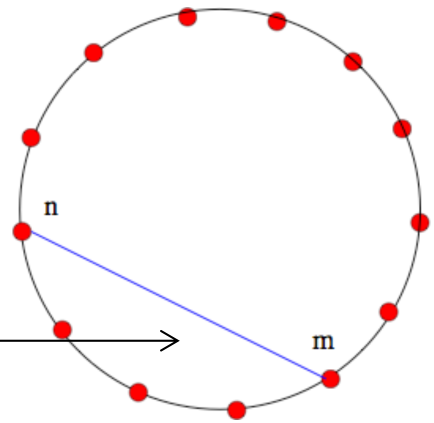
$$\textit{empty site } |0\rangle \leftrightarrow |\uparrow\rangle \textit{ spin up}$$

$$\textit{occupied site } |1\rangle \leftrightarrow |\downarrow\rangle \textit{ spin down}$$



$|\psi_{HS}\rangle$ is the ground state of the Hamiltonian

$$H \propto -\sum_{n<m} \frac{z_n z_m}{(z_n - z_m)^2} \vec{S}_n \cdot \vec{S}_m = C \sum_{i,n} \frac{\vec{S}_i \cdot \vec{S}_{i+n}}{\sin^2(\pi n / N)}$$



Relation between HS and iMPS

Take $$\alpha = \frac{1}{2}, \; z_n = e^{2\pi i n/N}$$

Using the hard core boson – spin map

$$\psi_{HS}(n_1, \ldots, n_{N/2}) \propto \psi_{CFT}(s_1, \ldots, s_N)$$

$$\prod_i z_{n_i} \prod_{1 \leq i < j \leq N/2} (z_{n_i} - z_{n_j})^2 \propto \prod_{i=odd} s_i \prod_{1 \leq i < j \leq N} (z_i - z_j)^{s_i s_j /2}$$

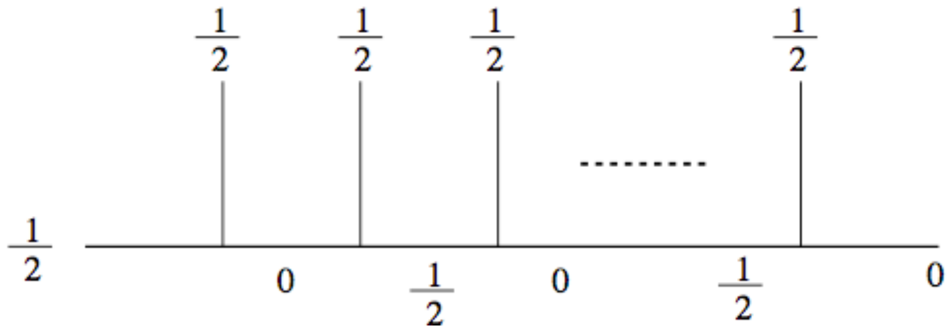
WZW model $SU(2)_{k=1}$

$$A_z(s) \propto e^{is/\sqrt{2}\,\varphi(z)} \rightarrow h = \frac{1}{4}$$

primary field $\phi_{1/2}(z)$

fusion rule: $\phi_{1/2} \times \phi_{1/2} = \phi_0$

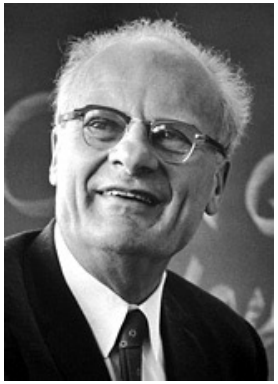
$$\psi_{CFT}(s_1, \ldots, s_N) =$$



The HS wave function is a conformal block

Algebraic Bethe Circuits

The Bethe Ansatz (1931)



Hans Bethe

Exact diagonalization of the Heisenberg Hamiltonian of spin 1/2

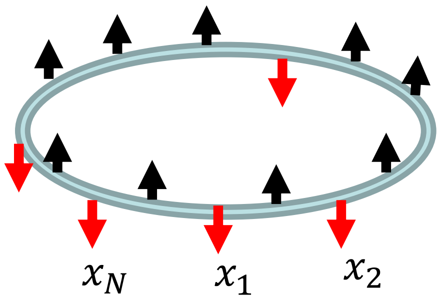
$$H = \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z$$

$$[H, S^z] = 0, \qquad S^z = \frac{1}{2} \sum_{i=1}^{N} \sigma_i^z$$

$$H\, |\psi_M\rangle = E |\psi_M\rangle \qquad S^z\, |\psi_M\rangle = \left(\frac{N}{2} - M\right) |\psi_M\rangle$$

M : number of down spins ↓

The wave function is a superposition of plane waves called magnons



$$|\psi_M\rangle = \sum_{1<x_1<\cdots<x_M\leq N} f(x_1,\cdots,x_M)\,|x_1,\cdots,x_M\rangle$$

$$f(x_1,\cdots,x_M)=\sum_P A_P e^{k_{P1}x_1+\cdots+k_{PM}x_M}$$

$$k_j, \qquad j=1,\cdots,M$$ quasi momenta

$$E = 2\sum_{i=1}^{M}\cos k_i$$

$$e^{ik_iN} = (-1)^{M-1}\prod_{i\neq j}^{M}\frac{1-2e^{ik_i}+e^{i(k_i+k_j)}}{1-2e^{ik_j}+e^{i(k_i+k_j)}}, \qquad i=1,\cdots,M$$

