EX UMBRA IN SOLEM

Homotopic approach to BK equation

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Outline

- Introduction
- BK and Geometric Scaling
- BK solutions
- Homotopic approach
- Summary and Outlook

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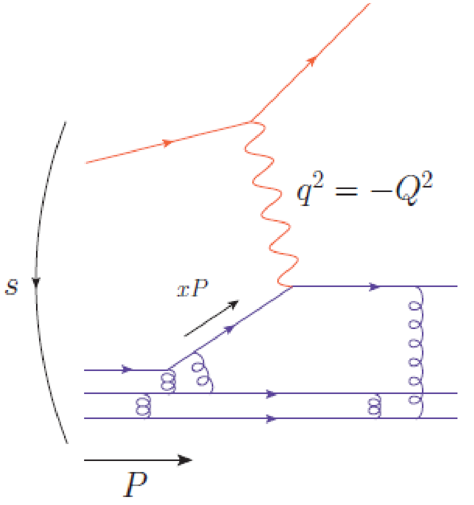
***Phys. Rev. D* 110 (2024) 5, 054045**
***Phys. Rev. D* 111 (2025) 9, 096025**

Deep Inelastic Scatering DIS

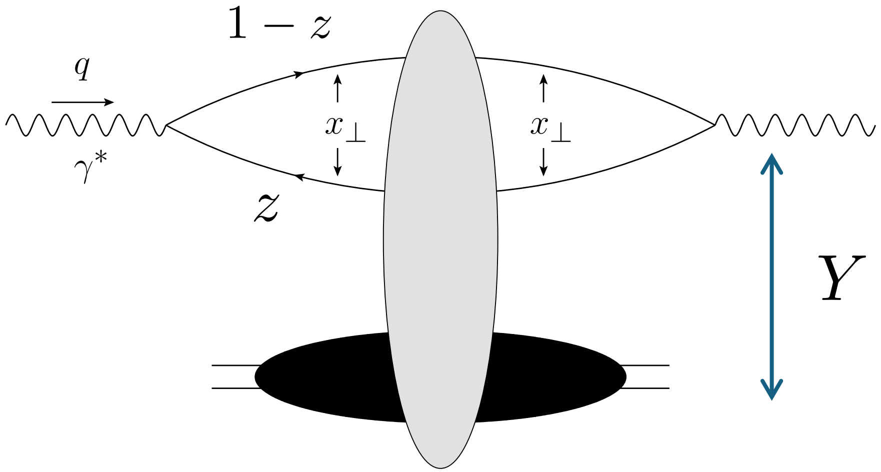
Mueller Dipole Approximation



The total DIS cross section is expressed in terms of the forward quark dipole amplitude $N(\vec{x}, \vec{b}, Y)$:

At small x, the dominant contribution to DIS structure functions comes from the dipole picture of DIS, where the virtual photon splits into a quark-antiquark pair, which then interacts with the target.

$$\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2\,\pi}\, d^2 b_\perp \int_0^1 \frac{dz}{z\,(1-z)}\ |\Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z)|^2\ N(\vec{x}_\perp, \vec{b}_\perp, Y)$$



$Y = \ln\frac{1}{x}$ is the rapidity

b is the impact parameter

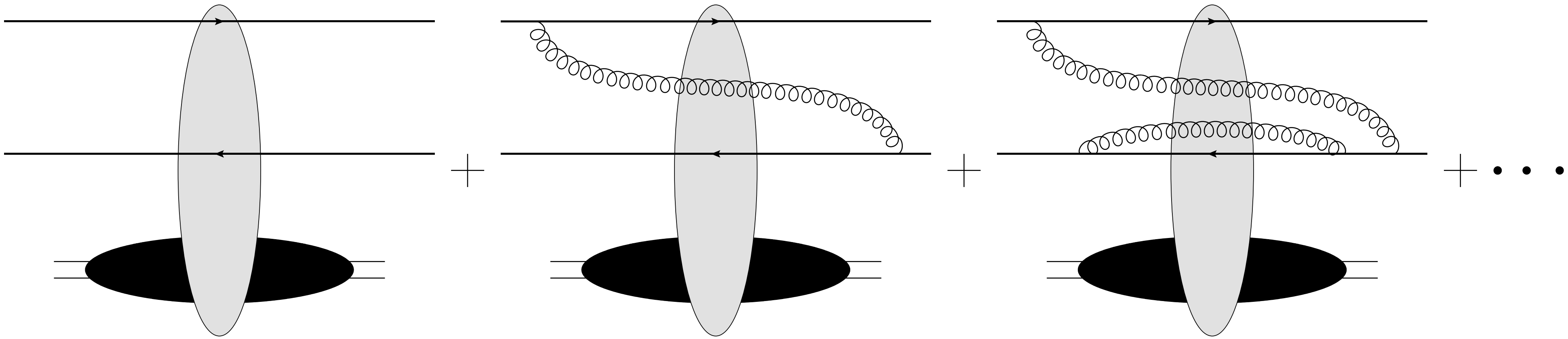
$x_\perp$ the dipole transverse size

$\Psi^{\gamma}{}_{q\bar{q}}\ (x_\perp, b, Y)$ Photon wave function in dipole approximation (known to NLO light cone)

Quantum Chromodynamics at high Energy, Kovchegov and Levin Cambridge 2012
Mueller 2002

Small-x evolution equations in the large N_c

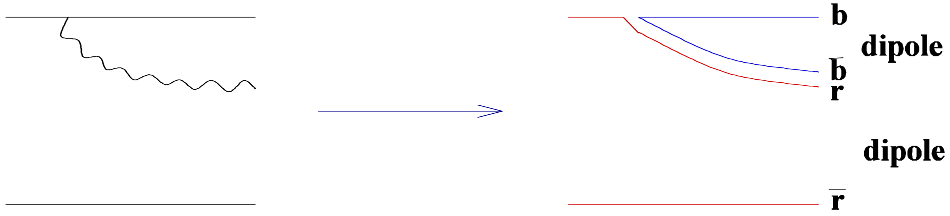
To include the quantum evolution in a dipole amplitude we can consider Emission of a small-x gluon taken in the large-N_C limit (A. H. Mueller in ’93-’94)

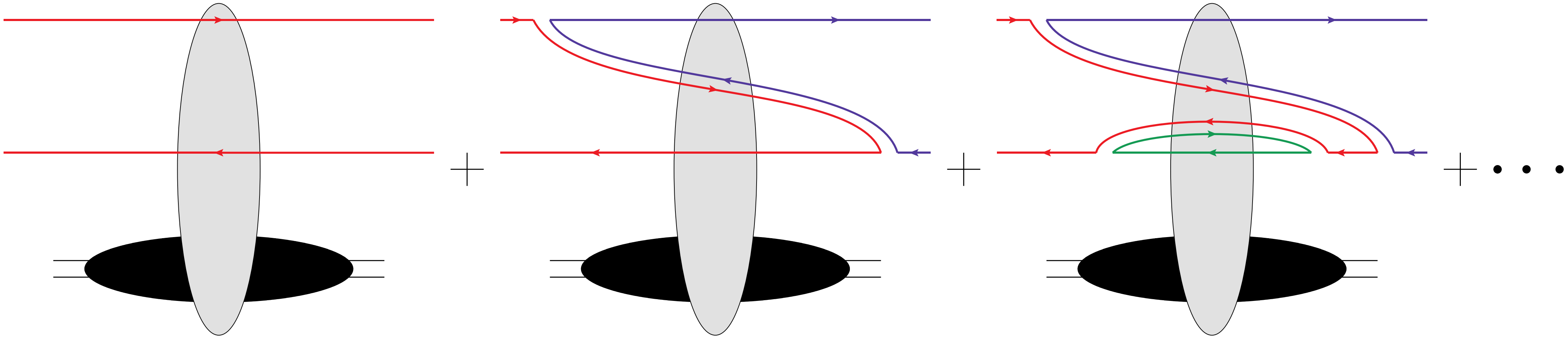


We need resum this cascade of gluons

In the large-Nc limit, each gluon becomes a quark-antiquark pair:

Gluon cascade becomes a dipole cascade

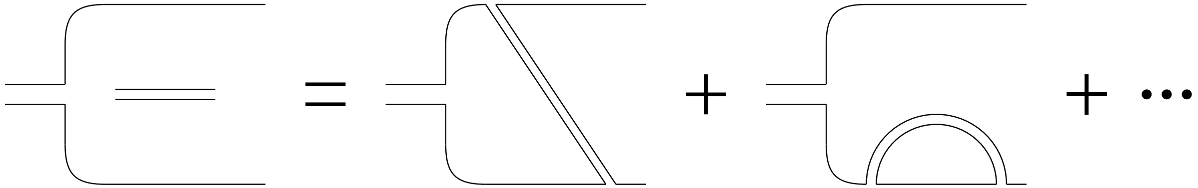




These extra gluons bring in powers of α_s ln s, such that when α_s << 1 and ln s >>1 this parameter is α_s ln s ~ 1 (leading logarithmic approximation, LLA).

$$\alpha_s \ln s \sim \alpha_s \ln \frac{1}{x} \sim 1$$

Representation for the Dipole



Nonlinear Evolution

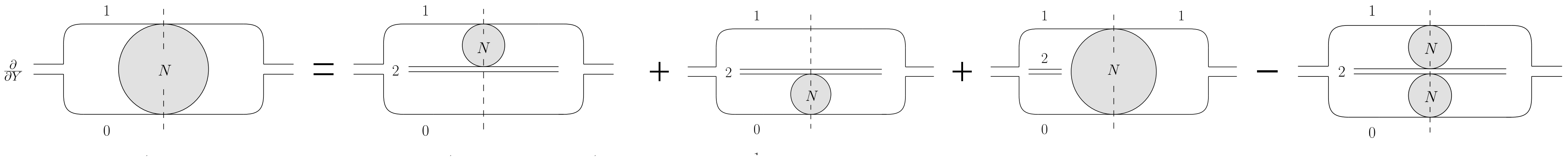
Summing the gluon cascade at large - N_C one can find the Balitsky – Kovchegov evolution equation for the dipole scattering amplitude N

$$Y = \ln\frac{1}{x} \sim \ln s$$

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2\,\frac{x_{10}^2}{x_{12}^2\,x_{02}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y) - N(x_{12},b,Y) * N(x_{20},b,Y)\right)$$

Balitsky ‘96, Yu.Kovchegov. ’99

Diagrammatic representation of the BK evolution equation for the forward dipole–nucleus scattering amplitude *N*, denoted by a shaded circle.



At fixed rapidity a colorless dipole with size x_{10} decays into two dipoles with sizes x_{21} and x_{20}.
Either one dipole proceeds to evolve and interact with the target while the other dipole remains a spectator or both dipoles evolve and interact with the target (the nonlinear term in BK).

Beyond the large-N_c limit we have to consider the JIMWLK functional evolution equation (Jalilian-Marian, Iancu, McLerran, Weigert Leonidov and Kovner 1997-2002)

The JIMWLK and BK equation was solved on the lattice and there is excellent agreement between the JIMWLK and BK simulations K. Rummukainen and H. Weigert ’04

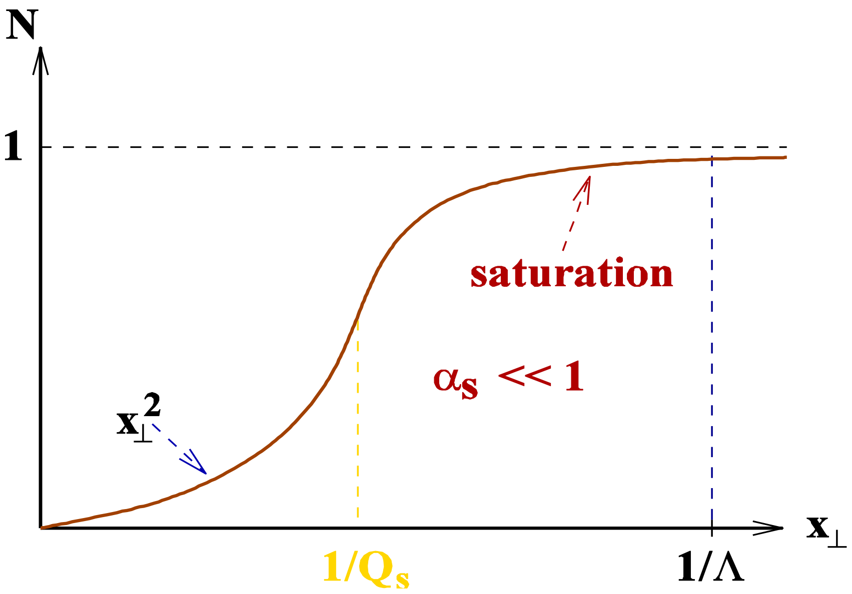
Solution of the BK nonlinear equation

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2\,\frac{x_{10}^2}{x_{12}^2\,x_{02}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y)\ - N(x_{12},b,Y) * N(x_{20},b,Y)\right)$$

Analyzing the behavior of the solution BK equation we can easily see that *N* = 1 and N = 0 are stationary solution of that equation.

The dipole-nucleus amplitude should be

The fact that the forward amplitude *N* goes to zero as *x*⊥ → 0 is based on a fundamental physical principle of **color transparency** in the zero-size dipole the color charges of the quark and the anti-quark cancel each other, leading to disappearance of the interactions with the target.



$N(x_{10},Y)$ grows with *x*⊥, but a very large *x*⊥ saturates to *N* → 1

Black disk limit,

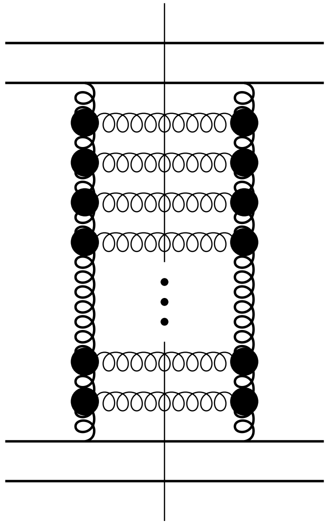
At the moment there is no exact analytical solution of the BK equation

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2\,\frac{x_{10}^2}{x_{12}^2\,x_{02}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y) - N(x_{12},b,Y) * N(x_{20},b,Y)\right)$$

But for $N(x_{10},Y) < 1$ at small side of the dipole $x_\perp < 1/Q_{so}$

$$\boxed{\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2\,\frac{x_{10}^2}{x_{12}^2\,x_{02}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y)\right)}$$

BFKL equation



BFKL equation: Balitsky, Fadin, Kuraev, Lipatov 1977-78.

