General Axion Electrodynamics

Start from general Lagrangian for EM coupled to ψ w/ axion

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e j^\mu A_\mu$$
$$-\frac{1}{2}(\partial_\mu\partial^\mu + m_a^2)a - \frac{g_{a\gamma}}{4} a F_{\mu\nu}\tilde{F}^{\mu\nu}$$

where $j^\mu = \bar{\psi}\gamma^\mu\psi \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$

axion EoM $\qquad (\Box + m_a^2)a = -\frac{g}{4}F\tilde{F} = g(E\cdot B)$

Maxwells Eqns (No axion, no sources)

$$\frac{\partial\mathcal{L}}{\partial A^\nu} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) = 0$$

$$\partial_\mu F^{\mu\nu} = 0 \rightsquigarrow \begin{matrix} \nabla\cdot E = 0 \\ \partial_t E = \nabla\times B \end{matrix}$$

(Bianchi Identity) $\quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$

or equiv. $\quad \partial_\gamma \tilde{F}^{\gamma\mu} = 0$

$\hookrightarrow \quad \nabla\cdot B = 0 \qquad \partial_t B = -\nabla\times E$

add sources $\qquad \partial_\mu F^{\mu\nu} = e j^\nu \qquad$ + EoM for ψ

add axions:

$$\mathcal{L}_{a\gamma\gamma} = -\frac{g_{a\gamma}}{4} a F\tilde{F}$$

$$\frac{\partial \mathcal{L}_{a\gamma\gamma}}{\partial(\partial_\mu A^\nu)} = -\frac{g_{a\gamma}}{4} a\, \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = -\frac{g_{a\gamma}}{2} a \tilde{F}$$

$$-\partial_\mu \left(\;\cdot\; \right) = g_{a\gamma} (\partial_\mu a) \tilde{F}^{\mu\nu} + \cancel{g_{a\gamma}\, a\, \partial_\mu \tilde{F}^{\mu\nu}} \quad \leftarrow 0$$

$\hookrightarrow$ Bianchi Identity

$$\partial_\mu F^{\mu\nu} + g_{a\gamma} (\partial_\mu a) \tilde{F}^{\mu\nu} = J^\nu$$

$$\nabla \cdot \vec{E} = \rho - g_{a\gamma} \nabla a \cdot \vec{B}$$

$$\nabla \times \vec{B} - \partial_t \vec{E} = \vec{j} + g_{a\gamma} \left(\partial_t a \cdot \vec{B} - \vec{E} \times \nabla a \right)$$

$$\rho_{\text{eff},a} = -g_{a\gamma} (\nabla a) \cdot \vec{B}$$

$$j_{\text{eff},a} = g_{a\gamma} \left(\partial_t a \cdot \vec{B} - \vec{E} \times \nabla a \right)$$

in turn induce EM fields

Lets get a bit of intuition on how axion EM works, which will prove useful in understanding how to search in astrophysics

$$\vec{B} = \vec{B}_0(g^0) + \vec{B}_1(g^1) + \cdots$$
$$\vec{E} = \vec{E}_0(g^0) + E_1(g^1) + \cdots$$

$$E_0 = 0 \qquad B_0(\vec{x}, t) = B(\vec{x}) = B_0 \hat{z}$$

- focus on leading order contribution and assume $E_1 \sim B_1 \sim e^{-i\omega t}$

- ∂_t(Amperes Law) $\quad \partial_t(\nabla \times B) - \partial_t^2 E = \partial_t j + \partial_t(\ldots)$
- use $\nabla \times E + \partial_t B = 0$ to eliminate B
- Drop $g \dot{B} \dot{a}$ and $g \partial_t (E \times \nabla a)$
- one gets wave Eqn

$$\partial_t(\nabla \times B) - \partial_t^2 E = \partial_t \vec{j} + g(\partial_t B \partial_t a + B \partial_t^2 a - \partial_t(E \times \nabla a))$$

$\partial_t(\nabla \times B) \to -\nabla \times \nabla \times E = -\nabla^2 E + \nabla(\nabla \cdot E)$

$\partial_t B \to \partial_t(B_0 + B_1) = \partial_t B_1 \sim \mathcal{O}(g)$, $\quad \partial_t(E \times \nabla a) \sim \mathcal{O}(g)$

$$\vec{j} = \overset{\leftrightarrow}{\sigma} \cdot \vec{E}$$

$$\partial_t j = \omega^2 (\mathbb{I} - \overset{\leftrightarrow}{\epsilon}) \cdot E$$

In linear response theory

$$\vec{j} = \overset{\leftrightarrow}{\sigma} \cdot \vec{E} = i\omega(\mathbb{I} - \overset{\leftrightarrow}{\epsilon}) \cdot \vec{E}$$

ϵ: extract dielectric tensor

Break down
- super strong E
- ultra fast times
- quantum regime . . .

$$-\nabla^2 \vec{E} + \nabla(\nabla\cdot\vec{E}) - \omega^2 \overleftrightarrow{\epsilon}\,\vec{E} = \omega^2 g_{\alpha\beta} B_0\, a$$

"Axion wave Eqn"

study wave Eqn in various limits

* Go to look at derivation of $\overleftrightarrow{\epsilon}$

lets describe bkg sources as collection of collisionless particles (take e^- for simplicity)

$$\rho = -e\, n_e \qquad \vec{J} = \rho\vec{v}$$

Eom for charges (non-rel. limit)

$$m_e \partial_t \vec{v} = -e(\vec{E} + \vec{v}\times\vec{B})$$

$$v_\parallel = \frac{-ie}{m_e\omega} E_\parallel \qquad v_{\perp,\pm} = \frac{-ie}{m_e\left(\omega \pm \frac{eB_0}{m_e}\right)} E_\pm$$

$$v_{\perp,\pm} = v_{\perp,1} \pm i v_{\perp,2}$$

$$\omega_p = \frac{e^2 n_e}{m_e}$$

$$j_\parallel = -e n_e v_\parallel = \frac{i e^2 n_e}{m_e\omega} E_\parallel = \frac{i\omega_p^2}{\omega} E_\parallel$$

$$j_{\perp,\pm} = \frac{i\omega_p^2}{(\omega \pm eB_0/m)} E_\pm$$

vacuum $j = 0$, $\overleftrightarrow{\epsilon} = 1$

$B\to 0$: $j = \frac{i\omega_p^2}{\omega} E$, $\overleftrightarrow{\epsilon} = (1 - \omega_p^2/\omega^2)$

Limit #1: Free Space $\frac{\epsilon_s}{\epsilon} = \mathbb{1}$

Take $B_0 = B_0 \hat{x}$, plane wave $a = a_0 e^{iky}$ $E(y)$

$$(\partial_y^2 + \omega^2) E_x = -\omega^2 g_{a\gamma\gamma} B_0 a$$

$$E_y = 0$$

$$(\partial_y^2 + \omega^2) E_z = 0$$

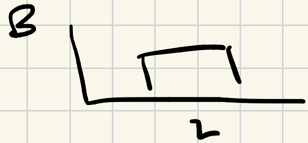
$\rightsquigarrow$ normal dispersion relation $\omega = k$

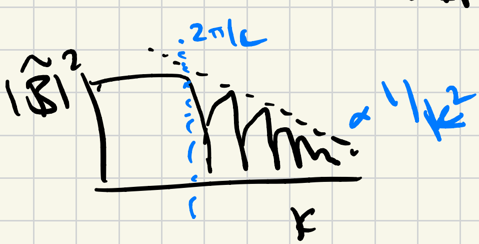
Solve via Green's functions

$$E_x = -\frac{i\omega g_{a\gamma}}{2} \int dy'\, B_0(y')\, a(y')\, e^{i\omega|y-y'|}$$

$$P_{a\to\gamma} = \frac{\langle S_y \rangle}{\langle T^{20} \rangle_a} = \frac{\frac{1}{2}|E_x|^2 (\omega/k)}{\frac{1}{2}\omega k_a a_0^2} \qquad E_x = -i\omega A_x$$

$$= \frac{\omega g_{a\gamma}^2}{4k} \Big| \underbrace{\int dy'\, e^{i\omega|y-y'|} e^{iky'} B_0(y')}_{|FT(B)|^2_{k = k_\gamma - k_a}} \Big|^2$$