XXZ model

Lieb, Wu 1972

$$H = \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta\, \sigma_i^z \sigma_{i+1}^z$$

anisotropy

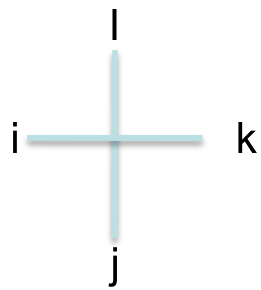
$$e^{ik_i N} = (-1)^{M-1} \prod_{j \neq i}^{M} \frac{1 - 2\Delta\, e^{ik_i} + e^{i(k_i+k_j)}}{1 - 2\Delta\, e^{ik_j} + e^{i(k_i+k_j)}}$$

If $\Delta = 0$ $\quad e^{ik_i N} = (-1)^{M-1}$

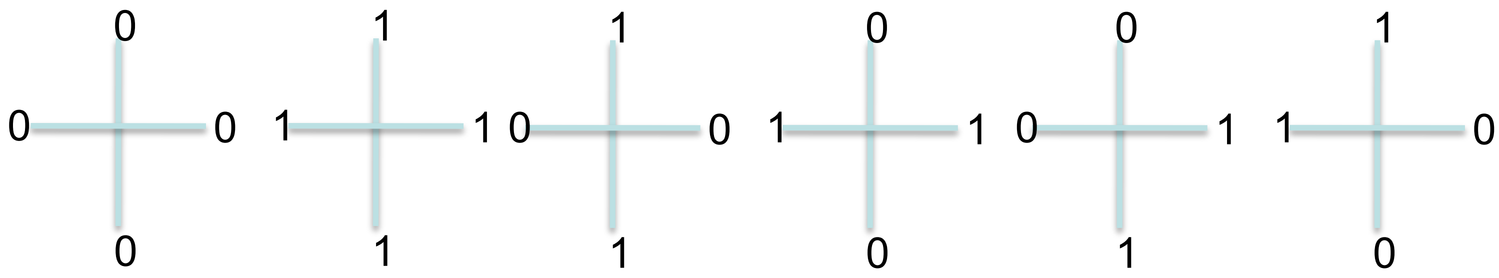
Jordan-Wigner

magnon → spinless free fermion

Algebraic Bethe Ansatz (six vertex model)

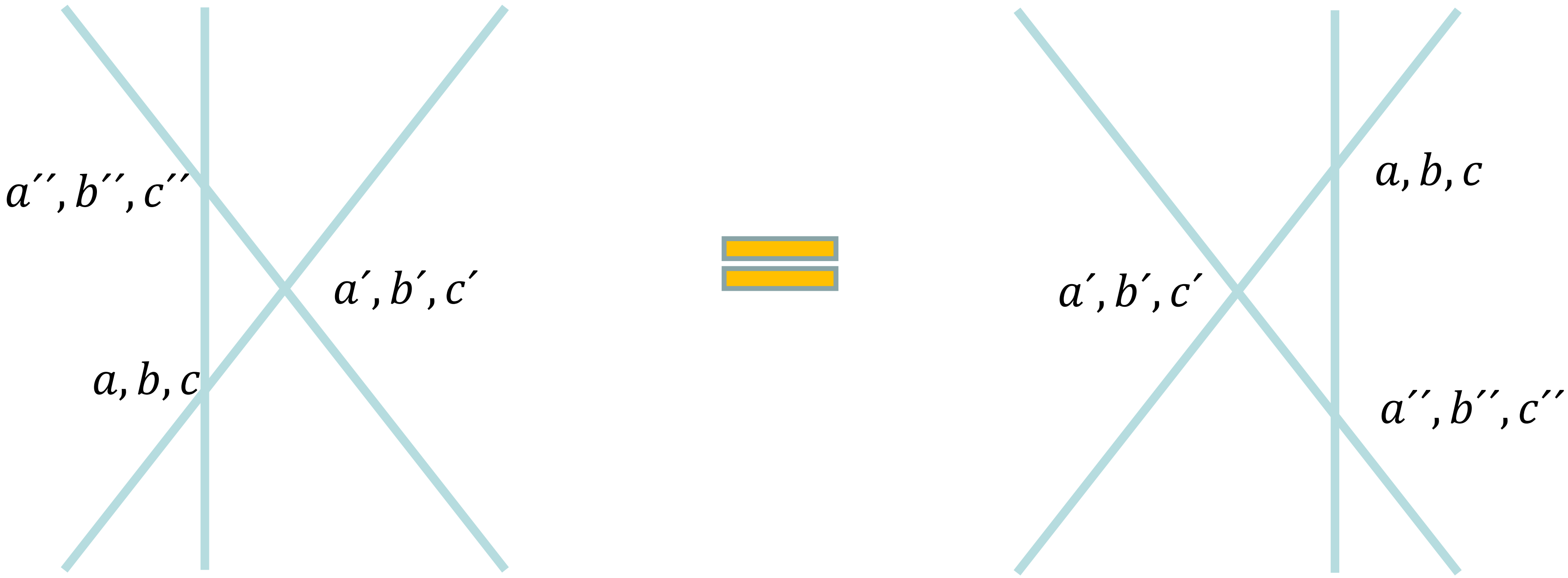
Boltzmann weights $R_{i\,j}^{k\,l} =$ 

$i + j = k + l$



a a b b c c

Yang-Baxter equation



anisotropy $\Delta = \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{a'^2 + b'^2 - c'^2}{2a'b'} = \dfrac{a''^2 + b''^2 - c''^2}{2a''b''}$