- One starts with a two-gluon exchange diagram and resum its radiative corrections.
- The leading high-energy contribution can be drawn as a ladder diagram, with the t-channel gluons being the special "reggeized" gluons and the thick dots representing effective Lipatov vertices.
- Each rung of the ladder brings in a power of α ln s

Nonlinear saturation effects become important when N ~ N^2 ⇨ N ~ 1. This happens at $k_T \sim Q_S$ and we define the saturation scale $Q_s(s)$

Typical partons have $k_T \sim Q_S$, so that their characteristic size is of the order $r \sim 1/k_T \sim 1/Q_S$
Typical parton size decreases with energy and $Q_s(y) = Q_{s0}(y)e^{\Delta\, y}$ with the rapidity $y = \ln\frac{1}{x}$

Analytical Solutions to the BFKL equation

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2\,\frac{x_{10}^2}{x_{12}^2\,x_{02}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y)\right)$$

First BFKL Pomeron Lipatov solution (1986)

$$\bar{\alpha}_s \equiv \frac{\alpha_s N_c}{\pi}.$$

We can consider that the solution in Mellin representation is

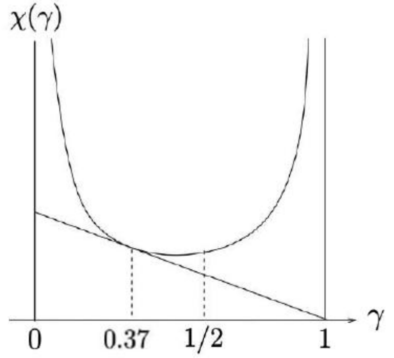
$N_{BFKL}(r,Y,b) = \int d\nu\, r^{2\gamma}\,\tilde{N}(\nu,Y,b)$ → $\frac{\partial \tilde{N}(\nu,Y,b)}{\partial Y} = \bar{\alpha}\,\chi(\gamma)\,\tilde{N}(\nu,Y,b)$ and $\tilde{N}(\nu,Y,b) = \tilde{N}_0\, e^{\bar{\alpha}\,\chi(\gamma)\,Y}$ See talk M. Nefedov

where $\chi(\gamma) = 2\psi(1) - \psi(1-\gamma) - \psi(\gamma) = \chi(0,\frac{1}{2} + i\,\nu)$; $\psi(z) = \frac{d\ln\Gamma(z)}{dz}$ Digamma Function $\gamma = \frac{1}{2} + i\,\nu$

Then $N_{BFKL}(r,Y,b) = \int d\nu\, r^{2\gamma}\,\tilde{N}_0\, e^{\bar{\alpha}\,\chi(\gamma)\,Y}$

Dominant contribution to high Y is

$$N_{BFKL}(r,Y,b) = \int \frac{d\gamma}{2\pi}\, r^{2\gamma}\,\tilde{N}_0 e^{\,\bar{\alpha}\,\chi(\gamma)\,Y} = \int \frac{d\gamma}{2\pi}\,\tilde{N}_0 e^{\,\bar{\alpha}\,\chi(\gamma)\,Y + \gamma\xi} \approx e^{\,\bar{\alpha}\,\chi(\gamma_{cr})\,Y + \gamma_{cr}\xi} \quad ; \ \xi = \ln r^2 Q_{so}^2$$



and using Saddle point $N_{BFKL}(r,Y,b) \approx e^{\,\bar{\alpha}\,\chi(\gamma_{cr})\,Y}(r\,Q_{so})^{\,2\,\gamma_{cr}}$

Saturation scale $Q_s(s)$ is happens at $r = 1/Q_s$ and we find that the saturation scale $Q_s(s)$ is

$$Q_s^2(Y) = Q_{s0}^2\, e^{\,\bar{\alpha}\,\frac{\chi(\gamma_{cr})}{\gamma_{cr}}\,Y} = e^{\,\bar{\alpha}\,\kappa\,Y} \quad \text{where } \kappa = \frac{\chi(\gamma_{cr})}{\gamma_{cr}}$$

BFKL and BK equation: Transition to saturation and Geometric Scaling

Stasto, Golec-Biernat and Kwiecinski

Phys. Rev. Lett. 86 (2001) 596.

have shown that the HERA data on DIS at low x , functions of two independent variables — the photon virtuality Q^2 and the Bjorken variable x , are consistent with scaling in terms of the variable τ

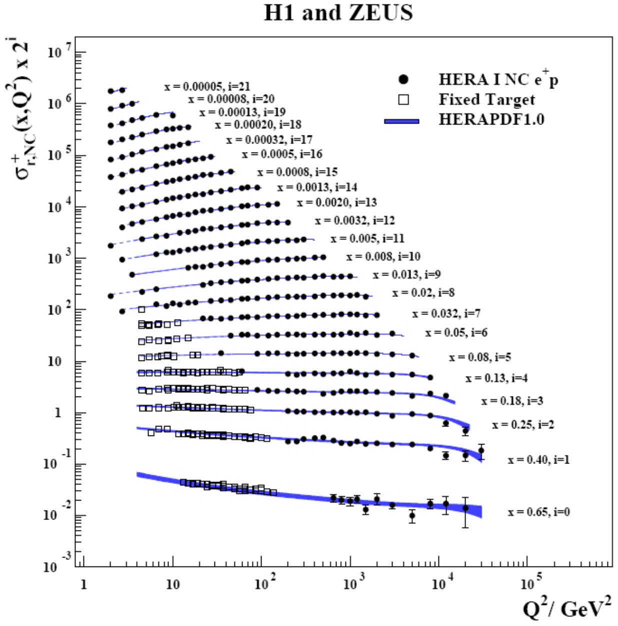
Saturation momentum: $Q_s^2 = Q_{s0}^2 e^{\lambda Y} = Q_{s0}^2 \left(\frac{x_0}{x}\right)^{\lambda} \rightarrow \tau = r^2 Q_s^2(Y);\ \ \tau = \frac{Q^2}{Q_s^2(Y)} = Q^2/Q_0^2 \left(\frac{x}{x_0}\right)^{\lambda}$ $\lambda = 0.3 - 0.4$

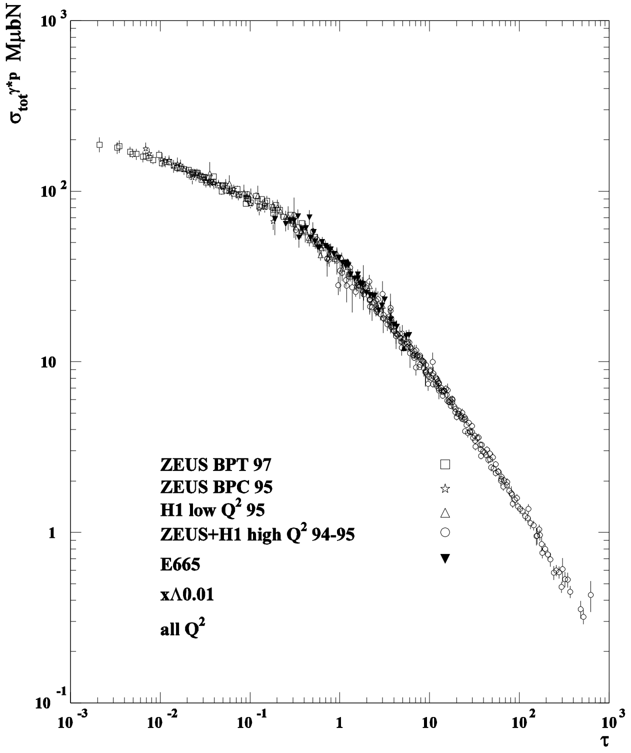
They plot the total DIS cross section, which is a function of 2 variables, Q^2 and x, as a function of just <u>one</u> variable

$$\sigma(x, Q) = \sigma(\tau)$$

Geometric Scaling

is to show that the scaling region for the various distribution functions is in fact much larger than the saturation region





Balistky Phys. Rev. D75, 014001 (2007)
I. Balitsky, Nucl. Phys. B463, 99 (1996)
Kovchgov, Iancu, Itakura, and Larry McLerran arXiv 0203.137
Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK)

BFKL and BK equation: Transition to saturation and Geometric Scaling

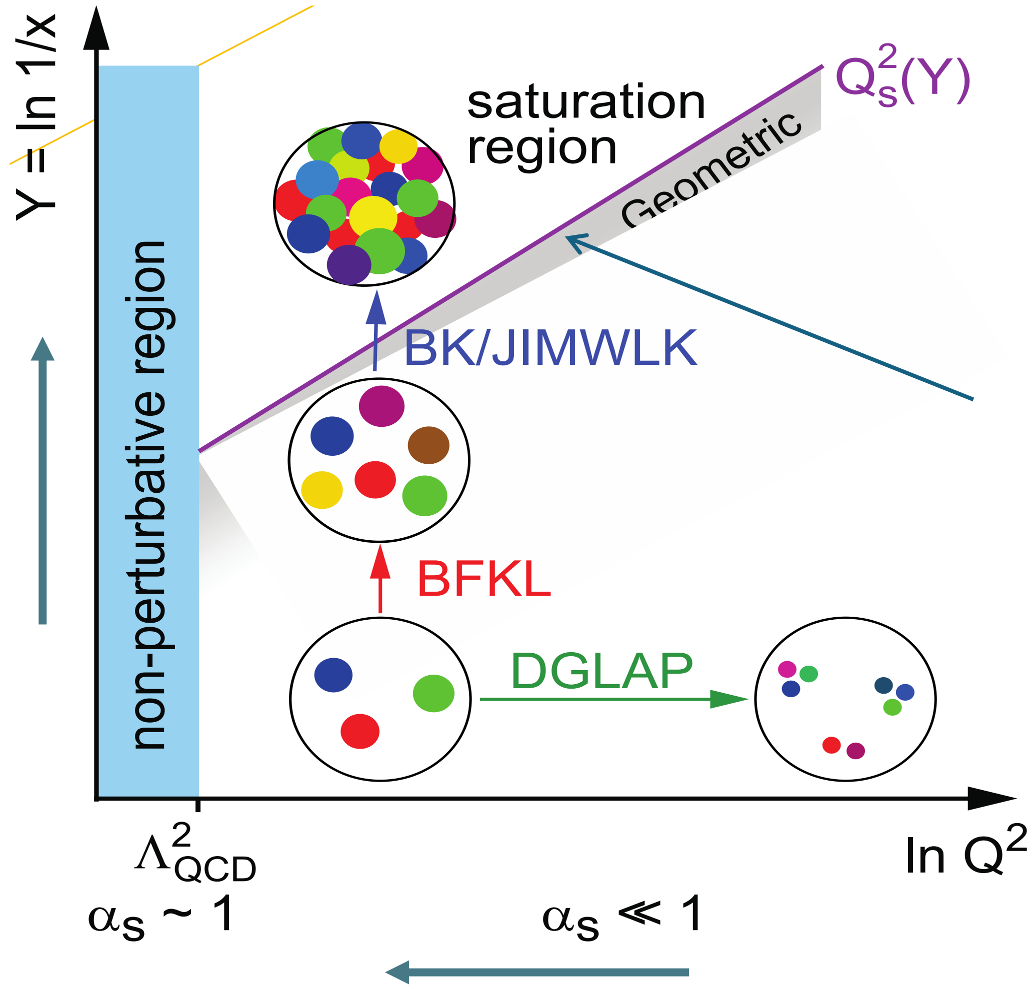
$N_{BK}(r,Y,b) \rightarrow N_{BK}(Q,Y,b)$

Nonperturbative analysis
BK – nonlinear

energy

for High energy or small x we have a high number of gluons leading a very dense system of gluons (CGC)

Saturation region and the transition from BFKL to BK is to $Q_s(Y)$ and Geometric scaling



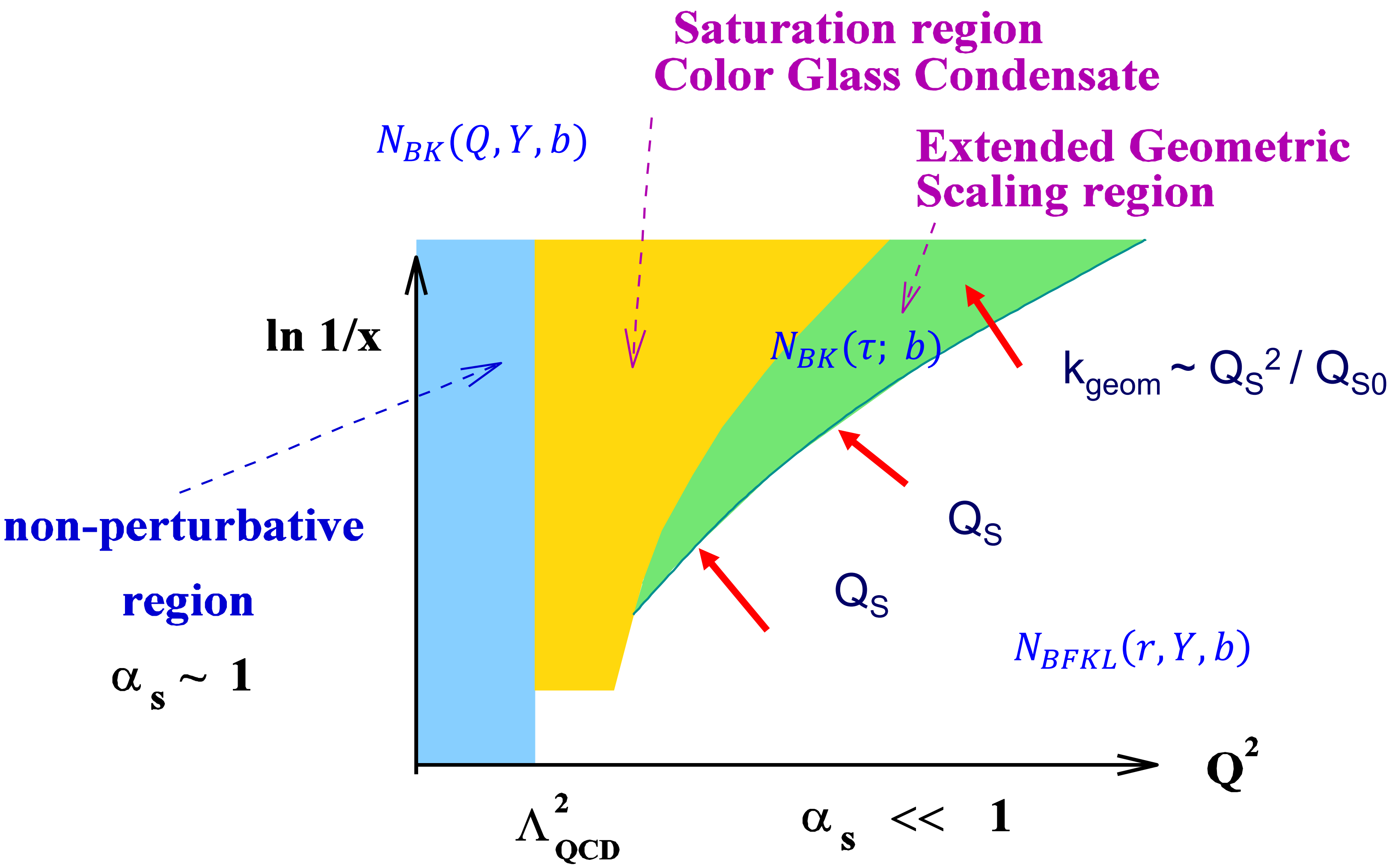
There is a huge number gluons at small-x

$$Q_s^2 \sim \left(\frac{1}{x}\right)^{\lambda}$$

Saturation Scale
grows with energy

Perturbative analysis
BFKL, linear equations

Evalution in Q^2:
$DGLAP\ equation$
Better resolution in the partons
Dokshitzer–Gribov–Lipatov–Altarelli–Parisi



Saturation physics is based on the existence of a large internal transverse momentum scale Q_S which grows with both decreasing Bjorken *x* and with increasing nuclear atomic number A

$$Q_s^2 \sim A^{1/3}\left(\frac{1}{x}\right)^{\lambda}$$

such that

$$\boxed{\alpha_S = \alpha_S(Q_S) << 1}$$

and we can use perturbation theory to calculate total cross sections, particle spectra and multiplicities, correlations, etc, from first principles.