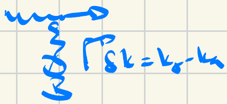




• could have guessed this result

$$P_{a\to\gamma}\,[\text{dimensionless}] \sim g^2 B^2 \times [\text{length}]^2$$

$\lesssim$ L_B or $(k_\gamma - k_a)^{-1}$ which is smaller



Limit 2: Unmagnetized Plasma

$\omega_c \ll \omega$

limit $\quad \omega_c \ll \omega, \omega_p$

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} 1-\omega_p^2/\omega^2 & & 0 \\ & 1-\omega_p^2/\omega^2 & \\ 0 & & 1-\omega_p^2/\omega^2 \end{pmatrix}$$

dispersion relation

$$\| n^2 \delta_{ij} - n_i n_j - \varepsilon_{ij} \| = 0$$

$\vec{n} = \vec{k}/\omega$

For $\vec{k} = k\hat{z}$ $\qquad \omega^2 = k^2 + \omega_p^2$ (transverse) $\qquad \omega_p = \omega$ (longitudinal)

$B = B_0(\cos\theta_B\,\hat{y} + \sin\theta_B\,\hat{x})$ $\quad a \sim a_0 e^{iky}$ $\quad E(x,y)$

$$\partial_y^2 E_x - \partial_x \partial_y E_y + (\omega^2 - \omega_p^2) E_x = -\omega^2 g_{a\gamma} B_0 \sin\theta_B\, a$$

$$\partial_x^2 E_y - \partial_y \partial_x E_x + (\omega^2 - \omega_p^2) E_y = -\omega^2 g_{a\gamma} B_0 \cos\theta_B\, a$$

No Greens Functions

Assume slow variation of B_0 (WKB limit)

axion $\quad$ y-Dir γ $\qquad E_x(y),\ E_y(y) \qquad \partial_x E_x, \partial_x E_y \simeq 0$

$$E_y = -\frac{\omega^2 g_{a\gamma} B_0 \cos\theta_B \, a}{\omega^2-\omega_p^2}$$

limiting cases

$$E_y \sim \begin{cases} -gBa & \omega_p \ll \omega \\ -\left(\frac{\omega}{\omega_p}\right)^2 gBa & \omega_p \gg \omega \end{cases}$$

(longitudinal excitation)

$$\left(\partial_y^2 + \omega^2-\omega_p^2\right)E_x = -\omega^2 g_{a\gamma} B_0 \sin\theta_B \, a$$

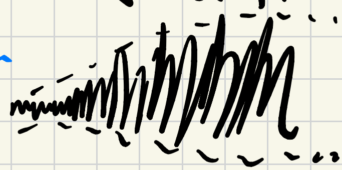
(axion couples to $B_\perp$)

(axion effects polarization!)

apply WKB

$E_x(y) = \tilde{E}_x(y)\, e^{iky}$ $\partial_y \tilde{E} \ll k\tilde{E}$

$2ik_a\partial_y \tilde{E} = -(m_a^2-\omega_p^2)\tilde{E} - \omega^2 g B \sin\theta \, a$



envelope + osc...

GF

$$E_x(y) = \frac{i\omega^2 g_{a\gamma} a_0}{2k}\int_{-\infty}^{y} dy' \, B_0(y')\sin\theta_B(y')\, e^{\underbrace{i\int_{-\infty}^{y'} dy''\left(\frac{m_a^2-\omega_p^2}{2k_a}\right)}_{i\phi}}$$

- $k_a \gg \omega_p, m_a$, $\phi \simeq 0$ $E_x \sim L_B$
- $k_a \ll m_a$, ϕ oscillating rapidly, resonant cont. at $m_a \simeq \omega_p$

↳ can analytically solve here using stationary phase

$$E_{x,\mathrm{res}} \sim \frac{i\,\omega^2 g_{a\gamma} a_0}{2k} B_0 \sin\theta_B \sqrt{\frac{2\pi}{\partial_y\left(\frac{m_a^2-\omega_p^2}{2k}\right)}}\;\Bigg|_{y=y_{res}}$$

$P \sim g^2 B^2 L^2$

$$L_{eff}^2 \sim \left|\frac{2\pi}{\partial_y\left(\frac{m_a^2-\omega_p^2}{2k}\right)}\right| \sim \left|\frac{1}{\partial_y \omega_p}\right|$$

$$\omega_p = \omega_{p,0}\left(\frac{r_0}{r}\right) \qquad L_{eff}^2 \sim \frac{r_{res}}{\omega_p}$$

Should be compared against L_B^2 and $\left(\frac{1}{k_\gamma - k_a}\right)^2$

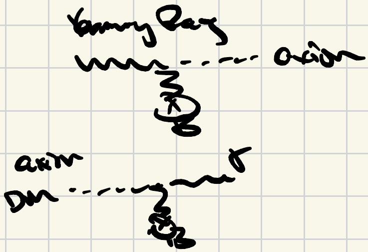
$$\sim \begin{cases} \frac{4\omega^2}{\omega_p^4} & \omega_p \gg m \\ \frac{4\omega^2}{m^4} & m \gg \omega_p \end{cases}$$

$$\frac{L_{res}^2}{L_{non\text{-}res}^2} \sim \frac{\frac{\omega}{\omega_p}\left(\frac{r_{res}}{\omega_p}\right)}{4\left(\omega^2/\omega_p^4\right)} \sim \frac{10^6}{3}\left(\frac{r_{res}}{100\text{ km}}\right)\left(\frac{\omega_p}{\mu eV}\right)$$

$r_{res} \sim O(\text{km})$ $\qquad \omega = \xi\,\omega_p \qquad \xi \gg 1$

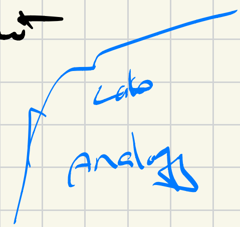
Non-Resonant wins for ultra-relativistic particles, low mass/low ω_p,

Large enhancement at resonant for non-relativistic axions



Non-Resonant

Resonant



Dark Photon E+M

$$\mathcal{L} = -\frac{1}{4} F^1_{\mu\nu} F^{1\mu\nu} - \frac{1}{4} F^2_{\mu\nu} F^{2\mu\nu} + J^\mu A^1_\mu + \cancel{J^{2\mu} A^2_\mu}$$

$$+ \frac{m_\gamma^2}{2} A^2_\mu A^{2\mu} - \frac{\chi}{2} F^1_{\mu\nu} F^{2\mu\nu}$$

[always work in $\chi \ll 1$ limit]

- one massless
- $J^{2\mu} = 0$

$$\begin{pmatrix} A^1_\mu \\ A^2_\mu \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\chi^2}} & 0 \\ \frac{-\chi}{\sqrt{1-\chi^2}} & 1 \end{pmatrix} \begin{pmatrix} B^1_\mu \\ B^2_\mu \end{pmatrix}$$

$$A^1_\mu \simeq B^1_\mu$$

$$A^2_\mu \simeq B^2_\mu - \chi B^1_\mu$$

$$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \ldots$$

$$\mathcal{L} = -\frac{1}{4} G^1_{\mu\nu} G^{1\mu\nu} - \frac{1}{4} \left[\partial_\mu (B^2_\nu - \chi B^1_\nu) - \partial_\nu (B^2_\mu - \chi B^1_\mu)\right]$$

$$[\quad] + \cancel{J^\mu (B^2_\mu + \chi B^1_\mu)} + \frac{m_\gamma^2}{2} \left[B^2_\mu - \chi B^1_\mu\right]\left[B^{2\mu} - \chi B^{1\mu}\right]$$

$$+ J^\mu B^1_\mu \qquad - \frac{\chi}{2} G^1_{\mu\nu} \left[\partial^\mu (B^{2\nu} - \cancel{\chi B^{1\nu}}) - \partial^\nu (B^{2\mu} - \cancel{\chi B^{1\mu}})\right]$$

$$= -\frac{1}{4} G^1_{\mu\nu} G^{1\mu\nu} + J^\mu B^1_\mu - \frac{1}{4} G^2_{\mu\nu} G^{2\mu\nu} + \frac{m_\gamma^2}{2} B^2_\mu B^{2\mu}$$

$$+ J^\mu B^1_\mu \qquad - \chi m_\gamma^2 B^2_\mu B^{1\mu} + O(\chi^2)$$

mass mixing

$m_\gamma \to 0$ decoupling!