R matrix for the 6 vertex model (XXZ)

$$R(\lambda) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} = \rho \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s_1 & s_2 & 0 \\ 0 & s_2 & s_1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

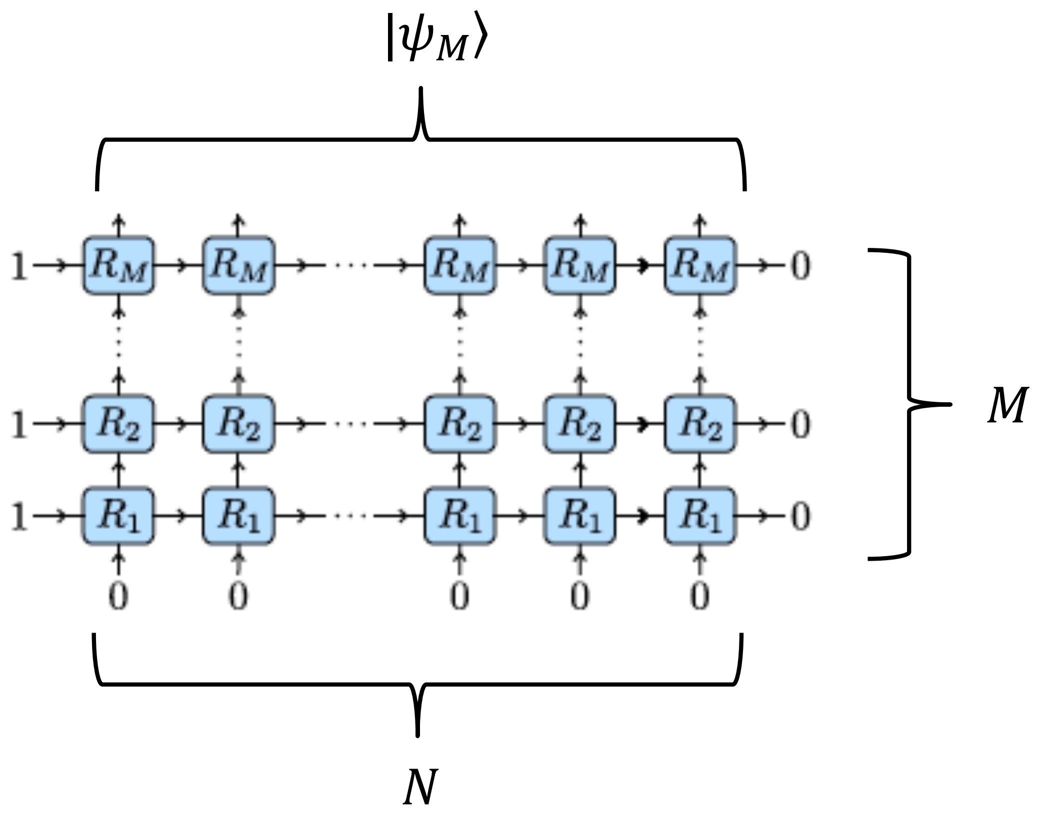
$$1 + s_2^2 - s_1^2 = 2 s_2 \Delta \qquad \cos\gamma = \Delta$$

$$s_1(\lambda) = \frac{\sinh i\gamma}{\sinh\left(\gamma\frac{\lambda+i}{2}\right)} \quad , \quad s_2(\lambda) = \frac{\sinh\left(\gamma\frac{\lambda-i}{2}\right)}{\sinh\left(\gamma\frac{\lambda+i}{2}\right)}$$

λ rapidity variable

Bethe states as tensor networks

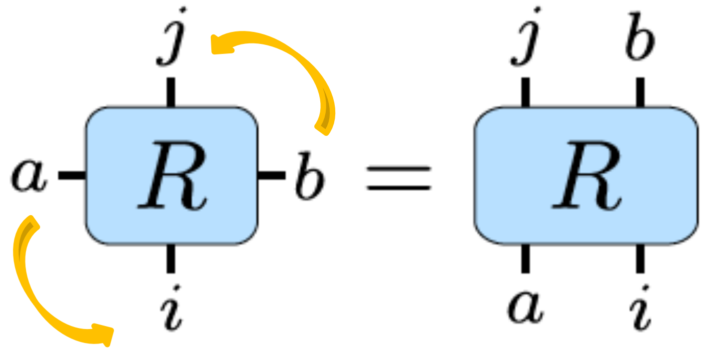
Alcaraz, Lazo 2003
Katsura, Maruyama 2010
Murg, Korepin, Verstraete 2012