BK evolution in coordinate space

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi} \int d^2x_2 \frac{x_{02}^2}{x_{12}^2\,x_{01}^2} \left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y) \; - N(x_{12},b,Y) * N(x_{20},b,Y)\right)$$

in principle need to solve the fully impact parameter dependent Balitsky Kovchegov (BK) equation

In this work $N(x_{10}, b, Y)$:

- The dependence on the impact parameter b which becomes an external parameter or can be absorb in Q_s
- satisfying the initial condition given by the BFKL Pomeron
- satisfying the Geometric Scaling inside the saturation region, Non-perturbative

Initial condition at Y_0 is McLerran -Venugopalan model MV:

$$\mathrm{N}(r_\perp, b, Y = Y_0) = 1 - \exp\left[-\frac{r_\perp^2 Q_{s0}^2}{4} \ln\left(\frac{1}{r_\perp \Lambda} + e\right)\right]$$

where Q_{s0} initial saturation scales:

$$Q_{s0}^2 = \begin{cases} Q_{s0}^2 & \text{for proton} \\ \mathrm{A}^{1/3} \times Q_{s0}^2 & \text{for nucleus} \end{cases}$$

A solution of BK equation is helpful to calculate F2 and compare with experimental data.

Kowalski, Lappi, Marquet, Venugopalan, PRC 78 (2008) 045201
Lappi, M¨antysaari, PRD 88 (2013) 114020

BK evolution in momentum space

Unfortunately finding an exact analytical solution it is a very difficult task, this eq. is non–linear and because of a complicated structure of the BFKL kernel χ(λ)

First approximation Perturbation theory for the linear part.

Y. Kovchegov arXiv 9905214 Perturbative approach

Analysis of the BK-equation in momentum space, by Fourier transforming N(r, b, Y), into N(q, b, Y), q being the Fourier conjugated of the dipole size.

$$\tilde{N}(k,Y) = \int \frac{d^2x}{2\pi x_\perp^2} e^{-i\mathbf{k}\cdot\mathbf{x}} N(x_\perp, Y) = \int_0^\infty \frac{dx_\perp}{x_\perp} J_0(kx_\perp) N(x_\perp, Y).$$

BK equation in momentum space, is given by the equation

$$\frac{\partial \tilde{N}(k,Y)}{\partial Y} = \frac{2\alpha N_c}{\pi} \chi\left(-\frac{\partial}{\partial \ln k}\right) \tilde{N}(k,Y) - \frac{\alpha N_c}{\pi} \tilde{N}^2(k,Y)$$

$$\chi(\lambda) = \psi(1) - \frac{1}{2}\psi\left(1 - \frac{\lambda}{2}\right) - \frac{1}{2}\psi\left(\frac{\lambda}{2}\right)$$

$$\tilde{N}_1(k,Y) = \exp\left[\frac{2\alpha N_c}{\pi} Y \chi\left(-\frac{\partial}{\partial \ln k}\right)\right] C(k), \qquad C(k) = \int \frac{d\lambda}{2\pi i}\left(\frac{k}{\Lambda}\right)^\lambda C_\lambda,$$

$$\tilde{N}_1(k,Y) = \int \frac{d\lambda}{2\pi i} \exp\left[\frac{2\alpha N_c}{\pi} Y \chi(-\lambda)\right] \left(\frac{k}{\Lambda}\right)^\lambda C_\lambda.$$

General Solution

$$\tilde{N}_n(k,Y) = -\frac{\alpha N_c}{\pi} \int_0^Y dy \exp\left[\frac{2\alpha N_c}{\pi}(Y-y)\chi\left(-\frac{\partial}{\partial \ln k}\right)\right] \left(\sum_{m=1}^{n-1} \tilde{N}_m(k,y)\,\tilde{N}_{n-m}(k,y)\right).$$

Motika and Sadzikowski arXiv 230602118 Hight Twist corrections

Result:

The perturbative solution is convergent, and we know the solution for N (k, Y) outside of the saturation region and the high energy asymptotics inside the saturation region

One need to have a better knowledge of the momentum space solution inside the saturation region.

We can obtain the solution in coordinate space N(x⊥, Y) using the Fourier transformation: To do that one would have to integrate over all values of the transverse momentum k from 0 to ∞.

If x⊥ < 1/Qs then the transverse momentum is effectively cut off and , therefore varies approximately from 0 to 1/x⊥ > Qs

If x⊥ > 1/Qs then the transverse momentum is in the saturation region and simply gets cut off by 1/x⊥ < Qs

Numerical solutions in momentum space: traveling wave

In the momentum space, the equation admits travelling-wave solutions.

- Geometric scaling
- Saturation Momentun $Q_s(Y)$
- N(q, Y) can satisface the Fisher, Kolmogorov-Petrovsky-Piscounov (F KPP) equation
- Numerical Solution

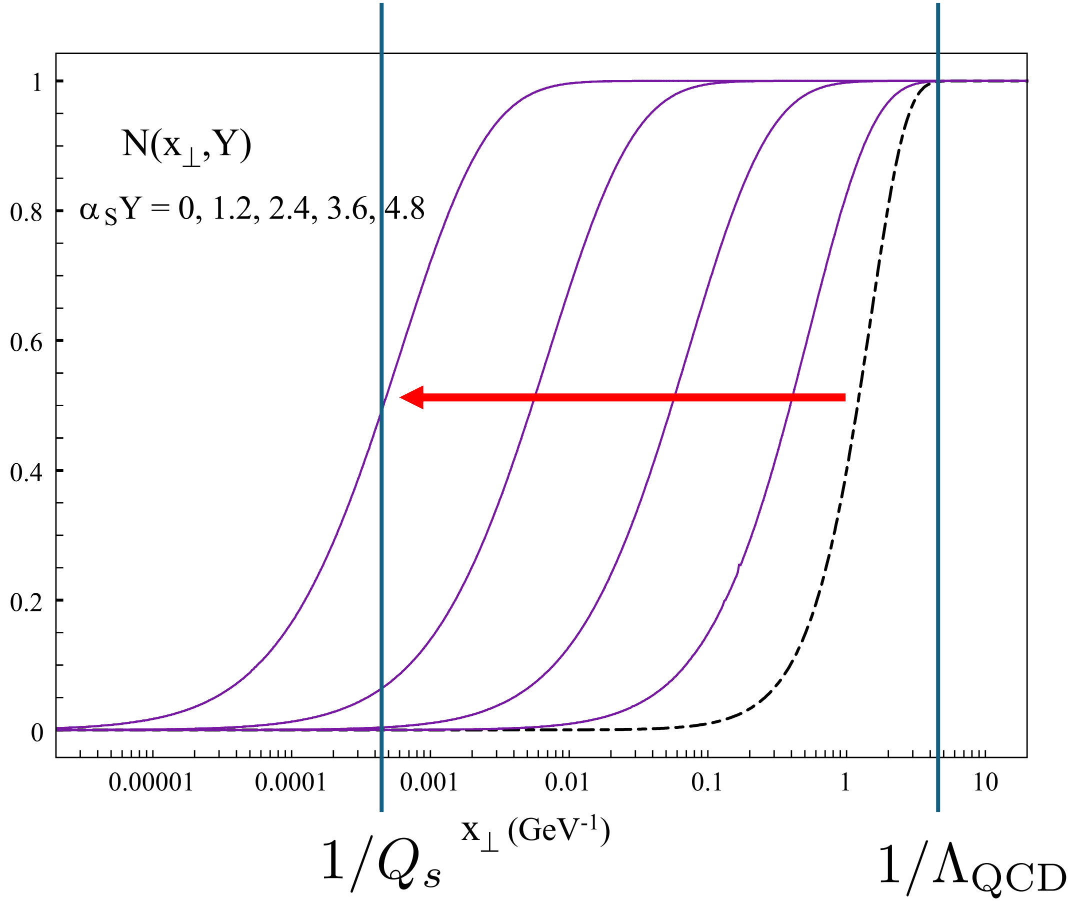
C. Marquet, G. Soyez arXiv:hep-ph/0504080
C. Marquet, R. Peschanski, G. Soyez arXiv:hep-ph/0509074
S. Munier, R. Peschanski arXiv:hep-ph/0310357 - arXiv:hep-ph/0309177

Numerical Solution of BK equation

We conclude that

$$Q_s^2 \sim \left(\frac{1}{x}\right)^{\lambda}$$

Energy increases → Q_s increases moving further away from Λ_{QCD}



numerical solution
by J. Albacete '03

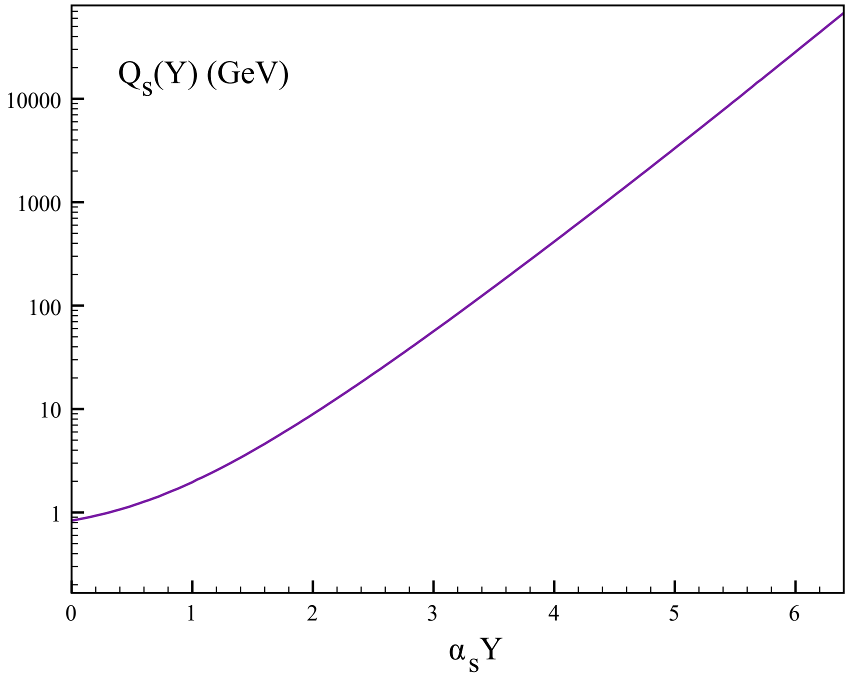
Albacete and V. Kovchegov 2007
Lappi et all. 2012- 2015

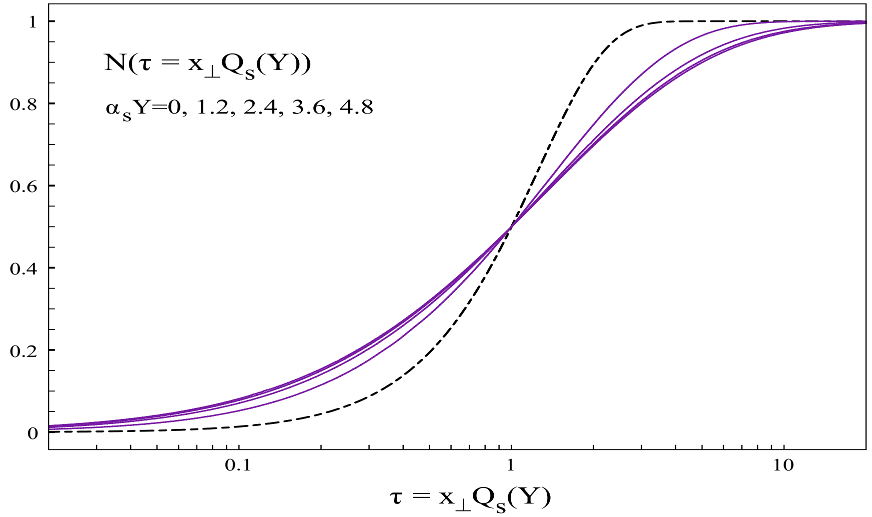
BK solution preserves the black disk limit, N < 1 always

$$\sigma^{q\bar{q}A} = 2 \int d^2b \, N(x_\perp, b_\perp, Y)$$

Saturation scale

Geometric Scaling





- One can see that, at small $x_\perp \ll \frac{1}{Q_s}$, we have N $\sim$ 0
- at large dipole sizes $x_\perp > \frac{1}{Q_s}$, the growth stops and the amplitude levels off (saturates) at N = 1.

J. Albacete (ca. 2006)

- The transition happens at around $x_\perp \sim \frac{1}{Q_s}$.