Diagonalize mass matrix

$$B^1_\mu = C^1_\mu - \chi C^2_\mu$$
$$B^2_\mu = C^2_\mu + \chi C^1_\mu$$

Mass terms:

$$\frac{m_{\gamma'}^2}{2} B^2_\mu B^{2\mu} - m_{\gamma'}^2 \chi B^1_\mu B^{2\mu}$$

$$= \frac{m_{\gamma'}^2}{2} C^2_\mu C^{2\mu} + \cancel{m_{\gamma'}^2 \chi C^1_\mu C^{2\mu}} - m_{\gamma'}^2 \chi \left[\cancel{C^1_\mu} C^{\mu 2} \right]$$

massive + massless

Current

$$J^\mu B^1_\mu = J^{1\mu}\left[C^1_\mu - \chi C^2_\mu\right]$$

massive field couples to current at χ

$$\Box A^\mu = J^\mu$$

Landau Gauge

$$\partial_\mu A^\mu = 0 = \partial_\mu A'^\mu$$

$$\Box A'^\mu + m_{\gamma'}^2 A'^\mu = \chi J^\mu$$

Assume e^- plasma w/ T come

$$\frac{d}{dt} \quad \frac{\partial \vec{p}_e}{\partial t} + (\vec{v}_e \cdot \nabla)\vec{p}_e = -e\left(\vec{E} + \chi \vec{E}' + \vec{v}_e \times (\vec{B} + \chi \vec{B}')\right) - \nu(\vec{p}_e - \vec{p}_i)$$

ignore

assume $\vec{p}_i \simeq 0$, $|v_e|$ small

$$\vec{v}_e = \frac{-e}{m_e(\nu - i\omega)}\left[\vec{E} + \chi \vec{E}'\right]$$

for simplicity $\nu = 0$

$$\vec{J} = e n \vec{v}_e$$

Can also define $\Pi^{\mu\nu} A_\nu = eJ^\mu$

FT

$$A(x,t) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, d\omega\, A(k,\omega)\, e^{ix\cdot k - i\omega t}$$

Gauge choice $E_i = i\omega A_i$

L/T Decomp

$$A_i(\vec{k},\omega) = A_L \frac{k_i}{|k|} + A_{T_i} \qquad A_{T_i} k_i = 0$$

$$\left[-\omega^2 + k^2 + \begin{pmatrix} \omega_p^2 & \chi_0 \omega_p^2 \\ \chi_0 \omega_p^2 & m_{\gamma'}^2 \end{pmatrix}\right]\begin{pmatrix} A_T \\ A_T' \end{pmatrix} = 0$$

$$\left[-\omega^2 + \begin{pmatrix} \omega_p^2 & \chi_0 \omega_p^2 \\ \chi_0 \omega_p^2 & \frac{m_{\gamma'}^2}{(1 - k^2/\omega^2)} \end{pmatrix}\right]\begin{pmatrix} A_L \\ A_L' \end{pmatrix} = 0$$

Presence of medium changes
what your propagating modes are...
(Define γ, γ' by dispersion relations!)

We want to know what couples to EM

in limit $\omega_p \ll M_{\gamma'}$

$$A_{obs} = A_{photon} + \chi A_{Dark\,P}$$

in limit $M_{\gamma'} \ll \omega_p$

$$A_{obs} = A_{photon} + \chi\left(\frac{M_{\gamma'}}{\omega_p}\right)^2 A_{dark}$$

$$\chi_{eff} \approx \chi_0 \left(\frac{M_{\gamma'}}{\omega_p}\right)^2$$

Resonant @ $M_{\gamma'} \approx \omega_p$

Note that for axion, suppression
goes as $\sim\left(\frac{\omega}{\omega_p}\right)^2$ <u>not</u> $\left(\frac{M_a}{\omega_p}\right)^2$

and is relevant for <u>$\omega \ll \omega_p$</u>

WIMP Nucleon Scattering

Take $\mathcal{L} \supset \bar{\chi}\chi \bar{q}q$

Solution to Dirac Eqn $\bar{\Psi}(\not{\partial} - m)\Psi = 0$

plane wave soln $\psi(x) = u(p) e^{-ip\cdot x}$

$$(\gamma^\mu p_\mu - m) u(p) = 0$$

Chiral rep $\quad u(p) = \begin{pmatrix} \sqrt{\sigma^\mu p_\mu}\, \xi \\ \sqrt{\bar{\sigma}^\mu p_\mu}\, \xi \end{pmatrix}$

$\sigma^\mu = (I_2, \sigma^i) \quad \bar{\sigma}^\mu = (I_2, -\sigma^i) \quad \xi = \mathbb{C}^2$ vector

NR limit $\quad u(p) = \frac{1}{\sqrt{4m}} \begin{pmatrix} (2m - \vec{p}\cdot\vec{\sigma})\xi \\ (2m + \vec{p}\cdot\vec{\sigma})\xi \end{pmatrix} + O(p^2)$

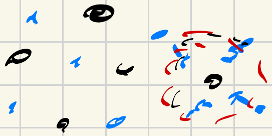
for bilinear $\bar{\Psi}\Gamma\Psi \quad \Gamma = I, \gamma^5\gamma^\mu, \ldots$

$$\bar{u}(p')u(p) = 2m\, \xi'^\dagger \xi$$

$$\bar{u}(p')\gamma^\mu\gamma^5 u(p) = \begin{pmatrix} \xi'^\dagger [\vec{P}\cdot\vec{\sigma}]\xi \\ 2m\, \xi'^\dagger \sigma \xi \end{pmatrix} \qquad P = p + p'$$

$$\sum_{s=1,2} \xi_s^\dagger \xi_s = 2 \qquad \hat{S}_x = \xi'^\dagger \frac{\sigma}{2} \xi$$

Comment on Debye Screening Scale



≈ neutralized by net neutral plasma

Electrostatic potential ψ of species i

$$\nabla \cdot D = \rho_f$$

$$\nabla \cdot (\epsilon E) = \rho_f \qquad \nabla \cdot E = \frac{\rho}{\epsilon}$$

$$E = -\nabla \psi$$

$$\nabla^2 \psi = -\rho/\epsilon = -\frac{1}{\epsilon} \sum_i z_i e n_i$$

thermal Boltzmann dist.

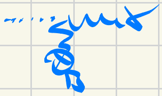
$$n_i \propto \exp\left(-\frac{z_i e \psi}{T}\right)$$

Taylor expand "Exp", charge neutrality implies

$$\sum n_i z_i = 0$$

$$\nabla^2 \psi_i = k^2 \psi_i \qquad k^2 = \frac{e^2}{\epsilon T} \sum n_i z_i^2$$

Overview lecture 1



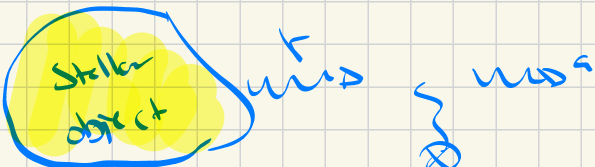
Vacuum (B⊥ only)

$$P_{a\to\gamma} \sim g^2 |FT(B)|^2 \qquad k = k_\gamma - k_a \approx \frac{m^2}{2\omega}$$

$= P_{\gamma \to a}$

[peaks when $\delta k \to 0$]

lim: ω large

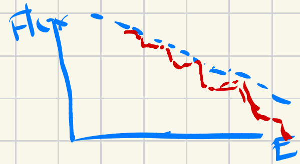


(or AGN)

stellar $\vec{B}$ fields,

cluster $\vec{B}$ fields,

B_{IGM}, EGM



- spectral oscillations,
- absorption
- polarization
- Re-appearance of photons
- Axion production in core

Compute $P_{a\to\gamma}$ in Plasma w/ $B_\parallel + B_\perp$

modify dispersion $\quad k_\gamma \sim \sqrt{\omega^2 - \omega_p^2}$

(e.g.)