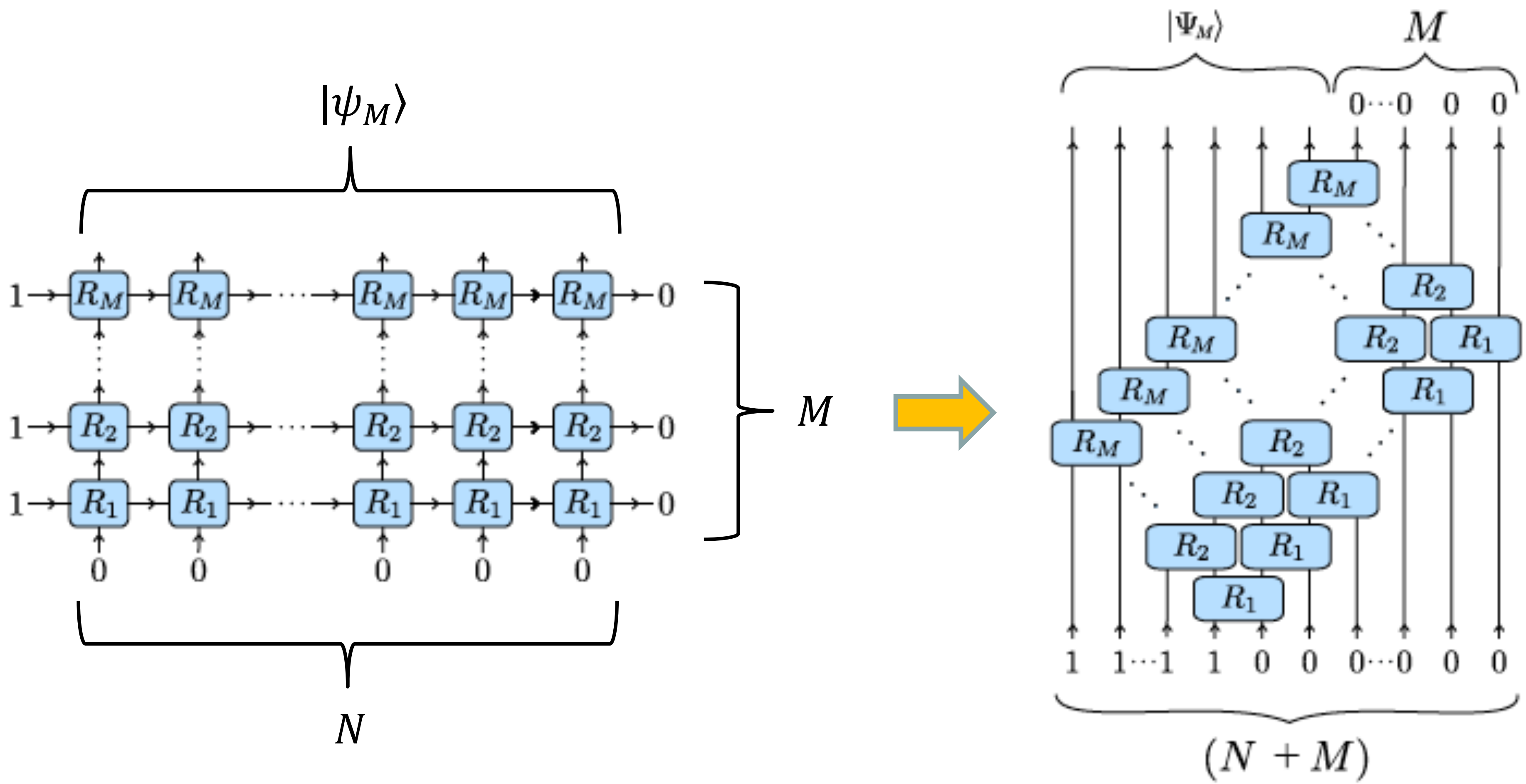


Goal : transform the Bethe tensor network into a quantum circuit

1st step:

$$R_{ai}^{jb} =$$





$|\psi_M\rangle$
N
M
$|\Psi_M\rangle$
$(N+M)$

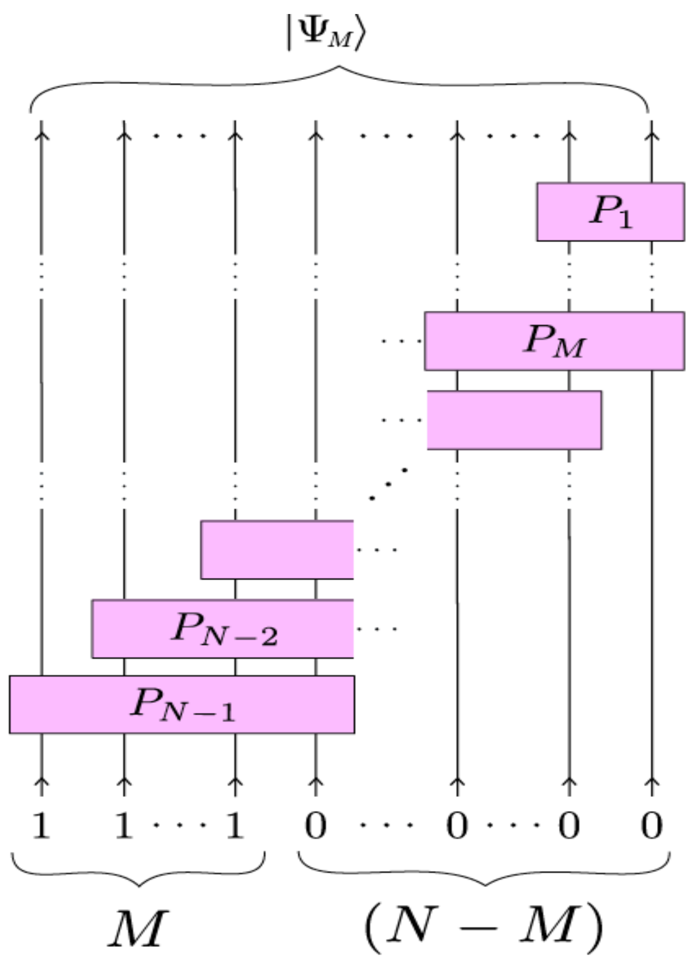
Problems:

- The R matrices are NOT unitary for the rapidities satisfying the Bethe equations
- Projection of the outgoing ancilla qubits renders the algorithm probabilistic

Both problems can be solved using QR decompositions iteratively

Matrix = Isometry x Upper Triangular

Algebraic Bethe Circuit (ABC)