Analytical Solutions to the BK equation in coodinate space

Levin – Tuchin (2000) Solutions BK eq. inside the saturation region: $Q_s(Y) \gg Q$

$$\frac{\partial(N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2\,\frac{x_{02}^2}{x_{12}^2\,x_{01}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y)\; - N(x_{12},b,Y) * N(x_{20},b,Y)\right)$$

If N$(r_\perp,b,Y) = 1$ - $\Delta(r_\perp,Y,b)$ and $r_\perp = |x_{10}| \gg \frac{1}{Q_s(Y)}$, then we can find a solution with the condition that:

$x_{12}^2 \sim x_{10}^2$ *and* $x_{10}^2 > x_{20}^2 \gg 1/Q_s^2$ one of the daughter dipole is smaller

$$\frac{\partial\Delta(r_\perp,Y,b)}{\partial Y} = -\frac{\bar{\alpha}}{2\,\pi}\int d^2x_{20}\,\frac{x_{10}^2}{x_{12}^2\,x_{20}^2}\,\Delta(x_{10},b,Y) = -\bar{\alpha}\,z\,\Delta(r_\perp,Y,b)$$

where

$$z = \ln r^2 Q_s^2(Y) = \ln\tau = \bar{\alpha}\,\kappa\,Y + \ln r^2 Q_s^2(Y_0) = \bar{\alpha}\,\kappa\,Y + \xi_0 \qquad \tau = r^2 Q_s^2(Y)\,;\,Q_s^2(Y) = Q_{s0}^2 e^{\lambda Y}$$

$$\Delta_{LT}(r_\perp,Y,b) = C\,e^{-\frac{z^2}{2\kappa}}$$

we can find a solution with Geometric Scaling in the saturation region

E. Levin and K. Tuchin, Nucl. Phys. B 573, 833 (2000) hep-ph/9908317;
Nucl. Phys. A 691, 779 (2001) [hep-ph/0012167]; 693, 787 (2001) hep-ph/0101275

Our Proposal to find solution in the Saturation Region
Homotopic Aproach

- One can see that a numerical solution of the BK equation does not allow us to introduce explicit dependence on Qs(Y).
- we need to have the analytical solution in which we can see explicitly the dependence of the scattering amplitude on Qs(Y; b).
- we suggest a procedure to find the solution to the BK equation as a sequence of iterations, based on Homotopy approach

Homotopy Method: consider a nonlinear differential equation

$$\mathcal{L}[u] \quad + \quad \mathcal{N}_{\mathcal{L}}[u] = 0 \rightarrow \ \mathcal{K}[p,u] = \mathcal{L}[u] \quad + p\ \mathcal{N}_{\mathcal{L}}[u] = 0$$

$\mathcal{L}[u]$ $\quad linear\ \ part$ $\qquad$ $\mathcal{N}_{\mathcal{L}}[u]$=Nonlinear Integral differential Operator

And $\mathcal{K}[p,u]$ is the Homotopic Function with $p \in [0,1]$ small parameter. The solution is

$$u_p(r,Y;b) = u_0(r,Y;b) + pu_1(r,Y;b) + p^2u_2(r,Y;b) + \cdots$$

$\mathcal{L}[u_0] = 0 \quad give\ de\ solution\ from\ linear\ \ part\ \ and$

$$u_1 \quad from\ the\ nonlinear\ part, \qquad u_2\ from\ recursive\ equation.$$

The convergency of the Homotopic approach has been proved by J. He (1999) , but it need to check to each solution.

Saikia arXiv 2204.10111

Homotopy solution to the BK evolution for heavy nuclei

Phys.Rev.D 107 (2023) 9, 094030

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \int d^2x_2 \frac{r_{10}}{\mathrm{r}_{12}\mathrm{r}_{20}}\left(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y) - N(x_{12},b,Y) * N(x_{20},b,Y)\right)$$

Kinematic region

$$x_{10}^2 Q_s^2(Y) < 1 \quad pQCD$$

$$x_{10}^2 Q_s^2(Y) \approx 1 \;\; Vicinity\ Saturation \rightarrow N_{10'}(z) = C_1 \left(x_{10}^2 Q_s^2(Y)\right)^{\bar{\gamma}}$$

$$x_{10}^2 Q_s^2(Y) > 1 \;\; saturation \;\; \rightarrow N_{10'LT}(z) = 1 - C\, e^{-\frac{z^2}{2\kappa}}$$

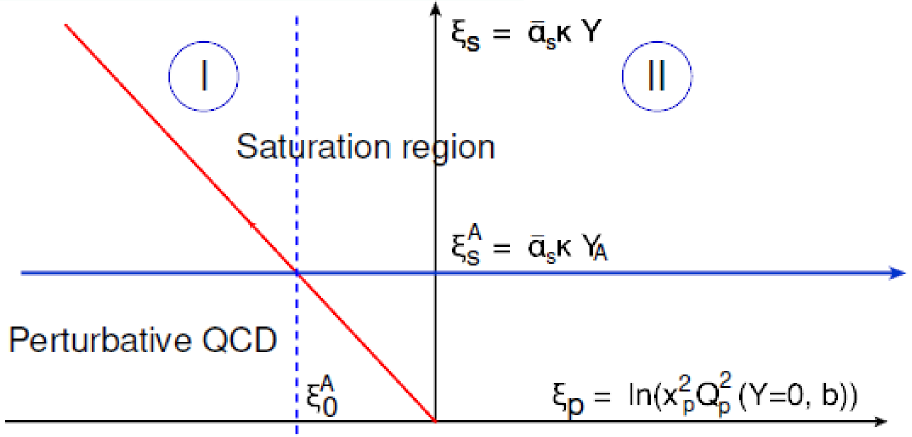
Variables:

$(\mathrm{r},\mathrm{Y}) \rightarrow\ \mathrm{z} = \ln\, r^2 Q_s^2(Y,b) = \bar{\alpha}\,\kappa\,(Y - Y_A) + \ln\, r^2 Q_s^2(Y_A,b)\,,$ $\qquad \ln(Q_{so}^2(Y_A,b)r^2) = \xi$

$Q_s^2(Y) = Q_{so}^2(Y = Y_A)\ e^{\bar{\alpha}\,\kappa\,(Y-Y_A)}$, and $\kappa = \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}$ and $\chi(\gamma_{cr})$ is the BFKL characteristic functon $\bar{\gamma} = 0.73$ $\kappa = 4.88$

$Y_A = \ln A^{\frac{1}{3}}$ where A is the number of nucleons in a nucleon

$N_I(z)$ $\qquad$ $N_{II}(z,\xi)$



Initial Condition: at Y_A for DIS with nuclei we consieder the McLerran-Venugopalan

$$N(x_{10},b,Y_A) = 1 - e^{-\frac{r^2 Q_s^2(Y=Y_A)}{4}} = 1 - e^{-\frac{1}{4}e^{\xi}}$$

Boundary Condition: at $x_{10}^2 Q_s^2(Y) = 1,\ \ z = 0\,,$ we have the following boundary condition

$$\mathrm{N}_{01}(z = 0) = N_0 \quad \text{and} \quad \frac{\mathrm{dN}_{01}(z)}{dz}\Big|_{z=0} = \bar{\gamma}\, N_0$$

BK in momentum representation

Transformation to Momentum space

$$N\left(r^2, b, Y\right) = r^2 \int \frac{d^2 k_\perp}{2\pi}\, e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{r}}\, \tilde{N}\left(k_\perp, b, Y\right) = r^2 \int_0^\infty k_\perp dk_\perp J_0\left(r k_\perp\right) \tilde{N}\left(k_\perp, b, Y\right)$$

$$\tilde{N}(k_\perp, b, \delta\tilde{Y}) = \int_0^\infty r dr \frac{J_0(r k_\perp)}{r^2} N(r, b, \delta\tilde{Y}) \qquad \frac{\partial \tilde{N}(k_\perp, b, \delta\tilde{Y})}{\partial k_\perp} = -\int_0^\infty dr J_1(r k_\perp) N(r, b, \delta\tilde{Y})$$

BK Momentum space

$$\frac{\partial \tilde{N}\left(k_\perp, b, Y\right)}{\partial Y} = \bar{\alpha}_S \left\{ \chi\left(-\frac{\partial}{\partial \tilde{\xi}}\right) \tilde{N}\left(k_\perp, b, Y\right) - \tilde{N}^2\left(k_\perp, b, Y\right) \right\}$$

$\lambda \to -\frac{\partial}{\partial \tilde{\xi}}$ is a differential operator acting on N(Y; k), which is a way of writing the integral kernel of the BK equation in coordinate space as the differential operator in the momentum space after a Fourier transform.

$$\tilde{z} = \ln(Q_s^2(Y,b)/k^2) = \bar{\alpha}\,\kappa\,(Y - Y_A) + \ln(Q_s^2(Y_A,b)/k^2) = \bar{\alpha}\,\kappa\,(Y - Y_A) + \tilde{\xi}_0$$

$$\frac{\partial^2 \tilde{N}\left(k_\perp, b, Y\right)}{\partial Y\, \partial \tilde{\xi}} = \bar{\alpha}_S \left\{ \chi_0\left(-\frac{\partial}{\partial \tilde{\xi}}\right) \frac{\partial \tilde{N}\left(k_\perp, b, Y\right)}{\partial \tilde{\xi}} + \tilde{N}\left(k_\perp, b, Y\right) - 2\frac{\partial \tilde{N}\left(k_\perp, b, Y\right)}{\partial \tilde{\xi}} \tilde{N}\left(k_\perp, b, Y\right) \right\}$$

$$\chi_0(\gamma) = \chi(\gamma) - \frac{1}{\gamma}$$

$$\frac{\partial \tilde{N}\left(k_\perp, b, Y\right)}{\partial \tilde{z}} = \frac{1}{2} + M\left(\tilde{z}, b, \delta\tilde{Y}\right) \quad \text{or} \quad \tilde{N}\left(\tilde{z}, b, \delta\tilde{Y}\right) = \frac{1}{2}\tilde{z} + \int_0^{\tilde{z}} d\tilde{z}' M\left(\tilde{z}', b, Y\right) + \Phi\left(\delta\tilde{Y}\right)$$

$$\delta\tilde{Y} = \overline{\alpha}_S\left(Y - Y_A\right)$$

we introduce a new function M, which approaches zero at large $\tilde{z}$, since 1/2 is the correct asymptotic behavior in the momentum representation. It corresponds to the amplitude in the coordinate representation, which is equal to unity deep in the saturation region.

In terms of the new variable

$$\frac{\partial M\left(\tilde{z},b,\delta\tilde{Y}\right)}{\partial\delta\tilde{Y}} + \kappa\,\frac{\partial M\left(\tilde{z},b,\delta\tilde{Y}\right)}{\partial\tilde{z}} = \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right)\,M\left(\tilde{z},b,\delta\tilde{Y}\right) - \tilde{z}\,M\left(\tilde{z},b,\delta\tilde{Y}\right) - 2\,\Phi\left(\delta\tilde{Y}\right)\,M\left(\tilde{z},\delta\tilde{Y}\right) - \frac{\partial A}{\partial\tilde{z}}$$

BK Momentum space

$$A(\tilde{z},b,\delta\tilde{Y}) = \left(\int_0^{\tilde{z}} M(z',b,\delta\tilde{Y})dz'\right)^2 \qquad M(\tilde{z},b,\delta\tilde{Y}) = k_\perp\int\limits_0^\infty dr J_1(rk_\perp)\left\{\frac{N(r,b,\delta\tilde{Y})\;-\;1}{2}\right\}$$

BK in momentum space Homotopic structure

$$\mathscr{L}[F] = \frac{\partial F}{\partial\tilde{Y}} + \kappa\frac{\partial F}{\partial\tilde{z}} - \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right)F + \tilde{z}F + 2\,\Phi\left(\delta\tilde{Y}\right)\,F\,, \quad \mathscr{N}_{\mathscr{L}}[F] = \frac{\partial}{\partial\tilde{z}}\left(\int\limits_0^{\tilde{z}} F dz'\right)^2$$

$$F\;\equiv\;M_p(\tilde{z},b,\delta\tilde{Y}) = \sum_{n=0}^{\infty} M_n(\tilde{z},b,\delta\tilde{Y})\,p^n$$

Homotopic structure to order n

$$\frac{\partial M_n}{\partial\tilde{Y}} + \kappa\frac{\partial M_n}{\partial\tilde{z}} = \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right)M_n - (\tilde{z}\;+\;\zeta)\,M_n - 2\sum_{j=0}^{n-1}\Phi_{n-j}\left(\delta\tilde{Y}\right)\,M_j - \frac{\partial A_{n-1}}{\partial\tilde{z}}$$

Homotopic structure order p = 0 in Saturation Region

$$\frac{\partial M_0}{\partial\,\delta\tilde{Y}} + \kappa\frac{\partial M_0}{\partial \tilde{z}} = \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right) M_0 - (\tilde{z}+\zeta)M_0$$

Initial condition in Saturation Region

$$M_0\left(\tilde{z},\delta\tilde{Y}\right) = \begin{cases} M_0^I\left(\tilde{z}\right) & \text{for } -\kappa\delta\tilde{Y} < \tilde{\xi} < \tilde{\xi}_0^A; \\ M_0^{II}\left(\tilde{z},\delta\tilde{Y}\right) & \text{for } \tilde{\xi} > \tilde{\xi}_0^A; \end{cases}$$

Saturation Region I

$$\kappa\frac{dM_0\left(\tilde{z},b\right)}{d\tilde{z}} = \chi_0\left(\frac{d}{d\tilde{z}}\right) M_0\left(\tilde{z},b\right) - \left(\tilde{z}+\zeta\right) M_0\left(\tilde{z},b\right)$$

Geometric Scaling solution $M_0^I(\tilde{z})$

Mellin Transformations

$$M_0\left(\tilde{z},b\right) = \int\limits_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}e^{\gamma(\tilde{z}+\zeta)}m_0\left(\gamma,b\right)$$

$$M_0^I\left(\tilde{z}\right) = \int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}e^{\gamma(\delta\tilde{Y}+\zeta)}\,m(0)\left(\frac{k_\perp}{Q_s(Y_A,b)}\right)^{-2\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)}e^{\kappa\gamma^2/2}e^{-2\psi(1)\gamma}$$