resonance $\quad k_\mu^\gamma \simeq k_\mu^a \qquad \omega_p \simeq m_a$

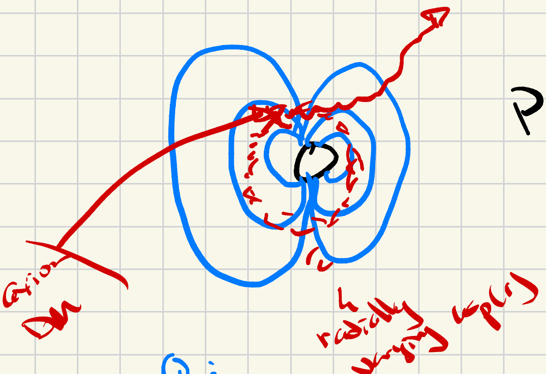
Resonant transitions dominate when axions are at ultra-relativistic (DM + $r_c < 10^5$)

$$E_x \propto \int dy\, \underline{\underline{B_\perp}}\, e^{i\phi(y)}$$

$$\phi(y) \to \begin{cases} (k_\gamma - k_a)\, y' \\ \int \frac{m^2 - \omega_p^2}{k}\, dy' \end{cases}$$

Neutron Stars

Radio Lines from Axion DM



$P \sim \rho_{DM}(r_c)\, r_c^2\, P_{a \to \gamma} \cdot v_c$ ~ $\mathcal{O}(0.5)$

$\times$ [Geometric factors] $\times$ [GR factors]
~ $\mathcal{O}(1)$ ~ $\mathcal{O}(1)$

Q:

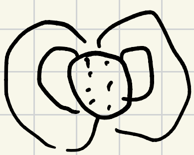
1.) where is resonance

2.) what is local phase space

3.) How efficient emission?

- Plasma dist. in magnetosphere

  - assume dipole aligned $\vec{B}$ in vacuum



  - NS is conductor, so inside + in co-rotating frame $\vec{E}_{c,in} = 0$

  - in pulsar frame $\vec{E}_{p,in} = -(\vec{\Omega} \times \vec{r}) \times \vec{B}$

  - continuity of $E_\perp$ @ boundary $E_{0,\theta}|_{r=R_*} = -\Omega R_* B_r$

$E_\phi|_{r=R_*} = 0$

implies @ surface $E \sim \Omega R_* B \sim 10^{-4} B$

$$M_i \gamma_i^2 + q\cancel{\Phi_i}^{\,0} = M_f \gamma_f + q \Phi_f$$

$$E = -\nabla \Phi \qquad \Phi_f \sim -E \cdot d$$

$$\gamma_f \sim \frac{qE \cdot d}{M_f} - 1 \simeq \frac{e\,\Omega R_* B \cdot D}{M_e}$$

$B \sim 10^{12}\,G$, $\Omega R_* \sim 10^{-4}$ $D \sim 1\,km$

$\gamma_f \sim 10^{10}$ $\quad \omega_\gamma \sim \frac{\gamma^3}{\rho_c}$

look for static configuration in which plasma sees $\vec{E}$ field

can also study Gauss' law in corotating frame

$\nabla \cdot E_c = \rho - \rho_{GJ}$

$$\vec{E} \cdot \vec{B} = 0$$

$$\vec{E} = -\Omega r \sin\theta\, \hat{\phi} \times \vec{B}_p$$

$$\rho = \nabla \cdot \vec{E} = \frac{-2\Omega_0 B}{[1 - (\Omega r)^2 \sin^2\theta]} = \rho_{GJ}$$

stable co-rotating charge density

• No acceleration of charge (

Goldreich-Julian law $J \cdot E = 0$ gives

$R_{*x} = \frac{2\mu}{R_x^3}$

$$\vec{E} \approx \frac{(\Omega R_x)}{2} B_0 \left(\frac{r_x}{r}\right)^4 \left[(1-3\cos^2\theta)\hat{r} - \sin(2\theta)\hat{\theta} \right]$$

$\mu = \frac{R_x^3 B_0}{2}$

why $E \neq 0$ can't be large

$\gamma_i m_e c^2 + q\Phi_i = q\Phi_f + \gamma_f m_e c^2$

$\vec{E} = -\nabla\Phi$

$\Delta\gamma \sim \frac{\Delta\Phi_f}{q m_e} \sim \frac{\Omega R_x B_0 h}{q m_e}$

$h \sim 1\,\mathrm{km}$, $\Omega R_x \sim 10^{-4}$

$\gamma_f \sim 10^{10}$

$E \cdot B \neq 0$, very large $\rightarrow$ γ e^- e^+ inevitable

"vacuum gap" not interesting

look for steady state w/ plasma

$E \cdot B = 0$

$$\vec{E} = \vec{E}_{cor} = -(\vec{\Omega} \times \vec{r}) \times \vec{B}$$

$$\vec{\nabla} \cdot \vec{E} = \rho \Rightarrow \rho \simeq \rho_{GJ} \sim -2\Omega \cdot B$$

$$\omega_p^2 = \sum \frac{q_i^2 n_i}{m_i} \qquad \rho = \sum_i n_i q_i$$

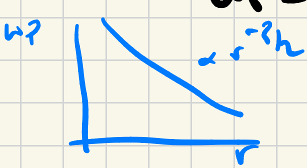
assume charge separated

then ω_p uniquely defined

$k_\mu^\gamma = k_\mu^a$ (strongly magnetized plasma)

$\omega_p \simeq m_a$

$$\omega_p \simeq 22\,\mu\mathrm{eV} \,|3\cos^2\theta - 1|^{1/2} \sqrt{\frac{B}{10^{13}\,G} \cdot \frac{1\,s}{P} \left(\frac{r_{NS}}{r}\right)^3}$$



$\Rightarrow$ good for probing $\mathcal{O}(10\,\mu\mathrm{eV} \rightarrow 0.1\,\mu\mathrm{eV})$ axions

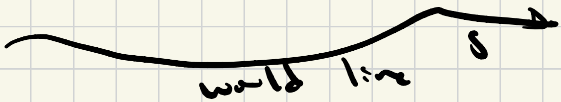
• Local phase space

asymptotically, $\lim_{r\to\infty} f_a(\vec{x},\vec{k}) = \frac{n_{a,\infty}}{(\pi k_0^2)^{3/2}} e^{-\vec{k}^2/k_0^2}$

$k_0 = m_a v_0 \qquad v_0 \sim 200 \text{ km/s}$

Liouville's theorem:

$$f_a(\vec{x}(s),\vec{k}(s)) = f(\vec{x}_\infty, \vec{k}_\infty)$$



$$n(\vec{x}) = \int d^3k\, f_a = \int d^3k\, \frac{n_{a,\infty}}{(\pi k_0^2)^{3/2}} e^{-|\vec{k}_\infty|^2/k_0^2}$$

Energy cons.

$$\left[|\vec{k}|^2 - \frac{2GM m_a^2}{r}\right] = |\vec{k}_\infty|^2$$

$$\frac{n(r)}{n_\infty} \sim \frac{v(r)}{v_0} \sim \frac{0.5}{10^{-3}} \sim 500$$

* asymp. homo + isotropy ⇒ locally as well

∘ Probability

$$P_{a\to\gamma} \sim g^2 B^2 L_{eff}^2$$

$$L_{eff}^2 \sim \frac{\pi}{2\times k_\gamma} \sim \frac{\pi}{\sqrt{2\times \omega_p'}} \quad \omega_p \sim \omega_{p,0}\left(\frac{r_{NS}}{r}\right)^{-3/2}$$

$$\sim \frac{r_c}{v\,\omega_p(r_c)} \sim \frac{10\,km}{10\,\mu eV}$$

$$L_{eff} \sim \mathcal{O}(10\,m)$$

$$P_{a\to\gamma} \sim \left(\frac{10^{-11}}{GeV}\cdot 10^{13}\,G\cdot 10\,m\right)^2 \sim 10^{-2}$$

Not just B field

$$L_{non\text{-}res} \sim \frac{\omega}{m^2} \sim \frac{\gamma}{m} \sim \frac{1}{m} \sim 10^{-2}\,m$$

$$\frac{(P_{a\to\gamma})_{res}}{(P_{a\to\gamma})_{non\text{-}res}} \sim \left(\frac{r_c}{v\cdot m}\cdot m\right)^2 \sim \underline{10^7 - 10^{12}}$$