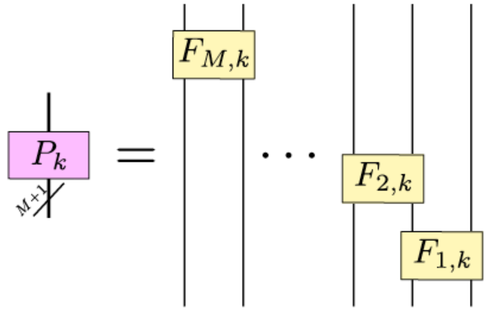


unitary gates

ABC for the XX model

All the unitary gates can be decomposed into the product F sim - gates

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta\, e^{i\alpha} & -\sin\theta\, e^{-i\beta} & 0 \\ 0 & \sin\theta\, e^{i\beta} & \cos\theta\, e^{-i\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Number of F- gates: $N\,M - \frac{M\,(M+1)}{2}$

Depth of circuit: $N + M - 1$

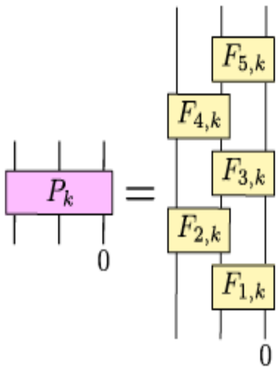
Related works

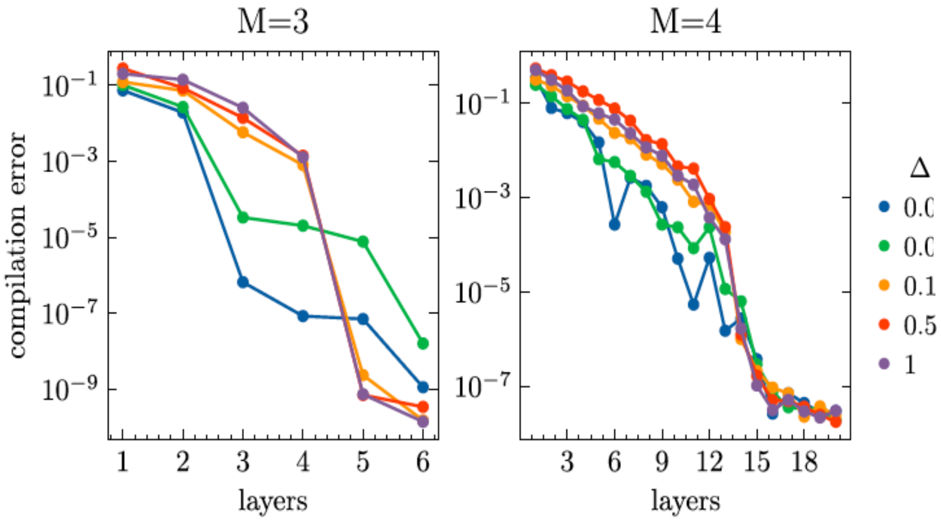
- XY circuits: number of local gates: N^2 , depth: $N \log N$
  Verstraete et al (2009)
- Scalings similar to circuits that prepate Slater determinants
  Kivlichan et al (2018); Jiang et al (2018); Arute et al (2020)
- Dimension of the Hamiltonian algebra for XX is polinomial with N
  Kökcü et al (2021)