Coordinate representation

$$N_0^I(z) = 1 + \frac{4m(0)}{\sqrt{2\pi\kappa}}\exp\left(-\frac{(z+\zeta-z_0)^2}{2\kappa}\right)$$

$$z_0 = 2(\psi(1)+\ln 2)$$

Saturation solution $M_0^{II}(\tilde{z},\delta\tilde{Y})$

$$\frac{\partial M_0}{\partial\,\delta\tilde{Y}} + \kappa\frac{\partial M_0}{\partial\tilde{z}} = \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right)M_0 - (\tilde{z}+\zeta)M_0$$

$$M_0^{II}\left(\tilde{z},\delta\tilde{Y}\right) = \int\limits_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}e^{(\tilde{z}+\zeta)\,\gamma}m_0^{II}\left(\gamma,\delta\tilde{Y}\right) \quad\rightarrow\quad \frac{\partial m_0^{II}}{\partial\delta\tilde{Y}} + (\kappa\gamma - \chi(\gamma) + 1/\gamma)m_0^{II} - \frac{\partial m_0^{II}}{\partial\gamma} = 0.$$

$$M_0^{II}\left(\tilde{z},b\right) = f_1\left(\delta\tilde{Y}\right)\int\limits_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}u_1^{-\gamma}\left(-\frac{1}{2}\frac{\Gamma\left(1+\gamma\right)}{\Gamma\left(1-\gamma\right)}\Gamma\left(-(\gamma+\delta\tilde{Y})\right)\right)$$

$$f_1\left(\delta\tilde{Y}\right) = \exp\left(-\frac{1}{2}\kappa(\delta\tilde{Y})^2 - \zeta\delta\tilde{Y} + 2\psi(1)\,\delta\tilde{Y}\right)$$

$$N_0^{II}(r,\delta\tilde{Y}) = 1 - f_1(\delta\tilde{Y})\left(\frac{r^2Q_s^2(Y_A,b)}{4}\right)^{-\delta\tilde{Y}}\exp\left(-\frac{r^2Q_s^2(Y_A,b)}{4}\right)$$

Matching $\xi = \xi_0^A$

$$\left.\frac{dN_0\,(z)}{d\,z}\right|_{z=0} = \bar{\gamma}\,N_0$$

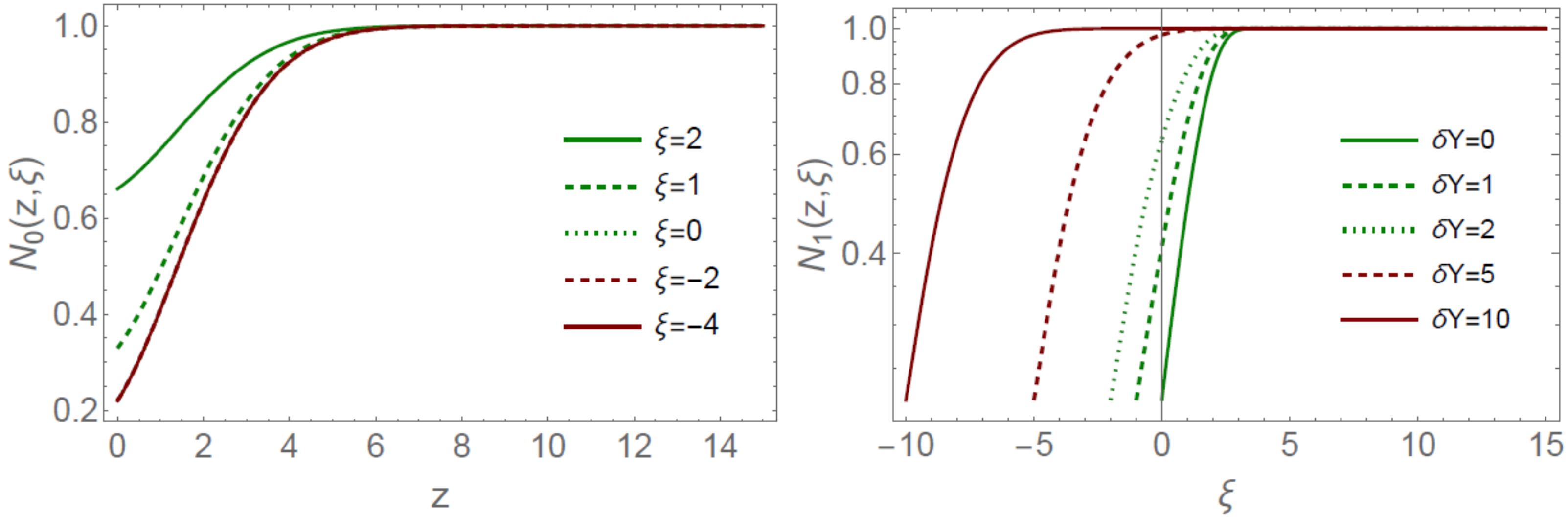
$$N_0^I\left(z = \kappa\delta\tilde{Y} + \xi_0^A\right) = N_0^{II}(\xi = \xi_0^A, z = \kappa\delta\tilde{Y} + \xi_0^A)$$

Coordinate representation:

$$N_0^{II}(r, b, \delta\tilde{Y}) = 1 - \exp\left\{-\frac{\Phi(z,\xi,b)}{2\kappa}\right\}$$

$$\Phi(z,\xi,b) = (z + \zeta - z_0)^2 - (\xi + \zeta - z_0)^2 + \kappa\frac{e^{\xi}}{2}$$

$$N_0^{II}(\xi, z) = 1 - G_0(\xi)\exp\left\{-\frac{(z+\zeta-z_0)^2}{2\kappa}\right\} \qquad G_0(\xi) = \exp\left\{\frac{(\xi+\zeta-z_0)^2}{2\kappa} - \frac{e^{\xi}}{4}\right\}$$



a

·b

Fig. -a: $N_0(z,\xi)$ versus z at fixed values of $\tilde{\xi}$. Fig. b: $N_0(z,\xi)$ versus ξ at fixed $\delta\tilde{Y}$ $(z = \kappa\delta\tilde{Y} + \xi)$. $N_0 = 0.22$; $\bar{\gamma} = 0.63$, $\zeta - z_0 = 0.867$, $C_1 = 0.842$, $\xi_0^A = 0$.

Homotopic structure order p = 1 in Saturation Region

$$\frac{\partial M_1}{\partial \delta\tilde{Y}} + \kappa\frac{\partial M_1}{\partial \tilde{z}} = \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right)M_1 - (\tilde{z}+\zeta)M_1 - 2\,\Phi_1\left(\delta\tilde{Y}\right)\,M_0 - \frac{\partial A_0}{\partial\tilde{z}}$$

$$A_0(\tilde{z},\delta\tilde{Y},b) = \left(\int_0^{\tilde{z}} M_0(z',\delta\tilde{Y})dz'\right)^2$$

Initial Condition in Saturation Region

$$\text{region } I:\ M_1(\tilde{z}=\tilde{\xi}_0^A) = 0; \quad \text{region } II:\ M_1(\tilde{z},b,\delta\tilde{Y}=0) = 0.$$

Region I

$$\kappa\frac{dM_1^I}{d\tilde{z}} = \chi_0\left(\frac{d}{d\tilde{z}}\right)M_1^I - (\tilde{z}+\zeta)M_1^I - \frac{dA_0^I}{d\tilde{z}}, \quad \text{with}$$

$$\frac{dA_0^I}{d\tilde{z}} = 2M_0^I\int_0^{\tilde{z}} M_0^I(z')dz'.$$

$$M_1^I(\tilde{z}) = \int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}e^{(\tilde{z}+\zeta)\gamma}m_1^I(\gamma,\delta\tilde{Y}),$$

$$\frac{dA_0^I}{d\tilde{z}} = \int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}e^{(\tilde{z}+\zeta)\gamma}a_0(\gamma)$$

$$M_1^I(\tilde{z}) = \frac{1}{2}\int_0^\infty dr' J_1(r'k_\perp)N_1^I(r')$$

$$N_1^I(z) \sim -\frac{c_1^I(\zeta)\,m(0)}{2\kappa\sqrt{2\pi\kappa}}\left(z - z_0^A\right)\Delta_{01}(z)$$

$$\Delta_{01}(z) = e^{-\frac{1}{2\kappa}(z+\zeta-z_0)^2}$$

$$\frac{\partial M_1}{\partial \delta\tilde{Y}} + \kappa\frac{\partial M_1}{\partial \tilde{z}} = \chi_0\left(\frac{\partial}{\partial \tilde{z}}\right) M_1 - (\tilde{z}+\zeta)M_1 - 2\,\Phi_1\left(\delta\tilde{Y}\right)\, M_0 - \frac{\partial A_0}{\partial \tilde{z}}$$

$$A_0(\tilde{z}, \delta\tilde{Y}, b) = \left(\int_0^{\tilde{z}} M_0(z', \delta\tilde{Y})dz'\right)^2$$

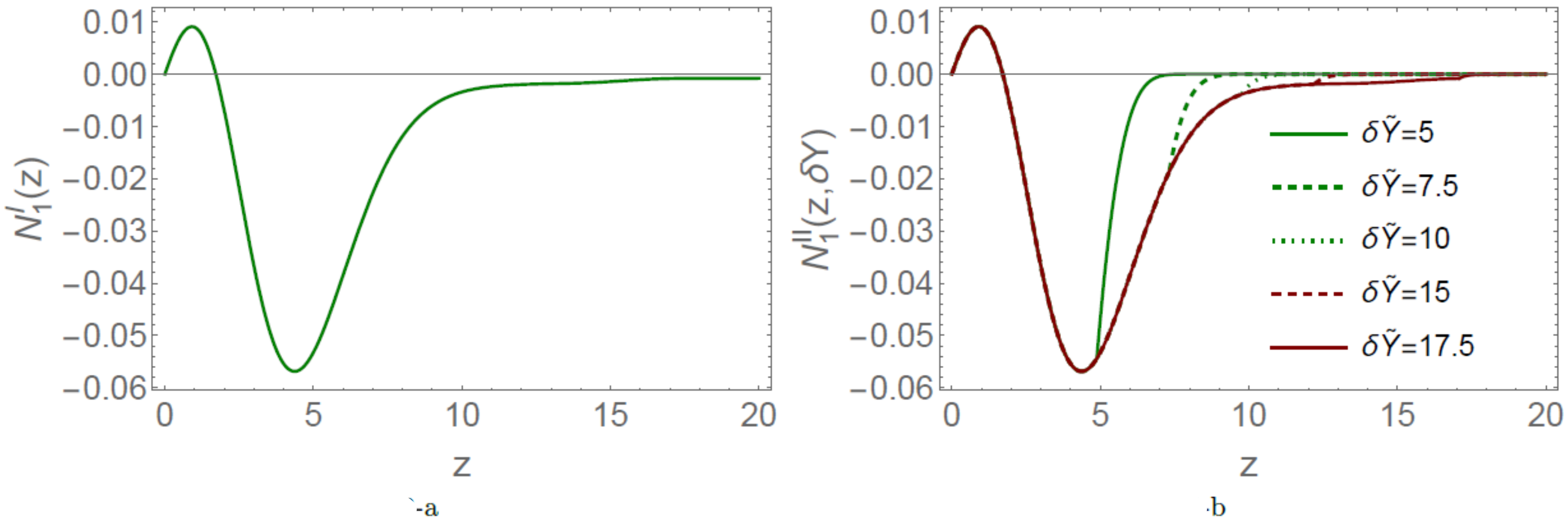
Region II

$$\text{region}\, I:\; M_1(\tilde{z} = \tilde{\xi}_0^A) = 0; \quad \text{region}\, II:\; M_1(\tilde{z}, b, \delta\tilde{Y} = 0) = 0.$$

$$N_1^{II}(z, \delta\tilde{Y}) = h(\delta Y)\,\exp\left(-\frac{1}{2}(z+\zeta-z_0)\delta\tilde{Y}\right)\exp\left(-\frac{1}{2}(\xi+\zeta-z_0)\delta\tilde{Y}\right)\exp\left(-\frac{r^2Q_s^2(Y_A,b)}{4}\right)(1+o(1))$$

$$h(\delta\tilde{Y}) = -\int_0^{\delta\tilde{Y}} d\delta\tilde{Y}'\tilde{\Phi}_1^{II}(\delta\tilde{Y}').$$

Numerical Estimates



a: $N_1^I\,(z)$ versus z. 3-b: $N_1\,(z, \delta Y)$ versus z at fixed $\delta\tilde{Y} = \bar{\alpha}_S(Y - Y_A)$. The value of $\xi_0^A = 0$.

From the analytical solution as well as from the numerical estimates in this Fig. one can see that the geometric scaling behavior preserves only in region I while in region II we see large scaling violation.

Summary at first calculation

We developed the homotopy approach for solving the non-linear evolution Balitsky-Kovchegov equation in the saturation region

First, we solved the linearized version of the BK equation in the momentum space deep in the saturation region. We found that this solution has the geometric scaling behavior for $\xi < \xi_0^A$

For $\xi > \xi_0^A$ we observe the violation of the geometric scaling behaviour in the saturation region.

This solution satisfies the boundary and initial conditions which are given perturbative QCD approach for $r\,Q_s < 1$ and by McLerran-Venugopalan for $Y = Y_A$.

The numerical part of the calculations is expressed through well converged integrals and can be easily estimated.

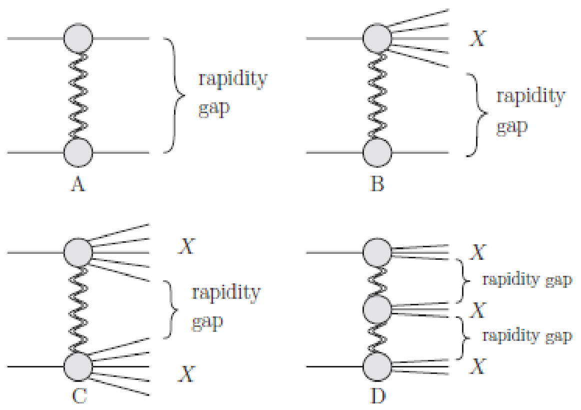
We found a way to study the BK equation in Momentum or coordinated representation, which can be extended to another process.