∘ Power

$$P \sim \rho_{DM}\cdot r_c^2 \cdot P_{a\to\gamma} \sim \left(10^2\cdot 10^4\,\frac{GeV}{cm^3}\right)(10\,km)^2 \cdot (10^{-2})$$

$$S_D \sim \frac{P}{4\pi d^2 \Delta f} \sim \mathcal{O}(10^6\,Jy) \sim$$

$S_{\mathcal{O}(10^{-5})}$

12 orders mag. above threshold

Static Axion Configurations

Einstein Eqns

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$\mathcal{L} \supset \bar{\Psi}_N (i\not\partial - m_N) \Psi_N + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi$
$- \frac{m^2}{2}\phi^2 - m_N \frac{\phi}{f_a} \bar{\Psi}\Psi$

$V_{eff,\phi} \sim \frac{1}{2} m^2 \phi^2 + m_N \frac{\langle \bar{\Psi}\Psi \rangle}{f_a} \phi$

$M_{N,eff} \sim M_N (1 + \phi/f_a)$

$$\mathcal{L}_{\psi\phi} = \sqrt{-g} \Big[\bar{\psi}(i\not D - m_N(\phi))\psi + \frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} m^2 \phi^2 \Big]$$

General Schwarzschild metric

$$g_{00} = e^{2\nu(r)} \qquad g_{\theta\theta} = -r^2$$

$$g_{rr} = -\frac{1}{\left[1 - \frac{2GM(r)}{r}\right]} \qquad g_{\phi\phi} = -r^2 \sin^2\theta$$

$$m_N(\phi) = m_N \left[1 + \frac{\phi}{f}\right]$$

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$

$$T^\mu{}_\nu = (T_{ideal})^\mu{}_\nu + (T_{grad})^\mu{}_\nu$$

$$\sum \begin{pmatrix} \rho & & & 0 \\ & -p & & \\ & & -p & \\ 0 & & & -p \end{pmatrix} \qquad \begin{aligned} \rho &= \rho(n_N, m_N(\phi)) + \frac{1}{2} m^2 \phi^2 \\ p &= p(n_N, M_N(\phi)) - \frac{1}{2} m^2 \phi^2 \end{aligned}$$

$$(T_{grad})^\mu{}_\nu \simeq (\partial_r \phi)^2 \left[\frac{1}{2}\delta^\mu_\nu - \delta^\mu_r \delta^r_\nu\right] \left(1 - \frac{2GM}{r}\right)$$

G_{tt} term

$$\frac{dM}{dr} = 4\pi r^2 \rho + \frac{1}{2}(\partial_r \phi)^2 \left[1 - \frac{2GM}{r}\right]$$

higher order

G_{rr} term

$$\frac{d\nu}{dr} = \frac{G(M + 4\pi r^3 p)}{r(r-2GM)}$$

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\frac{dp}{dr} = -(\rho+p)\frac{G(M+4\pi r^3 p)}{r(r-2GM)} - (\partial_r\phi)\left[m_\phi^2 \phi + \frac{\rho_N}{f_a}\right]$$

higher order

$$\phi''\left(1 - \frac{2GM}{r}\right) + \frac{2}{r}\phi'\left(1 - \frac{GM}{r} - 2\pi G r^2(\rho - p)\right) = m_\phi^2 \phi + \frac{\rho_N}{f_a}$$

focus on limit $m_\phi \gg 1/R_{NS}$

@ center of star

$$\phi'' \sim \frac{1}{R_{NS}^2}\phi \ll m_\phi^2 \phi$$

$$\phi_c \sim -\frac{\rho_N}{f_a}\frac{1}{m_\phi^2} \qquad \theta = \frac{\phi}{f_a} \sim -\frac{\rho_N}{f_a^2 m_a^2}$$

could have gotten this from $(\Box - m^2)\phi = \frac{\rho_N}{f}$

for $\Phi_c \sim \frac{\rho_n}{f_n m_\phi^2}$ what is effective m_n

$$m_n \sim m_{n,0}\left[1 + \frac{\phi}{f_n}\right]$$

$$1 \; ? \; \frac{\rho_n}{f_n^2 m_\phi^2}$$

keeping in mind $m_\phi > \frac{1}{R}$

approx $m_\phi \sim 10^{-9}$ eV , $\rho_0 \sim \frac{m_n}{f_m^3} \cdot 0.1 \sim 10^{33}\text{eV}^4$

$O(1)$ effect with

$$f_n \sim \sqrt{\frac{\rho_n}{m_\phi^2}} \sim 10^{16}\text{GeV}$$

low mass regime $(\Box - m^2)\phi = \rho/f_n$

$$\Box\phi \sim \frac{\Phi_c}{R^2} \sim \rho/f_n \qquad \Phi_c \sim \frac{\rho_n}{f_n} R_*^2$$

$$\sim \frac{M_*}{R_*^{\cancel{3}}} \cdot \frac{\cancel{R_*^2}}{f_n}$$

$$\Phi_c \sim \frac{M_*}{R_* f_n}$$

$m_\phi \ll 10^{-10}$ eV

$$f_n \sim \sqrt{\frac{M_*}{R_*}} \sim 10^{18}\text{GeV}$$

$$\frac{\Delta M_n}{M_n} \sim O(1) \text{ for } f_n \lesssim 10^{18}\text{GeV}$$

Plug in $f_n \sim 10^{12}$ GeV

$\Phi_c \sim 10^{31}$ eV $\gg f_n$

Alternative Sketch of static config

$$E \sim \int d^3x \left[\tfrac{1}{2}\dot{a}^2 + \tfrac{1}{2}|\nabla a|^2 + V_{eff}(a) \right]$$

$$d^3x \sim R^3$$

$$\nabla a \sim \frac{\delta a}{R} \sim \frac{a}{R} \qquad V_{eff}(a) = \frac{m_a^2}{2}a^2 + \lambda a \bar{N} N$$

$$n_n = \langle \bar{N} N \rangle$$

$$E \sim R^3 \left[\frac{1}{2}\frac{a^2}{R^2} + \frac{m_a^2}{2}a^2 + \lambda a n_n \right]$$

$$\frac{\partial E}{\partial a} = 0 \qquad -\lambda n_n = a \frac{1}{R^2}\left[\frac{R^2}{2R^2} + \frac{R^2 m_a^2}{2} \right]$$

$$n_n = \frac{M}{R^3 m_p}$$

$$a_{min} \sim \frac{-\lambda M_*}{R^{3} m_p} \cdot \frac{2R^2}{[1 + m_a^2 R^2]}$$

take $R \sim O(10\,km)$

$$a_{min} \sim \begin{cases} \dfrac{-2\lambda M_*}{m_p R} & \lesssim 10^{-10}\,eV \\[2ex] \dfrac{-2\lambda M_*}{m_p R\,(R\cdot m_a)^2} & m_a \gtrsim 10^{-10}\,eV \end{cases}$$

* compact objects key

$$\mathcal{L} \supset -\lambda a \bar{N} N - m_N \bar{N} N \quad \rightarrow \quad m_N \left(1 + \frac{a\lambda}{m_N}\right)\bar{N}N$$

$$\frac{\delta M_N}{M_N} \sim \lambda a_{min} \qquad (\mathcal{L}_{n,p}) \sim \text{TOV Eqns} + (\nabla a)$$

$$\mathcal{L}_c \sim \text{this } \underline{\text{check}}$$

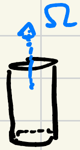
* can look for TOV mod, or (∇a)

Lecture # 4: Black Holes

The Zel'dovich Cylinder

Consider scattering a scalar wave Φ off a dissipative (absorbing) cylinder which rotates about $\hat{z}$ at Ω

$\Phi \sim f(r)e^{-i\omega t + im\phi}$



[asymptotically $f(r) = \frac{\text{const}}{\sqrt{r}} e^{-ikr}$]

- Incident wave behaves KG Eqn $\Box\Phi = 0$
- In rest frame of cylinder absorption characterized by α

$$\Box\Phi + \alpha\partial_t\Phi = 0$$

(damped oscillator)