ABC for the XXZ model

The unitaries can be prepared with modified F sim - gates

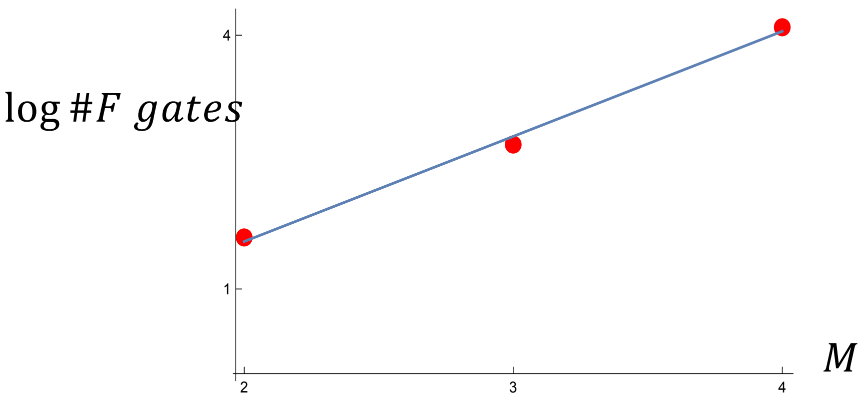
$M = 2$





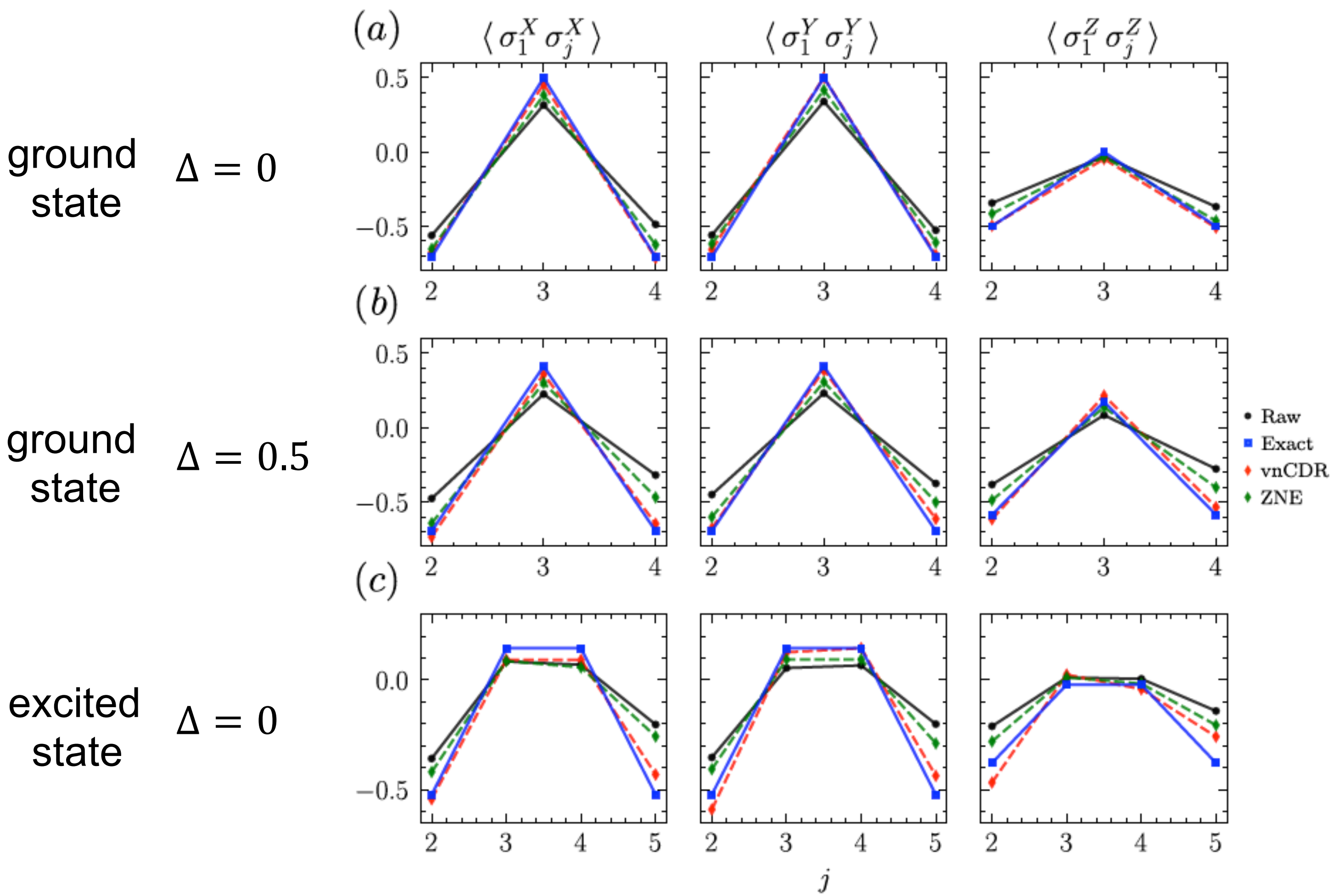
Compilation error

$$\epsilon = 1 - \left|\langle 0| \operatorname{Tr}(U^\dagger P_M)|0\rangle\right|^2 / 4^M$$



Exponential growth of F's gates with number of magnons

Two magnon states for N = 4 on IBM_Montreal and N=5 on IBM_Mumbai



Symmetry Resolved Entanglement

and

Equipartition of Entropy

$$H = \sum_n h_{n,n+1}$$

U(1) symmetry $\quad S^z = \sum_n s_n^z \qquad [H, S^z] = 0$

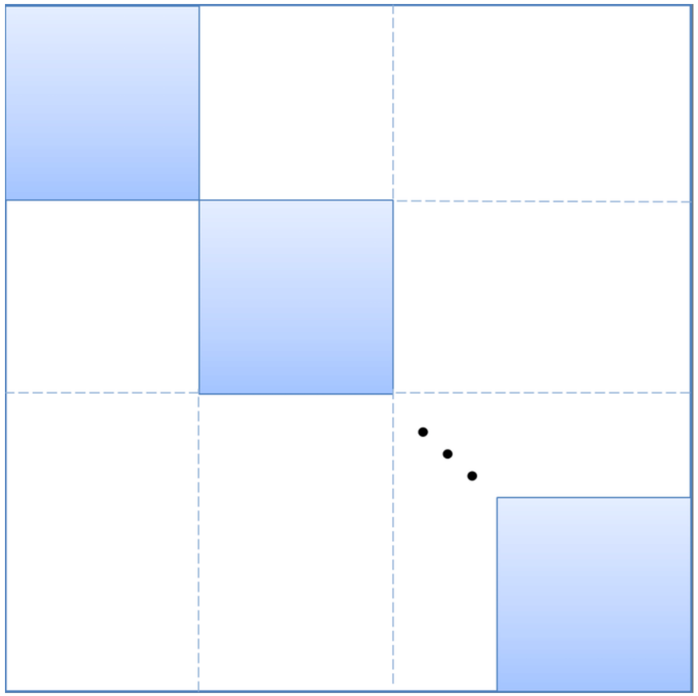
$$H\,|\psi\rangle = E\,|\psi\rangle \qquad S^z\,|\psi\rangle = s\,|\psi\rangle$$