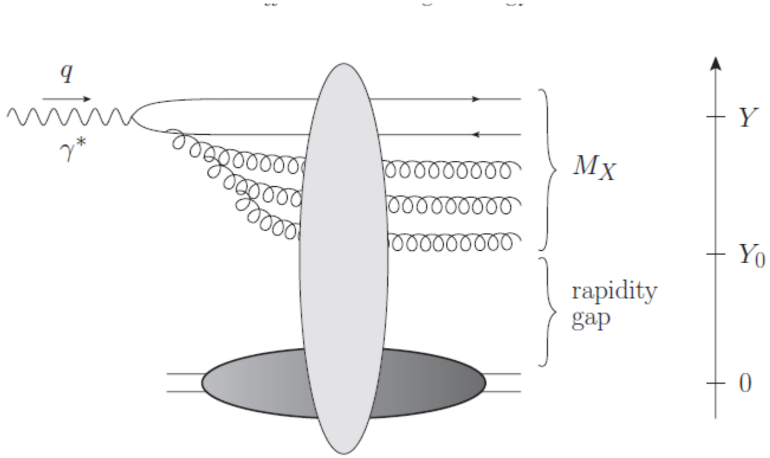
Applications:

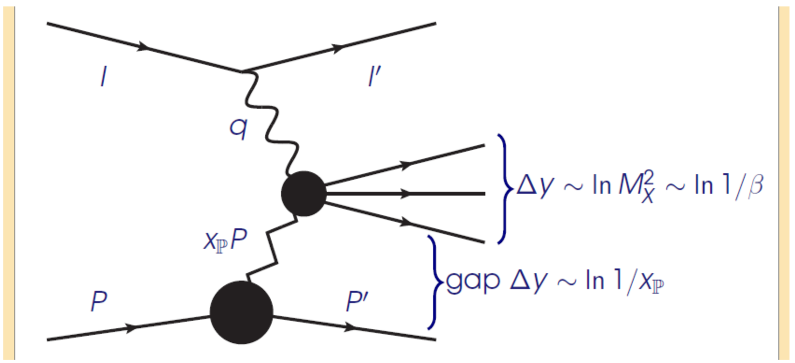
Diffractive Scattering
BK Equation with Running coupling constant

Diffraction at high energy

Single Diffraction







See T. Lappi talk

cross section of diffractive production with the rapidity gap larger than Y_0

$$\sigma^{\text{diff}}(Y, Y_0, Q^2) \;=\; \int d^2 r_\perp \int dz \, |\Psi^{\gamma^*}(Q^2; r_\perp, z)|^2 \, \sigma^{diff}_{\text{dipole}}(r_\perp, Y, Y_0)$$

where

$$\sigma^{diff}_{dipole}(r_\perp, Y, Y_o) = \int d^2 b \, N^D(r_\perp, Y, Y_0, b)$$

Evolution equation for Diffraction at high energy

Kovchegov-Levin (KL) equation (2000)

$$\frac{\partial N^D\left(Y, Y_0, r_{10}; b\right)}{\partial Y_M} =$$

$$\frac{\bar{\alpha}_S}{2\pi} \int d^2 r_2\, K\left(r_{10} | r_{12}, r_{02}\right) \left\{ N^D\left(Y, Y_0, r_{12}; b\right) + N^D\left(Y, Y_0, r_{20}; b\right) - N^D\left(Y, Y_0, r_{10}; b\right) \right.$$

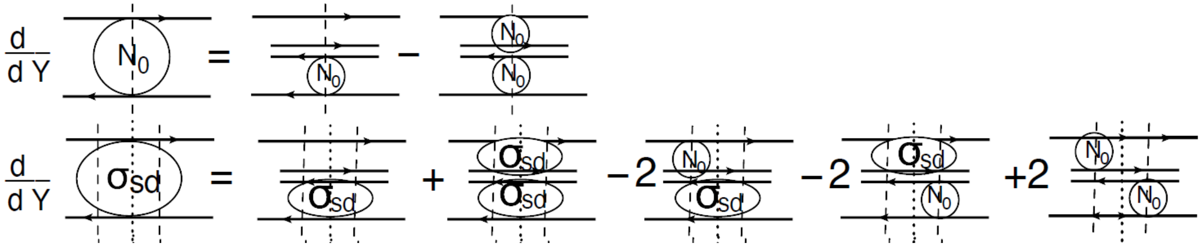
$$+ N^D(Y; Y_0, r_{12}; b) N^D(Y; Y_0, r_{20}; b) - 2\, N^D(Y; Y_0, r_{12}; b)\, N(Y; r_{20}; b)$$

$$\left. - 2\, N(Y; r_{12}; b) N^D(Y; Y_0, r_{20}; b) + 2 N(Y; r_{12}; b) N(Y; r_{20}; b) \right\}$$

N^D the evolution equation has been derived by Kovchegov and Levin in the leading log(1/x) approximation (LLA) of perturbative QCD

- This is a nonlinear evolution equation with the initial condition specified at $Y = Y_0$
- To solve it one has first to solve the BK equation to find the dipole amplitude N
- For this equation: N^D no analytic solution exists.
- We will use the Homotopic approach and the previous solution for N which satisfice the BK equation.

$$\bar{\alpha}_s = \frac{\alpha_s N_c}{\pi}$$



Diffraction at high energy

Introducing a new variables:

$$\boxed{\mathscr{N}(z,\delta\tilde{Y},\delta Y_0) = 2\,N(z,\delta\tilde{Y}) - N^D(z,\delta\tilde{Y},\delta Y_0)}$$

with the Initial Condition

$$\boxed{N^D(r,Y=Y_0,Y_0) = N^2(r,Y=Y_0)}$$

$$\frac{\partial \mathscr{N}_{01}}{\partial Y} = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi}\,\frac{x_{01}^2}{x_{02}^2\,x_{12}^2}\left\{\mathscr{N}_{02} + \mathscr{N}_{12} - \mathscr{N}_{02}\mathscr{N}_{12} - \mathscr{N}_{01}\right\}$$

$$\boxed{\boldsymbol{N(r_{10},Y) \rightarrow N(\xi,Y) \rightarrow N(z,Y)}}$$

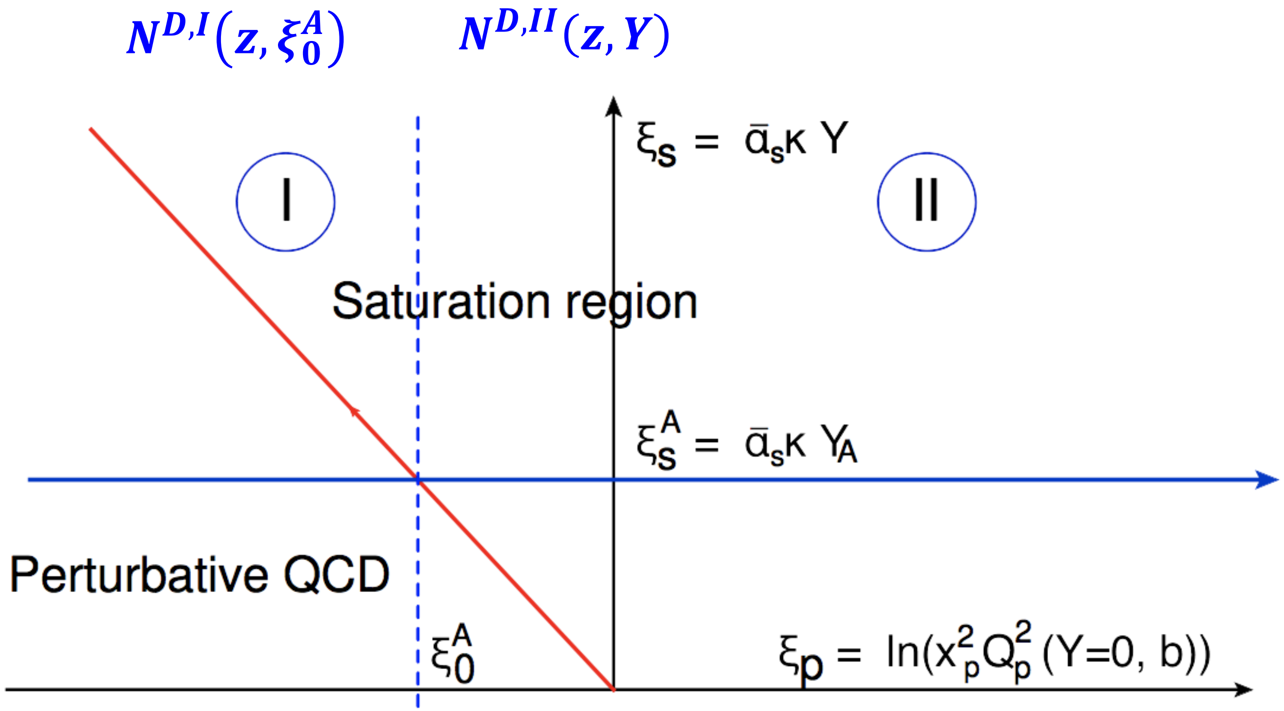
,

,

$$z = \ln\left(r^2\,Q_s^2\left(\delta\tilde{Y},b\right)\right) = \bar{\alpha}_S\,\kappa\,(Y - Y_A) + \xi \qquad \xi = \ln\left(Q_s^2\left(\delta\tilde{Y}=0\right)\,r^2\right)$$

$$Q_s^2\,(Y,b) = Q_s^2\,(Y=Y_A,b)\;e^{\bar{\alpha}_S\,\kappa(Y-Y_A)}$$

This equation takes the form of the Balitsky-Kovchegov (BK) equation



Saturation region of QCD for the scattering amplitude. The critical line (z=0) is shown in red.

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Diffraction at high energy

replacing $\mathcal{N}\left(z,\delta Y_0\right)$ by $\mathcal{N}\left(z,\delta Y_0\right) \;=\; 1 \;-\; \Delta^D\left(z,\delta Y_0\right)$

we get

$$\frac{\partial \Delta^D_{01}}{\partial Y} \;=\; \bar{\alpha}_S \int \frac{d^2\, x_{02}}{2\pi}\, \frac{x^2_{01}}{x^2_{02}\, x^2_{12}} \left\{ \Delta^D_{02} \Delta^D_{12} - \Delta^D_{01} \right\}$$

with initial conditions

$$\mathcal{N}(z \to z_0, \delta\tilde{Y} = \delta Y_0, \delta Y_0) \;=\; 2\, N(z_0, \delta Y_0) \;-\; N^2(z_0, \delta Y_0) \;= 1 - N^2(z_o, \delta Y_0)$$

$$\text{Region I}: \quad \Delta^D_{01}\left(z \to z_0, \delta Y_0\right) \;=\; C^2 \exp\left(-\frac{\left(z_0 - \tilde{z}\right)^2}{\kappa}\right)$$

$$\text{Region II}: \quad \Delta^D_{01}\left(z \to z_0, \delta\tilde{Y} \to \delta Y_0, \delta Y_0\right) = 1 \;-\; N_{in} \;=\; G^2\left(\xi\right) \exp\left(-\frac{\left(z_0 - \tilde{z}\right)^2}{\kappa}\right)$$

with $\quad G\left(\xi\right) \;= \exp\left(\frac{\left(\xi - \tilde{z}\right)^2}{2\,\kappa} \;-\; \frac{1}{4} e^{\xi}\right)$

and $\tilde{z} = 2(\ln 2 + \psi(1)) \;-\; \zeta$

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Modified homotopy approach

$$\frac{\partial \Delta^D_{01}}{\partial Y} = \bar{\alpha}_S \int \frac{d^2\, x_{02}}{2\pi} \frac{x^2_{01}}{x^2_{02}\, x^2_{12}} \left\{ \Delta^D_{02} \Delta^D_{12} - \Delta^D_{01} \right\}$$

$$\mathscr{L}[u] + \mathscr{N}_{\mathscr{L}}[u] = 0$$

The homotopy method we can use for the general equation:

$$\mathscr{H}\,(p,u) \;=\; \mathscr{L}[u_p] \,+\, p\,\mathscr{N}_{\mathscr{L}}[u_p] \;=\; 0$$

Solving, we reconstruct the function

$$u_p\,(Y,\boldsymbol{x}_{10},\boldsymbol{b}) \;=\; u_0\,(Y,\boldsymbol{x}_{10},\boldsymbol{b}) \;+\; p\,u_1\,(Y,\boldsymbol{x}_{10},\boldsymbol{b}) \,+\, p^2\,u_2\,(Y,\boldsymbol{x}_{10},\boldsymbol{b}) \;+\; \dots$$

We suggest to simplify the non-linear term replacing it by

$x^2_{12} \sim x^2_{10}$ *and* $x^2_{10} > x^2_{20} \gg 1/Q^2_s$ small dipole aproximation

$$\bar{\alpha}_S \int \frac{d^2\, x_{02}}{2\pi} \frac{x^2_{01}}{x^2_{02}\, x^2_{12}} \Delta^D_{02} \Delta^D_{12} \;\to\; \Delta^D_{01} \int\limits_0^z dz' \Delta^D_{02} \;=\; \Delta^D_{01} \left(\zeta - \int\limits_z^\infty dz' \Delta^D_{02} \right) \quad \text{with} \;\; \zeta = \int\limits_0^\infty dz' \Delta^D_{02}$$

with $\zeta = \int\limits_0^\infty dz' \Delta^D_{02}$

We modify the homotopic approach considering:

$$\mathscr{L}\left(\Delta_0^D\right) = \left(\frac{\partial}{\partial \tilde{Y}} + z - \zeta\right) \Delta_0^D + \Delta_0^D\left(z, \tilde{Y}, z_0\right) \int_z^{\infty} dz' \Delta_0^D\left(z', \delta\tilde{Y}, z_0\right)$$

$$\mathscr{N}_{\mathscr{L}}[\Delta^D] = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \frac{x_{01}^2}{x_{02}^2\, x_{12}^2} \Delta_0^D\left(x_{02}\right) \Delta_0^D\left(x_{12}\right) - \Delta_0^D \int^{x_{01}^2} \frac{dx_{02}^2}{x_{02}^2} \Delta_{02}^D$$

The first iteration (p = 0) gives

$$\mathscr{L}\left(\Delta_0^D\right) = 0; \quad \left(\frac{\partial}{\partial \tilde{Y}} + z - \zeta\right) \Delta_0^D + \Delta_0^D\left(z, \tilde{Y}, z_0\right) \int_z^{\infty} dz' \Delta_0^D\left(z', \delta\tilde{Y}, \boldsymbol{z}_0\right) = 0;$$

We need to solve $\boldsymbol{\Delta_0^D}$

Traveling Wave solution

Munier and Peschanski 2003

$$\kappa \frac{d\mathcal{N}_{01}\left(z,\xi_s\right)}{d\xi_s} = \left(1-\mathcal{N}_{01}\left(\xi,\xi_s\right)\right)\int\limits_{-\xi_s}^{\xi} d\xi'\,\mathcal{N}_{02}\left(\xi',\xi_s\right); \quad \text{with} \quad \xi_s = \kappa\,\delta\tilde{Y}$$

Introducing $\Delta^{(0)}\left(z,\xi_s\right) = 1-\mathcal{N}_{01}\left(z,\xi_s\right) = \exp\left(-\Omega^{(0)}\left(z,\xi_s\right)\right)$ we obtain:

$$\frac{\partial^2\Omega^{(0)}\left(z,\xi_s\right)}{\partial z^2} - \frac{\partial^2\Omega^{(0)}\left(z,\xi_s\right)}{\partial t^2} = \frac{1}{\kappa}\left(1-e^{-\Omega^{(0)}\left(z',\xi_s\right)}\right)$$

where $z = \xi_s + \xi$ and $t = \xi_s - \xi$.