- Normally, solve for reflected + transmitted waves by imposing $\Phi_i = \Phi_e + \Phi_r$ $\quad \partial_r\Phi_i = \partial_r(\Phi_e + \Phi_r)$

Go to corotating frame: $\phi \to \phi' + \Omega t$, which is like $\omega' = \omega - m\Omega$

$$\alpha\partial_t\Phi = \alpha(-i)(\omega - m\Omega)\Phi$$ # sign flip if $\omega < m\Omega$

Potential Exercise:

assume dissipation only at $r=R$, solve for reflected + transmitted amplitudes

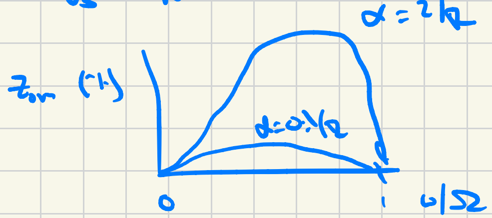
$$\frac{1}{r}(r\phi')' + \left(\omega^2 - i\alpha(\omega - m\Omega)\delta(r-R) - \frac{m^2}{r^2}\right)\phi = 0$$

$$\phi = \begin{cases} C_0 J_m(\omega r) & r<R \\ C_1 J_m(\omega r) + C_2 Y_m(\omega r) & r>R \end{cases}$$

$$r\to\infty,\quad \phi \sim \frac{I e^{-i\omega r}}{(r\omega)^{1/2}} + \frac{R e^{i\omega r}}{(r\omega)^{1/2}}$$

Define amp factor $Z_{om} = \frac{|R|^2}{|I|^2} - 1$

plot vs ω/Ω



Another Ex:

Note change in entropy $dS = \frac{1}{T}(dE - \Omega dJ)$

for incident wave $dJ = \frac{m}{\omega} dE$

if incident wave fully absorbed $dS \sim dE\left(1 - \frac{m\Omega}{\omega}\right)$

$\omega < m\Omega$ violates 2nd law thermodynamics

Resolution: Consider reflected wave

See Zel'dovich 1972,
Review by Brito, Cardoso, Pani

Black Hole Superradiance

Goal: Solve $(\Box - \mu^2)\Phi = 0$

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left(a\, dt - (r^2+a^2)d\phi\right)^2$$

$$\Delta \equiv r^2 - 2Mr + a^2 \qquad \Sigma \equiv r^2 + a^2\cos^2\theta$$

$G \equiv 1$ $\qquad a \equiv 2Mr_+\Omega_H$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right) - \mu^2\phi = 0$$

- Note that Eqn is separable

$$\Phi(\vec{r},t) = R(r)\, S(\theta)\, e^{-i\omega t} e^{im\phi}$$

(angular) $$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta S) + \left[a^2(\omega^2-\mu^2)\cos^2\theta - \frac{m^2}{\sin^2\theta} + \Lambda\right]S(\theta) = 0$$

(radial) $$\partial_r(\Delta\partial_r R) + \left[\frac{\omega^2(r^2+a^2)^2 - 4Mam\omega r + m^2a^2}{\Delta} - (\omega^2a^2 + \mu^2r^2 + \Lambda)\right]R = 0$$

Lets look at a few things:

Slowly spinning limit w/ non-rel bound states
($\omega \simeq \mu$)

(angular) $$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta S) + \left[\frac{m^2}{\sin^2\theta} + \Lambda\right] S = 0$$

$S = Y_{\ell m}$ w/ $\Lambda = \ell(\ell+1)$

+ limit $r \gg M$ for $\ell \geq |m|$

(radial) $$\left[-\frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{2M\mu^2}{r} + \frac{\ell(\ell+1)}{r^2} + (\mu^2-\omega^2)\right]R = 0$$

Radial Eqn for é in hydrogen atom!

with $\alpha_{EM} \to M\mu \sim \frac{r_g}{\lambda_c}$

$$R(r) \sim r^{\pm\nu} e^{\mp(2\sqrt{\mu^2-\omega^2}\,r)} \qquad \nu \equiv \frac{M\mu^2}{\sqrt{\mu^2-\omega^2}}$$

Bound states, $R(r) \underset{r\to\infty}{\sim} 0$

$$R_0^{far} \sim C_0 e^{-x/2} x^\ell U(\ell+1-n, 2\ell+2, x)$$

$$x \equiv 2\sqrt{\mu^2-\omega^2}\,(r - r_+)$$

$n \geq \ell+1$

Key differences come at small r

Go back to general Eqn for $R(r)$, and solve in limit $r \ll 1/\mu$

$$z(z+1)\partial_z\left[z(z+1)\partial_z R\right] + \left[P^2 - l(l+1)z(z+1)\right]R = 0$$

($z \to 0$ is horizon, $z \to \infty$ is ∞) $\quad z = \dfrac{(r-r_+)}{(r_+-r_-)}$ $\qquad r_\pm = M(1 \pm \sqrt{1-a^2})$

$$P = \frac{am - 2Mr_+\omega}{r_+ - r_-}$$

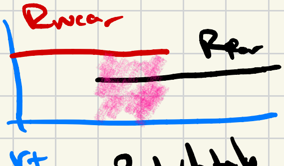
impose purely ingoing boundary conditions at $z \to 0$

$$R^{near}(r) \sim \left(\frac{z}{z+1}\right)^{iP} {}_2F_1(-l, l+1, 1-2iP, 1+z)$$

Perform matched asymptotics

Expand $R^{Near}(r)$ $r \to \infty$

$R^{far}(r)$ $r \to 0$



Perturbatively match

* only possible for discrete energies

$$\omega = \omega_R + i\Gamma$$

$$\omega_R \simeq \mu\left(1 - \frac{\alpha^2}{2n^2} + \dots\right)$$

$$\Gamma \sim \mu\,\alpha^{4l+4}(m\Omega_H - \omega_R) \qquad \alpha = \mu M$$

Aside, perturbative approach is one avenue ....

Can also expand $R(r)$ (intelligently) and obtain recursion relation

$$R(r) = (r - r_+)^{-i\sigma} (r - r_-)^{i\sigma + \chi - 1} e^{qr} \sum_{n=0}^{\infty} a_n \left(\frac{r - r_+}{r - r_-}\right)^n$$

$$\sigma = \frac{2 r_+ M(\omega - m\Omega)}{r_+ - r_-}$$

$$q \equiv \pm \sqrt{\mu^2 - \omega^2}$$

$$\chi = M \frac{\mu^2 - 2\omega^2}{q}$$

Recurrence relation relating a_n which can only be solved for discrete ω_R, Γ.

What have we learned? Quasi bound states are discrete, and grow exponentially when $\omega_{\mathrm{Re}} < m\Omega$

$$\Gamma_{211} \sim \left(\frac{\alpha}{0.4}\right)^9 \mathcal{O}(\text{hours})$$

Exercise to student, repeat for Proca Eqn + show growth rates are amplified

Self-Interactions in Superradiance

So far we've been talking about $(\Box-\mu^2)\Phi=0$

$$\mathcal{L} = \mu^2 f_a^2\left[1-\cos(a/f_a)\right]$$

$a \to a+2\pi$ invariant

$$\frac{\mu^2 a^2}{2} - \frac{1}{4!}\frac{\mu^2}{f^2}a^4 + \frac{\mu^2}{6!\,f^4}a^6 - \cdots$$

"natural" to have self-int. with well defined scale

$$(\Box-\mu^2)\Phi = -\frac{\lambda}{6}\Phi^3 \qquad \lambda = \frac{\mu^2}{f^2}$$

What does source term do?

$$(\Box-\mu^2)\Phi^{(0)} = 0$$

$$(\Box-\mu^2)\Phi^{(1)} = -\frac{\lambda}{6}\Phi^{(0)}$$

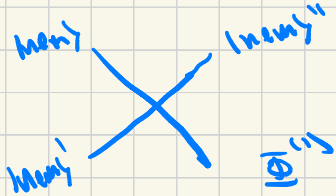
$$\Phi^{(0)} = \sum_{\{nlm\}} c_{nlm}\psi_{nlm}e^{-i\omega_{nlm}t} + c.c.$$

