

$$27 \otimes 27$$

F. Cougoulic, with S. Peigné. Results are based on [2504.11362]

High energy QCD: from the LHC to the EIC, Benasque, 2025.

$$27 \otimes 27 = 729$$

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Context: Soft gluon radiation a colorful puzzle.

High energy collisions ?

Compute the cross section ? \rightarrow In general, a mix of all scales \rightarrow Hard problem

Sketch of a collision illustrated within QCD (collinear) factorization [Collins, Soper, Sterman in the 1980's]

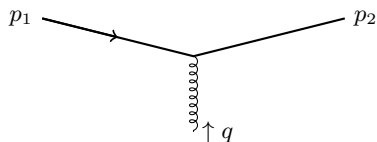
$$\sigma(A + B \rightarrow Y + X) \sim$$

(1)

Context: Soft gluon radiation a colorful puzzle - Back to the basics

Partonic cross section

- High energy collision \Rightarrow hard scale $Q^2 \Rightarrow$ Perturbation Theory : pQCD
- From your preferred textbook, use Feynman rules to compute the Born level cross section. Consider our first example $q + g \rightarrow q$ scattering:



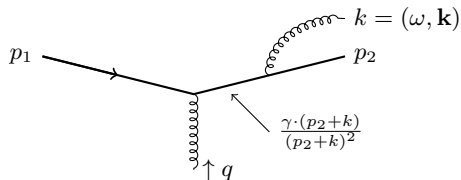
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What about the next order in the strong coupling α_s ?

- Loops and co. in the UV \Rightarrow Renormalization : not Today's topic
- Collinear and/or soft singularity \Rightarrow **Today's motivation**

Context: Soft gluon radiation a colorful puzzle - Soft gluons everywhere

What are those soft gluons ? (In the mass less limit)



Why are they important ?

$$\text{Denominator} \sim \omega E_2 (1 - \cos\theta)$$

(3)

- Collinear limit $\theta \rightarrow 0$, Soft limit $\omega \rightarrow 0$.
- Produce large logarithms due to kinematics.
- Appears at each order in the series!
- $\alpha_s \ll 1$ but $\alpha_s \log^2 \sim \mathcal{O}(1)$

Context: Soft gluon radiation a colorful puzzle - Soft gluons everywhere 2

Already in QED, the solution to this problem is a *resummation* [See Weinberg Vol.1].
It look like this:

$$\sum_{n=0}^{\infty} \text{---} \begin{array}{c} \text{\textit{k}}_1 \\ \text{\textit{k}}_2 \\ \vdots \\ \text{\textit{k}}_n \end{array} \text{---} \longrightarrow \text{Exp} \left[\text{---} \overset{\text{---}}{\curvearrowright} \text{---} \right] \quad (4)$$

The Kernel is related to the soft (photon) radiation spectrum.

Building up a Sudakov form factor. $\mathcal{M}_{Born}^2 \rightarrow \mathcal{M}_{Born}^2 \cdot \prod_i F_i$, for each charged external parton i .

⇒ **What is new in Quantum Chromodynamics ?**

Context: Soft gluon radiation a colorful puzzle - Soft gluons everywhere 3

What is new in **Quantum Chromodynamics** ? \rightarrow QCD is a non-abelian theory

$$(e_1 e_2)_{QED} = (e_2 e_1)_{QED} \rightarrow (t_1 t_2)_{QCD} \neq (t_2 t_1)_{QCD} \quad (5)$$

What does it mean for our initial example ?



Is it possible to bypass this difficulty ? \rightarrow Color conservation

$$t_1 + t_g = t_2 \rightarrow 2 t_1 t_2 = (t_1)^2 + (t_2)^2 - (t_g)^2 \quad (6)$$

Quadratic Casimirs operators \Rightarrow proportional to the identity in color space !