Region I: Geometric Scaling $\Omega(z,\xi) \rightarrow \Omega(z)$

$$\kappa\frac{\partial^2\Omega^{(0)}\left(z,\xi_s\right)}{\partial\xi_s\,\partial z} = 1-e^{-\Omega^{(0)}\left(z,\xi_s\right)}$$

$$\frac{d\Omega^{(0)}\left(z\right)}{dz} = p\left(\Omega^{(0)}\right) \qquad \frac{1}{2}\kappa\frac{dp^2}{d\Omega^{(0)}} = 1-e^{-\Omega^{(0)}(z)}$$

$$p = \frac{d\Omega^{(0)}}{dz} = \sqrt{\frac{2}{\kappa}\left(\Omega^{(0)} + \exp\left(-\Omega^{(0)}\right) - 1\right) + C_1}$$

(region I)

we solve this, getting the following implicit solution

$$\int_{\Omega_0^{(0)}}^{\Omega^{(0)}} \frac{d\Omega'}{\sqrt{\Omega' + \exp(\Omega') - \Omega_0}} = \sqrt{\frac{2}{\kappa}}(z - \tilde{z})$$

We define

$$\Omega_0^{(0)} = (z_0 - \tilde{z})^2 / (2\,\kappa) \;-\; 2\ln(C)$$

with

$$\mathscr{U}\left(\Omega^{(0,I)}, \Omega_0^{(0)}\right) \;=\; \sqrt{\frac{2}{\kappa}}\,(z - \tilde{z})$$

For finding $\Omega^{(0,I)}(z)$ in a general case we have to find the inverse function for $\mathscr{U}$

Then we obtain the following result

$$\Delta_0^{(0,I)}(z) \;=\; \Delta_{LT}(z)\exp\left(-a + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-1)!!}{2^k k\, k!}\left(\frac{2\kappa}{(z-\tilde{z})^2}\right)^{k+\frac{1}{2}} \Delta_{LT}^k(z)\right)$$

$$\text{where } \Delta_{LT}(z) \;=\; \exp\left(-\frac{(z-\tilde{z})^2}{2\,\kappa}\right)$$

Region II: General solution $\Omega(z,\xi)$

$$\frac{\partial^2\Omega^{(0)}\left(z,\xi_s\right)}{\partial z^2}-\frac{\partial^2\Omega^{(0)}\left(z,\xi_s\right)}{\partial t^2}=\frac{1}{\kappa}\left(1-e^{-\Omega^{(0)}\left(z',\xi_s\right)}\right)$$

where $z=\xi_s+\xi$ and $t=\xi_s-\xi$.

Solving this equation, we obtain

$$\int\limits_{\Omega_0}^{\Omega^{(0)}(z,\xi_s)}\frac{d\Omega'}{\sqrt{-\Omega_0\;+\;\Omega'+\exp\left(-\Omega'\right)}}=\sqrt{\frac{2}{\kappa}}\left(\left(1+\nu\right)\;z\;+\;\nu\,t\;-\;\tilde{z}-2\nu\,\xi_{0,s}\right)$$

where $\xi_{0,s}=\kappa\delta Y_0$ and $\Omega_0\;=\;\frac{\left(z_0-\tilde{z}\right)^2}{2\,\kappa}-\frac{\left(z_0-\xi_{0,s}-\tilde{z}\right)^2}{\kappa}+\frac{1}{2}\exp\left(z_0-\xi_{0,s}\right)$

$$\mathscr{U}\left(\Omega^{(0,II)},\Omega_0\right) = \sqrt{\frac{2}{\kappa}}\left(\left(1+\nu\right) z + \nu\, t - \hat{z}\right) \qquad \text{where } \hat{z} = \tilde{z} + 2\nu\xi_{0,s}$$

We repeat the same steps of before an obtain:

$$\Delta_0^{(0,II)}\left(z,\delta\tilde{Y}\right) = \tilde{\Delta}_{LT}\left(z,\xi_s\right)\exp\left(-\Omega_0 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2k-1)!!}{2^k k\, k!}\left(\frac{2\kappa}{\left((1+\nu)z+\nu t-\hat{z}\right)^2}\right)^{k+\frac{1}{2}}\tilde{\Delta}^k_{LT}\left(z,\xi_s\right)\right)$$

$$\tilde{\Delta}_{LT}\left(z,\xi_s\right) = \exp\left(-\frac{\left((1+\nu)z+\nu t-\hat{z}\right)^2}{2\,\kappa}\right)$$

where

$$\Delta_0^{(0,II)}\left(z,\xi_s\right) = \exp\left(-\frac{\left((1+\nu)z+\nu t-\tilde{z}\right)^2}{2\,\kappa} - \frac{\left(z_0-\tilde{z}\right)^2}{2\,\kappa} + \frac{\left(z_0-\xi_{0,s}-\tilde{z}\right)^2}{\kappa} - \frac{1}{2}\exp\left(z_0-\xi_{0,s}\right)\right)$$

$$\nu = 0$$

Matching on the line $\boldsymbol{\xi = \xi_0^A}$

$$\Delta_0^{(0,II)}\left(z_A,\delta\tilde{Y}\right) =$$

$$\underbrace{\exp\left(-\frac{((1+\nu)z_A+\nu t_A-\tilde{z}-2\nu\,\xi_{0,s})^2}{2\,\kappa}-\frac{(z_{0,A}-\tilde{z})^2}{2\,\kappa}+\frac{(z_{0,A}-\xi_{0,s}-\tilde{z})^2}{\kappa}-\frac{1}{2}\exp\left(z_{0,A}-\xi_{0,s}\right)\right)}_{region\quad II}$$

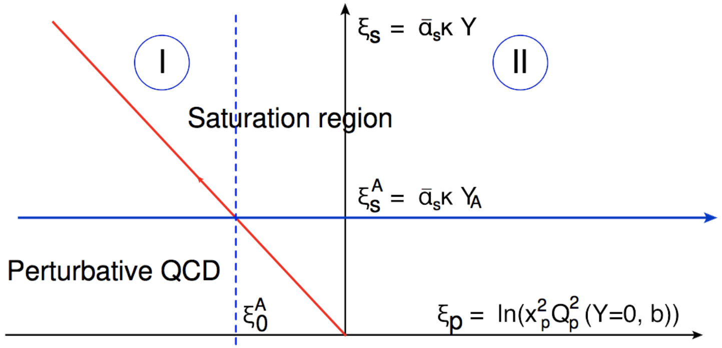
$$= \underbrace{\Delta^{(0,I)}(\eta(z_A)) = C^2\exp\left(-\frac{(z_A-\tilde{z})^2}{2\,\kappa}\;-\;\Omega_0^{0,I}\right)}_{region\quad I}$$

we find

$$\nu = 0$$

and

$$C^2 = \exp\left(\frac{\left(\xi_0^A-\tilde{z}\right)^2}{\kappa}-\frac{1}{2}e^{\xi_0^A}\right)$$



First order correction in the modified Homotopic Approach

$$\left(\kappa\frac{\partial}{\partial z} + z - \zeta\right)\Delta_1^D(z,z_0) + \underbrace{\Delta_0^D(z,z_0)\int_z^\infty dz'\Delta_1^D(z',\boldsymbol{z}_0) + \Delta_1^D(z,z_0)\int_z^\infty dz'\Delta_0^D(z',\boldsymbol{z}_0)}_{\sim(\Delta^0)^3} = -\underbrace{\mathcal{N}_{\mathscr{L}}[\Delta_0^D(z)}_{\sim(\Delta^0)^2}$$

Taking into account only terms of the order of $\left(\Delta^0\right)^2$

$$\left(\kappa\frac{\partial}{\partial z} + z - \zeta\right)\Delta_1^D(z,z_0) = -\mathcal{N}_{\mathscr{L}}[\Delta_0^D(z)$$

The particular solution can be written as follows

$$\Delta_1^D(z,z_0) = -\Delta_0^D(z,z_0)\int_z^\infty dz'\frac{1}{\Delta_0^D(z',z_0)}\mathcal{N}_{\mathscr{L}}[\Delta_0^D(z')]$$

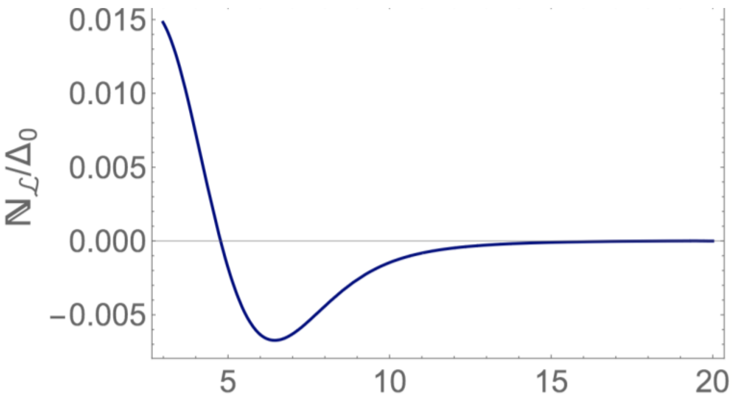
$$\Delta_0^D(z,\xi) = \exp\left(-\frac{(z-\tilde{z})^2}{2\kappa} + \phi^I(\zeta,z_0)\right)$$

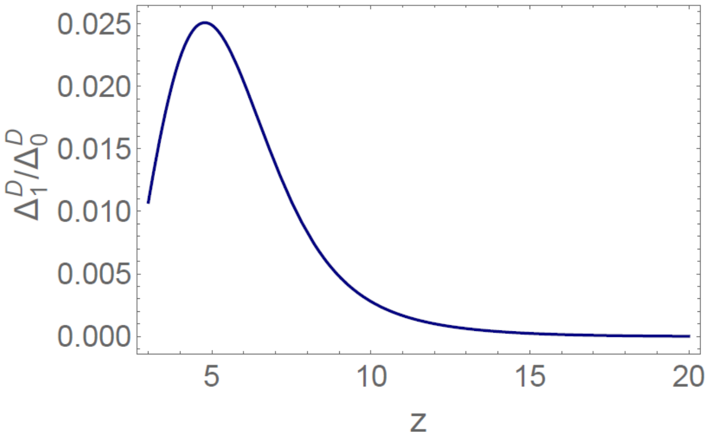
numerical estimation

Indeed, this correction turns to be small for region I.

$$\left(\kappa\frac{\partial}{\partial z} + z - \zeta\right)\Delta_1^D(z,z_0) = -\mathscr{N}_{\mathscr{L}}[\Delta_0^D(z)$$

$$\Delta_1^D(z,z_0) = -\Delta_0^D(z,z_0)\int\limits_z^\infty dz'\frac{1}{\Delta_0^D(z',z_0)}\mathscr{N}_{\mathscr{L}}[\Delta_0^D(z')]$$





For region II.

numerical estimation

$$\Delta_0^D(z,\xi,z_0) = \exp\left(-\frac{z^2}{2\kappa} + \phi^{II}(\xi,z_0)\right)$$

$$\exp\left(\phi^{II}(\xi,z_0)\right) = \exp\left(\frac{z_0^2}{2\kappa}\right) G^2(\xi) \exp\left(-\frac{(z_0-\tilde{z})^2}{\kappa}\right)$$

$$\phi^{II}(\xi,z_0) = \frac{z_0^2}{2\kappa} - \frac{(z_0-\tilde{z})^2}{\kappa} + \frac{(\xi-\tilde{z})^2}{\kappa} - \frac{1}{2}e^{\xi}.$$

$$\mathcal{N}_{\mathcal{L}}[\Delta_0^{(0,II)}] = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \Delta_0^{(0,II)}(x_{02}) \Delta_0^{(0,II)}(x_{12}) - \Delta_0^{(0,II)} \int^{x_{01}^2} \frac{dx_{02}^2}{x_{02}^2} \Delta_0^{(0,II)}$$

$$\Delta_0^D(z,\xi,z_0) = \exp\left(-\Sigma\left(z(\xi),\xi,z_0(\xi)\right) - \frac{1}{2}e^{\xi}\right) \text{ with } \Sigma(z,\xi,z_0) = \frac{z^2}{2\kappa} - \frac{z_0^2}{2\kappa} + \frac{(z_0-\tilde{z})^2}{\kappa} - \frac{(\xi-\tilde{z})^2}{\kappa}$$

Modified homotopy approach

Introducing $\boldsymbol{x}_{02} = \frac{1}{2}\boldsymbol{r} + \boldsymbol{x}$, $\boldsymbol{x}_{12} = \frac{1}{2}\boldsymbol{r} - \boldsymbol{x}$

$$\begin{aligned}
\mathscr{N}_{\mathscr{L}} &= 2\,\bar{\alpha}_S e^{-\xi_r} \exp\left(-2\Sigma\left(z(\xi_r), \xi_r, z_0(\xi_r)\right) - e^{\xi_r}\right)\\
&= 2\,\bar{\alpha}_S e^{-\xi_r} \exp\left(-2\left(\frac{z(\xi_r)^2}{2\kappa} - \frac{z_0(\xi_r)^2}{2\kappa} + \frac{(z_0(\xi_r) - \tilde{z})^2}{\kappa} - \frac{(\xi_r - \tilde{z})^2}{\kappa}\right) - e^{\xi_r}\right)\\
&= \bar{\alpha}_S \exp\left(-\frac{\left(z - 2\ln 2\right)^2}{\kappa} + \tilde{\phi}\left(\xi_r, z_0\right)\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
\kappa\,\Delta^D_1\left(z, \xi, z_0\right) &= \Delta^D_0\left(z, \xi, z_0\right) \int_z^\infty dz' \frac{\mathscr{N}_{\mathscr{L}}[\Delta^D_0\left(z'\right)]}{\Delta^D_0\left(z, \xi, z_0\right)}\\
&= \Delta^D_0\left(z, \xi, z_0\right) \int_z^\infty dz' \exp\left(-\frac{\left(z' - 2\ln 2\right)^2}{\kappa} + \frac{\boldsymbol{z}'^2}{2\,\kappa} + \left(\tilde{\phi}\left(\xi_r, z_0\right) - \phi^{II}\left(\xi_r, z_0\right)\right)\right)\\
&= \Delta^D_0\left(z, \xi, z_0\right) \sqrt{\frac{\pi\kappa}{2}} e^{\frac{4\ln^2 2}{\kappa}} \operatorname{erfc}\left(\frac{z - 4\ln 2}{\sqrt{2\kappa}}\right) e^{\left(\tilde{\phi}(\xi_r, z_0) - \phi^{II}(\xi_r, z_0)\right)} \xrightarrow{z \gg 1} \frac{\kappa}{z}\left(\Delta^D_0\left(z, \xi, z_0\right)\right)^2
\end{aligned}$$

Homotopy approach allows us to solve analytically and asymptotically.