$(\Phi^{(0)})^3$

Term:

- $\psi_{nlm}^3 e^{-3i\omega_{nlm}t}$ $\{nlm\} \to$ free
- $\psi_{nlm}^2\psi_{nlm}^* e^{-i\omega_{nlm}t}$ (self-energy correction)

$\cdot\ \psi_{nlm}\,\psi_{(nlm)'}\,\psi^*_{(nlm)''}\, e^{-i(\omega_{nlm}+\omega_{(nlm)'}-\omega_{(nlm)''})t}$



Growth of these states is <u>not</u> independent

Try to solve for rate of energy transfer induced by $2 \leftrightarrow 2$ scattering

Basis expansion: $\Phi^{(1)} = \sum_{|nlm\rangle} C_{nlm}\psi_{nlm} + \underbrace{\sum_{lm}\int dk\, C_{lm}(k)\,\psi_{klm}}_{\text{free states}}$

$$\mathcal{O}' = -\nabla^2 - \frac{2\alpha}{r} + \frac{l(l+1)}{r^2} \qquad \mathcal{O}'\psi_{nlm} = -\frac{\alpha^2\mu^2}{n^2}\psi_{nlm}$$

$$\mathcal{O}'\psi_{klm} = k^2\psi_{klm}$$

$$\sum_{nlm}\left(\mu^2-\omega^2-\frac{\alpha^2\mu^2}{n^2}\right)C_{nlm}\psi_{nlm} + \sum_{lm}\int dk\, C_{lm}(k)\,\psi_{klm}\left(\mu^2-\omega^2+k^2\right)$$

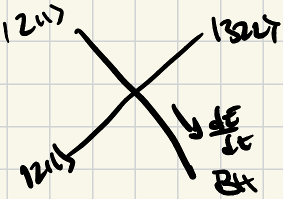
$$= -\frac{\lambda}{6}\,\Phi^{(0)3}$$

Project onto ψ^*_{nlm}, integrate + use orthonormality

$$C_{nlm} = \frac{1}{\left(\mu^2 - \omega^2 - \frac{\alpha^2\mu^2}{n^2}\right)} \int d^3r \, \psi_1 \psi_2 \psi_3^* \psi_{nlm}^*$$

Energy losses through horizon or @ infinity

$$\frac{dE}{dt} = \int_{\partial\Sigma} T_0^{\ \mu} n_\mu \, d\Omega$$



Energy flow almost perfectly uniform

$$\frac{dE}{dt} \sim (\quad) \mu \alpha^{11} \left(\frac{M_p}{f_a}\right)^4 N_1 N_2 N_3$$

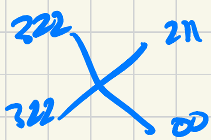
Key notes:

∘ $\partial_t E_{211} \propto \Gamma_{SR,211} N_{211} - \gamma N_{211}^2 N_{322}$

$$\left(\Gamma_{SR,211} - \gamma N_{211} N_{322}\right) N_{211}$$

* large occupation numbers lead to quenched growth!

* enhanced growth rate of 322

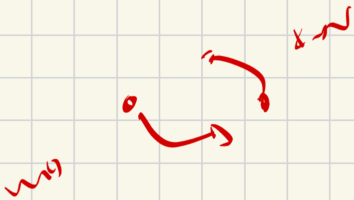


Equilibrium configuration

- At small f_a, this leads to smaller occupation numbers, delayed spin down less GWs .....

- Non-decoupling of levels means one must study "coupled evolution"

Dynamical Friction



$m_a \ll 1/\Omega$

$$g^{\mu\nu} = \begin{pmatrix} -(1-2\psi) & & & 0 \\ & 1+2\psi & & \\ & & 1+2\psi & \\ 0 & & & 1+2\psi \end{pmatrix}$$

$$(\Box - \mu^2)\Phi = 0$$

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\right) = -(1-2\psi)\partial_t^2 + 4\dot{\psi}\partial_t + (1+2\psi)\nabla^2$$

$$\psi = \frac{GM}{r}$$

Ansatz $\quad \Phi = \frac{1}{\sqrt{2\mu}}\phi e^{-i\mu t}$

WKB $\quad \dot{\phi} \ll \mu\phi \;,\; \psi$ small

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{\nabla^2}{2\mu} + V\right)\psi$$

$$V = -\frac{GM\mu}{2}\left[\frac{1}{|r - R(t)|} + \frac{1}{|r + R(t)|}\right]$$

$$\vec{R}(t) = (R, \pi/2, \Omega t) \qquad \Omega^2 = \frac{GM}{8R^3}$$

$$\Psi(r,t) = \sum_{nlm} \frac{u_{nlm}(r)}{r} Y_{lm}(\theta,\phi) e^{-i\omega t}$$

intuitively, one can understand $\langle \Psi' | V | \Psi \rangle$

$$\sum_{\Omega} e^{i\Omega t}$$

$$e^{i\omega t - im\phi} = e^{i\omega t - im\phi} e^{i\Omega t}$$

Didn't have enough time to finish...

See Tomasello (2025)

Recap

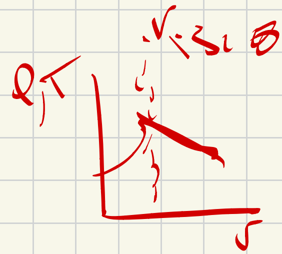
(Energy transfer..)

- SR requires $\omega < m\Omega$, dissipation → controls rate (confinement)
- Apply to context of BHs, we find they like axions can spin down BHs! [modeling stellar, coupling, errors...]
  - how do we measure $\tilde{a}$?
    - GW (not worrying now)
    - X-ray spectral fitting [- Continuum fitting
      - F_e line]
    - TDE
- Also interesting systems b/c ρ_a super high! [GW, Rel. axions] SELECTED
- General idea of coupling (w/ new physics) to contribute stellar energy drift
  - ★ stellar objects + cooling

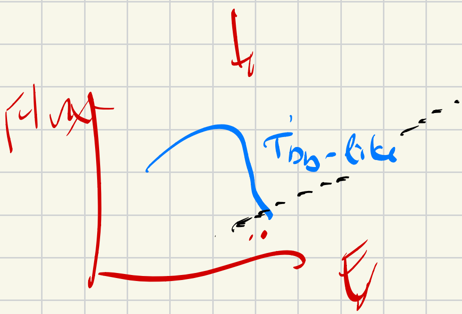
- Trapping → stellar binary
- Rotational → Ω, axion clouds
- Nuclear → sourcing axion field from nuc matter
- $\vec{B}$ → (something Jon Hunter about)

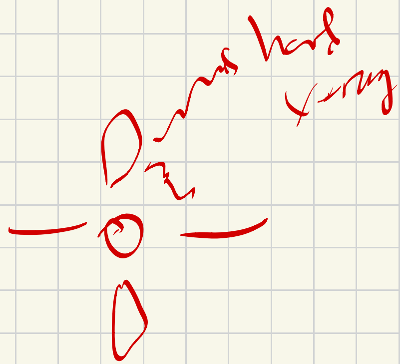
• Continuum Fitting

$T(r, \tilde{a})$

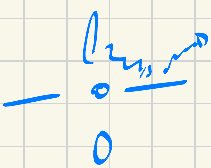


r_{isco} changes by $O(5)$ with $\tilde{a}$





• Iron Line



• Iron not fully ionized
• can de-excite producing an asymmetric broadened line

TDEs

→ if tidal force exceeds self-gravity

you destroy

$$a_{tidal} \sim \frac{G M_{BH} R_*}{r^3}$$

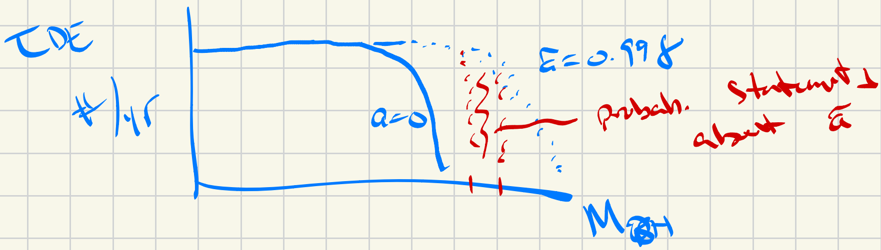
$$a_{self} \sim \frac{G M_*}{R^2}$$

$$r_t \sim R_* \left(\frac{M_{BH}}{M_*}\right)^{1/3} \xi_{GR}$$

$$\frac{r_t}{G M_{BH}} \sim \left(\frac{R_*}{G M_{BH}}\right)\left(\frac{M_{BH}}{M_*}\right)^{1/3} \xi_R \propto \frac{1}{M_{BH}^{2/3}}$$

small M_{BH} $\quad \frac{r_t}{r_g} \gg 1$

for $M \sim 10^8 M_\odot$ $\quad \exists\, M_{max}$ for which TDE "Hills Mass"



(Superradiance around stars?)