As in QED, building up Sudakov form factors : $\mathcal{M}_{Born}^2 \rightarrow \mathcal{M}_{Born}^2 \cdot \prod_i F_i$

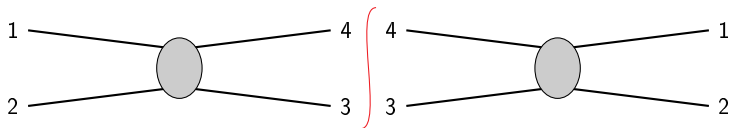
Where is the puzzle ?

Context: Soft gluon radiation a colorful puzzle - Beyond color triviality

Where is the puzzle ?

Previous example was too “simple” (w.r.t. color structure).

Let's change to a richer process: $q_1 + q_2 \rightarrow q_3 + q_4$



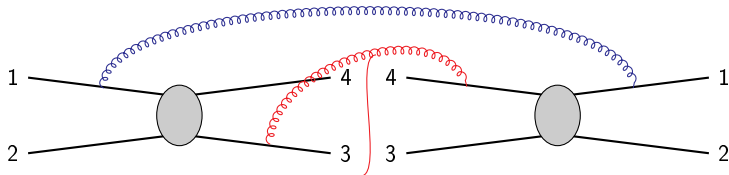
Color conservation reads

$$t_1 + t_2 = t_3 + t_4 \quad (7)$$

Square it \Rightarrow Only quadratic Casimir operators $(t_i)^2 \Rightarrow$ Not possible anymore:

$$2t_1t_2 = (t_3)^2 + (t_4)^2 - (t_1)^2 - (t_2)^2 + 2t_3t_4 \quad (8)$$

Implications



- Contributions $\propto (t_i)^2$: Ready to be exponentiated into F_i 's
- Contributions remaining $\propto t_i t_j$ with $i \neq j$.

P1: The color Puzzle

How do we take care of the remaining “cross terms” ?

→ Needs some tools for the color algebra

Soft anomalous dimension matrix - Solution to the puzzle

Full calculation was done by

- Y. Dokshitzer and G. Marchesini. *Soft gluons at large angles in hadron collisions*. JHEP, 01:007, 2006. → Referred here as DM

It involves the color matrix called “soft anomalous dimension matrix” \mathcal{Q} .

Formal definition

$$\mathcal{Q}(b) = \frac{1}{2N_c} [(T_t^2 + T_u^2) + b(T_t^2 - T_u^2)] \quad (9)$$

b function of kinematic logarithms of Mandelstam variables. For $t \rightarrow u$, $b \rightarrow 0$

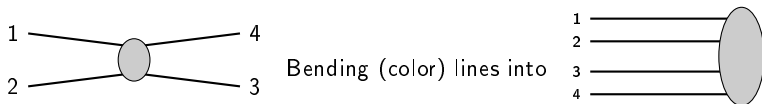
$$T_t^2 = (t_1 - t_4)^2 = (t_2 - t_3)^2, \quad T_u^2 = (t_1 - t_3)^2 = (t_2 - t_4)^2 \quad (10)$$

\mathcal{Q} contains non diagonal part → can rotate between irreps.

⇒ What does it look like ?

sADM - Solution to the puzzle 2

Understanding the color content in Q ;



The operator T_t^2 acting on the r.h.s. tensors is

$$\left[\begin{array}{c} \text{red wavy line} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{red wavy line} \end{array} - 2 \begin{array}{c} \text{---} \\ \text{---} \\ \text{red wavy line} \\ \text{---} \end{array} \right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \text{ blob} \quad (11)$$

Decompose the “Blob”-tensor into a sum over irreps

\Rightarrow Replace the square bracket by the eigenvalue of \hat{C}_2 acting on $R_1 \otimes \bar{R}_4$.

Define $\{s, t, u\}$ -channel projectors as

$$\mathcal{P}_\alpha^{(s)} \rightarrow \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{ blob } \alpha \quad \mathcal{P}_\beta^{(t)} \rightarrow \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{ blob } \beta \quad \mathcal{P}_\gamma^{(u)} \rightarrow \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{ blob } \gamma \quad (12)$$

$$Q(b) = \frac{1}{2N_c} [(T_t^2 + T_u^2) + b(T_t^2 - T_u^2)]$$

Understanding the color content in Q ;

Simple interpretation : T_t^2 and T_u^2 are given by the diagonal matrix of Casimir eigenvalues in their respective projector basis.