$$\rho_A = tr_B\,|\psi\rangle\langle\psi|$$

$$S^z = S_A^z + S_B^z$$

$$[S_A^z, \rho_A] = 0$$

$\rho_A =$

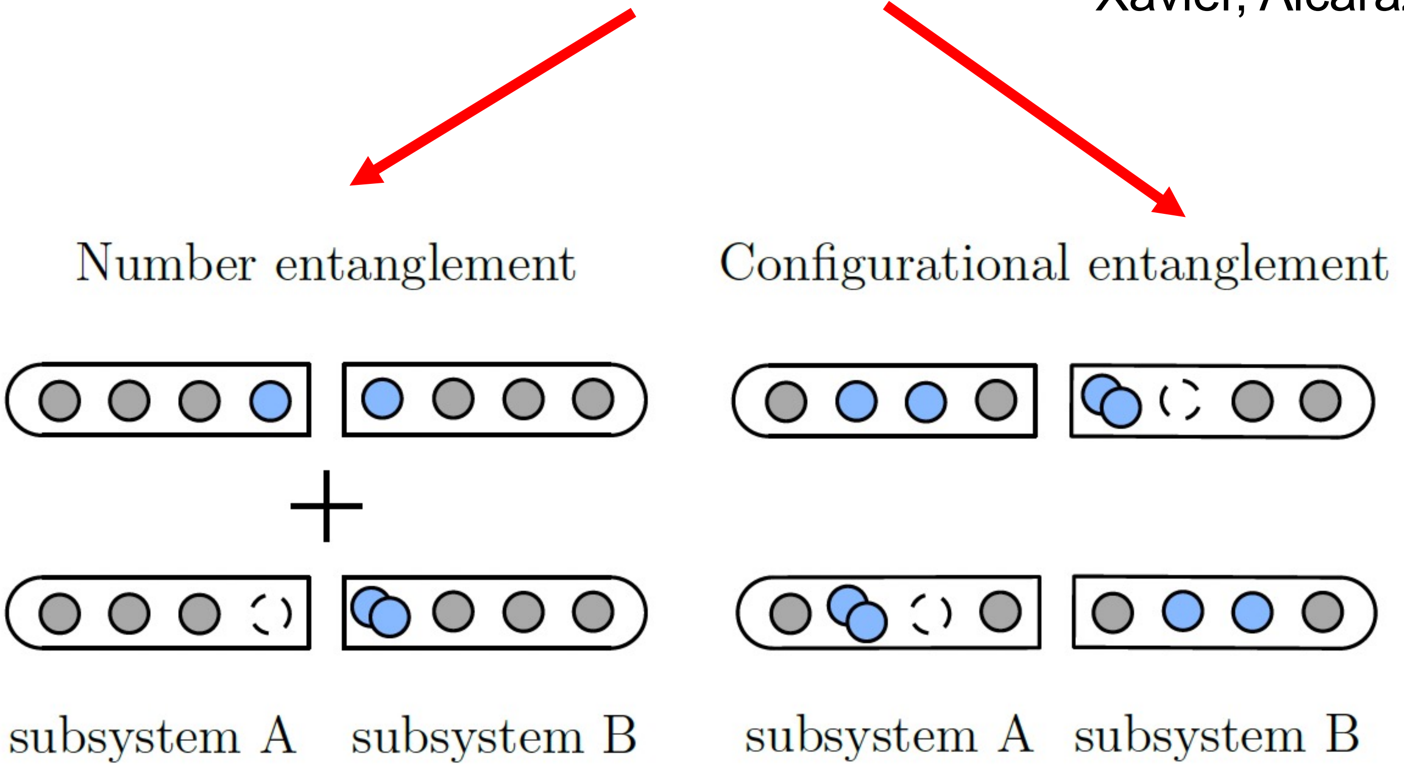


$$\rho_A = \bigoplus_m p_{A,m}\, \rho_{A,m}$$

$p_{A,m}$ Probability of obtaining m when measuring S_A^z in A

$$S_A = S_A^n + S_A^c$$

Goldstein, Sela 2017
Xavier, Alcaraz, Sierra 2018

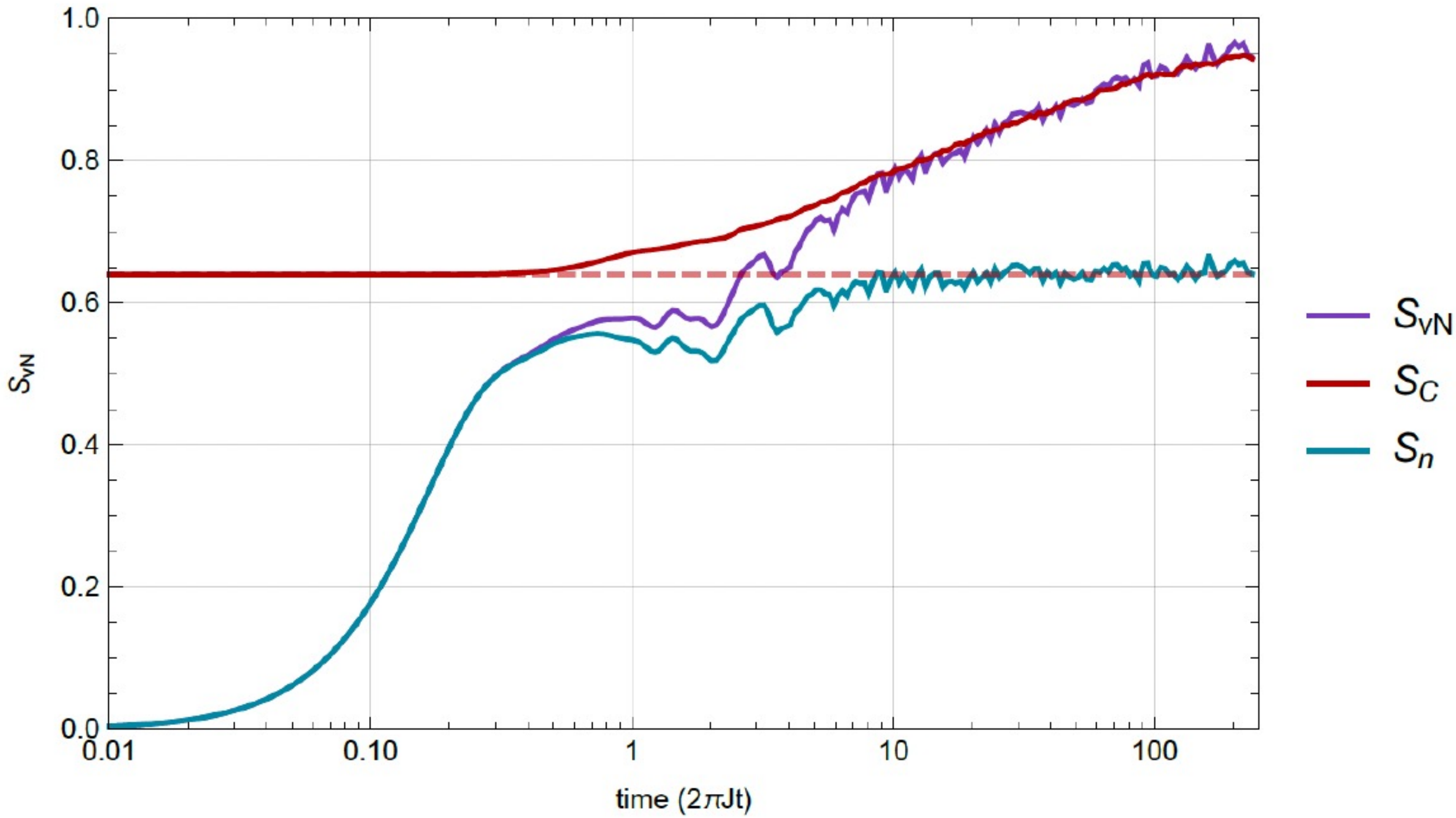


$$S_A^n = -\sum_m p_{A,m} \log p_{A,m}$$

Lukin, et al 2018

$$S_A^c = \sum_m p_{A,m} \, S_{A,m}$$

Evolution of entanglement entropy in a random system with strong disorder (Lukin et al 2018)



Equipartition of entanglement entropy

Xavier, Alcaraz, Sierra 2018

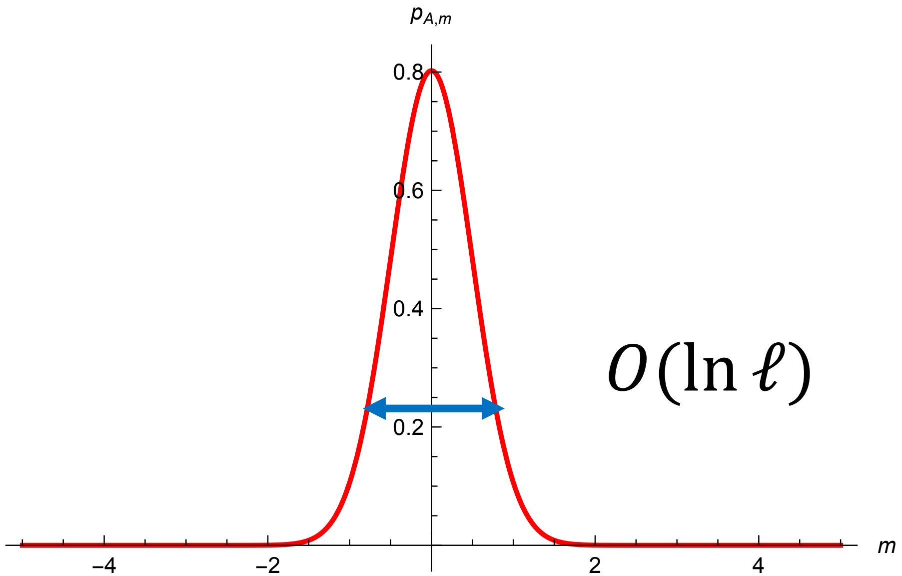
$S_{A,m}$: approximately independent on m

$$S_{A,m} = S_A - S_A^n$$

Example : XXZ model

$$H = \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta\, \sigma_i^z \sigma_{i+1}^z$$

$p_{A,m}$



$O(\ln \ell)$

$$S_{A,m} = \frac{1}{3}\ln \ell - \frac{1}{2}\ln(K \ln \ell) + \cdots$$

Luttinger parameter

→ Equipartition of entanglement

Rainbow spin chain

Javier Rodríguez Laguna
Giovanni Ramírez
Silvia Santalla
Nadir Samos Saéz de Buruaga
Pasquale Calabrese
Eric Tonni
Jêrome Dubail
Vincenzo Alba
Paola Ruggiero
Sudipto Singha Roy
Begoña Mula
Lucy Byles
Jiannis Pachos

Symmetry Resolved Entanglement

Jose Xavier
Francisco Alcaraz

Algebraic Bethe Circuits

Esperanza López
Alejandro Sopena
Max Hunter Gordon
Diego García Martín
Roberto Ruíz

Infinite MPS – Haldane Shastry

Ignacio Cirac
Anne Nielsen
Hong-Hao Tu
Benedikt Herweth
Ivan Glasser
Sourav Manna
Julia Wildeboer
Antoine Tilloy
Albert Gassull

Thanks for your attention