- Can we play the "SR" game w/ neutron stars?

Requirements

- Rot. SR requires $\omega < m\Omega$

$$\frac{\Omega_{MSP}}{\Omega_{BH,max}} \sim \mathcal{O}(0.1)$$

$$\alpha^{max} = M\mu = 10^{-2}(\alpha_{BH})$$ [little less E^{Rot}]

- Bound state

(No problem, no dissipation from horizon)

Γ too long · Dissipation (no Ergo region!)

E.g. $\cdot\, d_{\mu a} \bar{N} \gamma^\mu \gamma^5 N$

assume $(\Box - \mu^2)\phi + \Gamma \partial_t \phi = 0$

$$\phi = \psi(r) Y^{lm}(\theta, \phi) e^{-i\omega t}$$ [slow spin limit]

$$r^2\psi'' + 2r\psi' + [r^2(\omega^2 + i\omega\Gamma_\phi) - l(l+1)]\psi = 0$$

inside star $\psi \sim j_l(r\sqrt{\omega^2 + i\omega\Gamma_\phi}) + y_l(\ldots)$ irregular, drop out

outside $\psi \sim c_1 h_l^{(1)}(\omega r) + c_2 h_l^{(2)}(\omega r)$

spherical Hankel funcs

$$\phi(r\to\infty) \sim \phi_{out}\frac{e^{i\omega r}}{r} + \phi_{in}\frac{e^{-i\omega r}}{r}$$

$$\phi(r=R)_{out} = \phi(r=R)_{in}$$

+ derivative

$$Z_{lm} = \frac{|\phi_{out}|^2}{|\phi_{in}|^2} - 1$$

$$= -\frac{4\pi^2\omega(R\omega)^{2l+1}}{(1+2l)!!\,(2l+3)!!}\cdot\Gamma_\phi$$

(Non Rotating)

(Rotating)

Axions tapping Rot. E of pulsars

- Rotating $\vec{B}$ induces $E^{\text{quad}}\big|_{r=R_{NS}} \sim \Omega R B$

- Typically, plasma screens $(E\cdot B) = 0$

This can't happen everywhere

- observationally we see emission from near field

- $\nabla \times B = j$

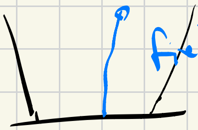
$(\nabla\times B)_{\text{open}} \neq 0 \qquad (\nabla\times B)_{\text{closed}} \simeq 0$

Can we supply j with $E\cdot B = 0$?

large $\vec{B}$ sends $v_e \parallel$ to B

$\boxed{\text{Go to } *}$

Toy 1D Example of axion Back-Reaction



field line almost straight

Look for static stable configurations

Gauss' Law $\nabla \cdot E = \rho$

co-rotating frame $\nabla \cdot \vec{E} = \rho + (\Omega \times r) \times B$

$\partial_x E = \rho - \rho_{GJ}$

Ampere's law $\nabla \times B - \cancel{\partial_t \vec{E}}^{0} = j$

So dipolar fields $\nabla \times B = 0$

Deviation from dipole due to currents

(constraint → free parameter)

Go here *

$\rho = n_c \cdot e \quad j = n e \cdot e\,v = e n_c \frac{\sqrt{\gamma^2 - 1}}{\gamma} = \rho \frac{\sqrt{\gamma^2 - 1}}{\gamma}$

$\rho = \frac{\gamma j}{\sqrt{\gamma^2 - 1}}$

Energy conservation

$q \Phi_i + \gamma_i m = q \Phi_f + \gamma_f m \qquad \gamma_i = 1 \qquad E = -\nabla \Phi$

$m(\gamma_f - 1) + q(\Phi_f - \Phi_i) = 0 \qquad \Phi_i = 0$

$\frac{\partial}{\partial x^2}\left[m(\gamma - 1) + q\Phi \right] = 0$

$$m \frac{\partial \gamma}{\partial x^2} + q \frac{\partial \Phi}{\partial x^2} = 0$$

$$\frac{\partial \gamma}{\partial x^2} = \frac{e}{m}\rho - \frac{e}{m}\rho_{GJ} = \frac{e\rho_{GJ}}{m}\left[\frac{\rho}{\rho_{GJ}} - 1\right]$$

$$= \frac{e\rho_{GJ}}{m}\left[\frac{J}{\rho_{GJ}}\frac{\gamma}{\sqrt{\gamma^2-1}} - 1\right]$$

$$\omega_p^2 = \frac{e^2 n}{m} = \frac{e\rho}{m}$$

Change variables $\lambda = \frac{1}{\omega_p}$

$$\frac{d^2\gamma}{d\lambda} = \left(\frac{J}{\rho_{GJ}}\right)\frac{\gamma}{\sqrt{\gamma^2-1}} - 1$$

$\gamma \to 1$ $\quad \left(\frac{J}{\rho_{GJ}}\right)\left(\frac{\gamma}{\sqrt{\gamma^2-1}}\right) \to \infty$

$d^2\gamma / d\lambda > 0$ accelerating

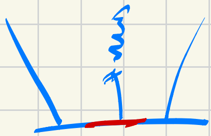
$\gamma \to \infty$ $\quad \frac{d^2\gamma}{dx} \sim \left(J/\rho_{GJ}\right) - 1$

$\left(J/\rho_{GJ}\right) > 1$ acceleration

$\left(J/\rho_{GJ}\right) < 1$ deceleration

So can we compute $\left(J/\rho_{GJ}\right)$?

Yes, for active pulsars $\left(J/\rho_{GJ}\right) > 1$ in part of open field lines

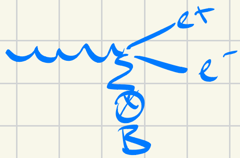


particles can't screen E!

- Accelerating is emit γ-rays

$$\langle E_\gamma \rangle \sim \frac{3\gamma^3}{2\rho_c}$$

$$\rho_c \sim \Theta(10^2\ \mathrm{km})$$



$$\tau \sim \int d\psi\ \alpha_{EM}\, m_e \sin\psi \left(\frac{B}{B_Q}\right) e^{-\frac{8 B_Q m_e}{3 E_\gamma B \sin\psi}}$$

$$\psi = \cos^{-1}(\hat{B} \cdot \hat{k}_\gamma)$$

Exp Factor $-\left(\frac{B_Q}{B}\right)\left(\frac{m_e}{E_\gamma}\right)\frac{1}{\sin\psi} \sim -\frac{10^{18}}{\gamma^3}\frac{1}{\sin\psi}$

$\sin\psi \sim$ small, $\psi \sim 10^{-3}$, need $\gamma \gtrsim 10^7$

- achievable on $D \sim \Theta(10\ \mathrm{m})$

Dense plasma screens E, but is relativistic + thus free streaming

Scrunny $\to (E\cdot B)(t)$

$$(\Box - \mu^2)\phi = gE\cdot B = j(x)$$

free

$$\phi_0 = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega(k)}}\left[a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right]$$

ω) source

$$a_k \to a_k + \frac{i}{\sqrt{2\omega(k)}}\tilde{j}(k) \qquad a_k^\dagger \to a_k^\dagger - \frac{i}{\sqrt{2\omega(k)}}\tilde{j}(-k)$$

$$\tilde{j} = \int d^4x\, e^{ik\cdot x} j(x)$$

$$\frac{dN}{dt\,d^3k} = \frac{\langle 0|a_k^\dagger a|0\rangle}{(2\pi)^3\, T} \propto |\tilde{j}(k)|^2$$