We obtained that nonlinear terms, which include the remains of the nonlinear corrections, lead to small contributions and can be treated in the perturbation approach.

Homotopic approach to nonlinear Balitsky-Kovchegov (BK) evolution equation with running QCD coupling

The NLO BK: The running-coupling corrections to small-*x* evolution are obtained considering contribution of the quarks loops into de gluon lines

$$\frac{\partial(N(x_{10},b,Y)}{\partial Y} = \int d^2x_2\, K_{rc}\,(r,r_1,r_2)(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y) - N(x_{12},b,Y) * N(x_{20},b,Y))$$

We have two prescriptions:

- Balistsky (2007) $K_{rc}^{Bal}\,(r,r_1,r_2)$

$$K_{rc}^{Bal}\,(r;r_1,r_2) = \bar{\alpha}_S\,(r^2)\left\{\frac{r^2}{r_1^2\,r_2^2} + \frac{1}{r_1^2}\left(\frac{\bar{\alpha}_S\,(r_1^2)}{\bar{\alpha}_S\,(r_2^2)} - 1\right) + \frac{1}{r_2^2}\left(\frac{\bar{\alpha}_S\,(r_2^2)}{\bar{\alpha}_S\,(r_1^2)} - 1\right)\right\}$$

$Kovchegov\ and\ Weigert\ (2007)\quad K_{rc}^{KW}\,(r,r_1,r_2)$

$$K_{rc}^{KW}\,(r;r_1,r_2) = \bar{\alpha}_S\,(r_1^2)\left\{\frac{1}{r_1^2} - 2\,\frac{\bar{\alpha}_S\,(r_2^2)}{\bar{\alpha}_S\,(R_0^2)}\,\frac{\boldsymbol{r}_1\cdot\boldsymbol{r}_2}{r_1^2 r_2^2} + \frac{1}{r_2^2}\,\frac{\bar{\alpha}_S\,(r_2^2)}{\bar{\alpha}_S\,(r_1^2)}\right\}$$

$$R_0^2\,(r;r_1,r_2) = r_1 r_2\left(\frac{r_2}{r_1}\right)^{\frac{r_1^2+r_2^2}{r_1^2-r_2^2} - 2\,\frac{r_1^2 r_2^2}{\boldsymbol{r}_1\cdot\boldsymbol{r}_2}\,\frac{1}{r_1^2-r_2^2}}$$

Both prescriptions neglect different contributions. However, numerical studies indicate that the Balitsky prescription provides a closer result to the full answer, where no approximations are made. Therefore, we adopt the Balitsky prescription in our analysis

Albacete and Y. V. Kovchegov (2007) arXiv:0704.0612

For Numerical calculation with running-coupling corrections to the BK/ JIMWLK evolution equations

See

Lappi and Mäntysaari: JIMWLK equation 1212.4825; 1502.0240

Albacete and Kovchegov (2007) 0704.0612

Analytic solution are calculated using the homotopy approach for solving the nonlinear Balitsky-Kovchegov (BK) evolution equation with running QCD coupling. *Phys.Rev.D* 111 (2025) 9, 096025

Such corrections slow down the growth of the saturation scale with energy, putting the predictions of saturation physics more in line with the experimental data.

In our approach:

- In the vicinity of the saturation scale it is considered: $r^2 \approx r_1^2 \approx r_2^2 \approx \frac{1}{Q_s^2(Y)}$ in those terms with running coupling corrections :
- $\alpha_s(r^2) = \frac{4\pi}{b_0\, Ln\, 1/r^2\Lambda_{QCD}^2}$, $b_0 = \frac{11N_c}{3} - \frac{2\mathrm{N_f}}{3}$
- We can show that the Saturations Scale $Q_s^2(Y)$ is $$Q_s^2(Y,b) = Q_s^2(Y = Y_0, b)\, e^{\sqrt{\frac{8N_c}{b_0}\frac{\chi(\gamma_{\rm cr})}{1-\gamma_{\rm cr}}Y}}.$$

We introduce a new variable which allow to perform the Homotopic approach:

$$r^2 \approx r_1^2 \quad and \quad r^2 > r_2^2 \quad ; \; r_2^2 > \frac{1}{Q_s^2(Y)}$$

We obtain the equation:

$$\frac{\partial N(r,Y;b)}{\partial Y} = \tilde{N}(r,Y;b)(1 - N(r,Y;b)). \quad \longrightarrow$$

$$N(r,Y;b) = 1 - e^{-\Omega(r,Y;b)}$$

we obtain the following equation

$$\frac{\partial^2 \Omega(r,Y;b)}{\partial Y \partial l} = 1 - e^{-\Omega(r,Y;b)}.$$

with

$$\tilde{N}(r,Y;b) = \int_{1/Q_s^2}^{r^2} dr'^2 \frac{\bar{\alpha}_S(r'^2)}{r'^2} N(r',Y;b) \quad \longrightarrow$$

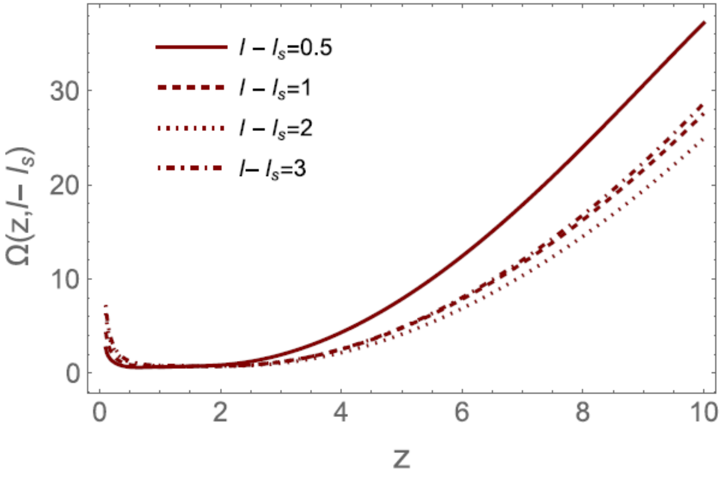
$$l = \int^{r^2} dr'^2 \frac{\bar{\alpha}_S(r'^2)}{r'^2} = -\frac{4N_c}{b_0}\ln(4N_c/(b_0\bar{\alpha}_S(r^2)))$$

$$= -\frac{4N_c}{b_0}\ln(\bar{\xi}) \qquad \text{with } \bar{\xi} = -\ln\left(r^2\Lambda_{\rm QCD}^2\right) \equiv -\xi$$

We found the analytic function for the scattering amplitude, which satisfies the initial and boundary conditions.

We find a good convergency.

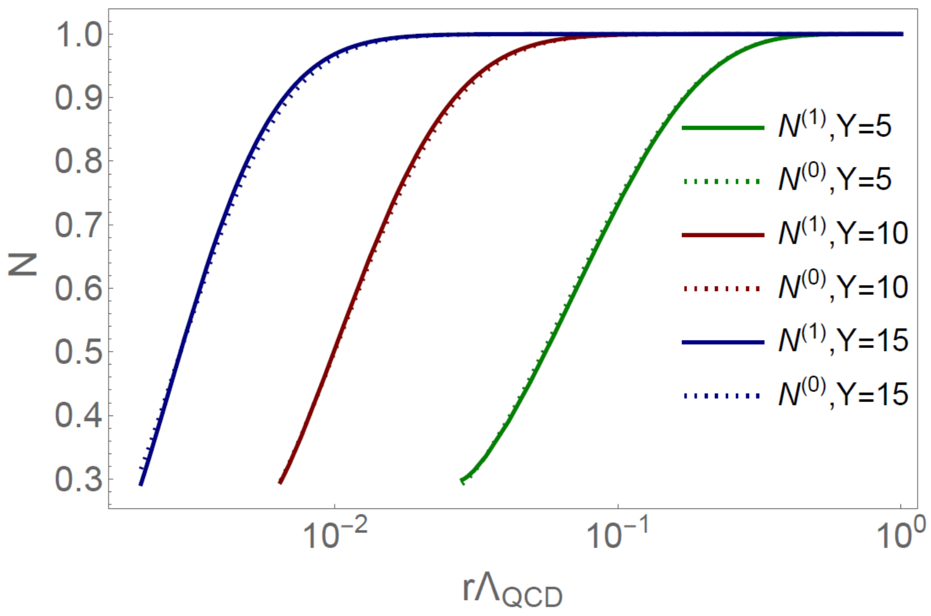
The geometric scaling behavior could be in the vicinity of the saturation scale. This Fig. shows an approximate z-scaling behavior in the limited region of z.



$$l - l_s = \int_{1/Q_s^2}^{r^2} dr'^2 \, \frac{\bar{\alpha}_S(r'^2)}{r'^2} = -\frac{4N_c}{b_0} \ln\left(\frac{-\xi}{\xi_s}\right).$$

We plot in Fig. the first N^0 and the second N^1 iterations in the homotopy approach. One can see that the homotopy approach works quite well for the model

Results



Summary and Outlook

Using the Homotopic Approach for three different amplitudes in the small x we were able to find Analytic Solution in the Saturation region

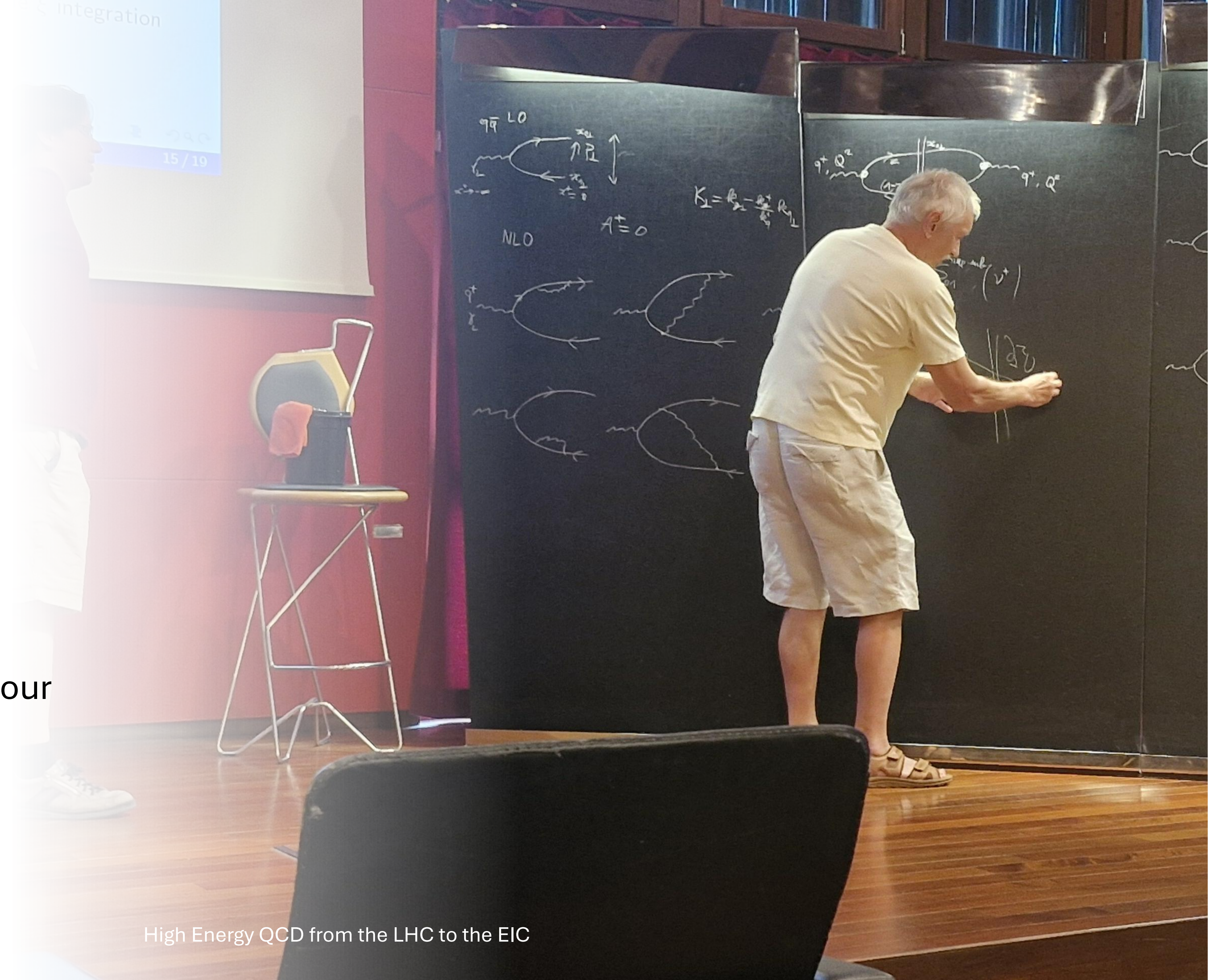
Numerical estimation indicates that the solutions converge in the Saturation region

Next Steps

We are in conditions to apply our solutions for determinate different scattering process

As for example: to attempted to describe the combined set of the inclusive diffractive cross sections measured by H1 and ZEUS collaboration at HERA.

HAPPY
BIRTHDAY
IAN



Thank you very much for your attention

WONPAQCD 2025 is the 6th international Workshop on Non-Perturbative Aspects of QCD. It will be held from the 1st to the 5th of December 2025 at the Universidad Técnica Federico Santa María (UTFSM), Valparaíso, Chile.

6th Workshop on Nonperturbative Aspects of QCD (01-5 December 2025)