What about $T_t^2 + T_u^2$?

Recall that $s + t + u = \sum_{i=1}^4 m_i^2$, the same relation holds for the color part:

$$T_s^2 + T_t^2 + T_u^2 = \sum_{i=1}^4 C_{R_i} \quad (13)$$

\Rightarrow Color-Kinematics Duality (conjecture)

Up to some Casimir of independent parton, $T_t^2 + T_u^2$ is related to T_s^2 .

Choosing a preferred basis for $Q \Rightarrow s$ -channel projector basis.

sADM - A mysterious symmetry

In their paper, Y. Dokshitzer and G. Marchesini computed the $gg \rightarrow gg$ process, the calculation is quite involved for two main reasons:

- The gluon gluon decomposition reads $\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \bar{\mathbf{10}} \oplus \mathbf{27}$.
⇒ Requires the construction the projection operators for a large vector space.
- It involves the computation the “rotation” matrix from $\{t, u\}$ -channel to the s -channel.
⇒ Related to the calculation of Wigner $6j$ coefficients.

When trying to diagonalize \mathcal{Q} they remark that out of the six eigenvalues, three have a surprising symmetry:

$$\lambda_{4,5,6}(x, b) = \lambda_{4,5,6}(b, x) \quad x = 1/N_c \quad (14)$$

This is unexpected, quoting the two experts on the subject:

“We note an unexpected mysterious property of the equation [equation for the eigenvalues] ... which is symmetric with respect to the transformation that interchanges parameters characterising external (scattering angle) and internal (colour group) degrees of freedom.”

“Giving the complexity of the expressions involved, such a symmetry being accidental looks highly improbable.”

Motivation

- Explore several cases systematically
⇒ Is this symmetry fortuitous or not ? (Academic)
- Relative importance of this matrix: Form factors / soft gluon spectrum
⇒ Does this symmetry has an impact for HE collisions ? (Phenomenology)
- Formal relation to multiple scattering in the Gaussian approximation.

Naive remarks about $gg \rightarrow gg$

- $aa \rightarrow aa$: same partons elastic process.
- $a\bar{a} \rightarrow a\bar{a}$: the gluon is self conjugate.

Is there something relevant in this list?

and so, the exploration began...

Overview of the rest of the talk

Part 2: Introduction to birdtracks

- Notation - Building blocks
- Some manipulation
- Projection and Quadratic Casimir
- Few references

Part 3: $27 \otimes 27$

- Setup and tensor basis
- Using the Casimir to filter irrep
- Irrep degeneracy
- Soft anomalous dimension matrix

Part 2: Introduction to birdtracks

Introduction to Birdtracks - Notations

Building blocks

Indices in Roman letter

- Fundamental $\{i, j, \dots\} = \{1, 2, \dots, N_c\}$.
- Adjoint $\{a, b, \dots\} = \{1, 2, \dots, N_c^2 - 1\}$.

Indices in Greek letter

- Irreducible representation (irrep) labels α, β, \dots

Tensor	Usual notations	Birdtracks
Kronecker delta (fundamental rep)	δ^i_j	$i \longrightarrow j$
Kronecker delta (adjoint rep)	δ^a_b	$a \rightsquigarrow b$
Color Generator (fundamental rep)	$(t^a)^i_j$	$i \longrightarrow j$ $\quad \downarrow$ $\quad a$
Color Generator (adjoint rep)	$i f_{abc}$	$a \rightsquigarrow c$ $\quad \downarrow$ $\quad b$

The color tensor in a Feynman diagram is just the diagram itself in birdtracks

Introduction to Birdtracks - Manipulations

How does one compute using birdtracks ?

Few things to remember:

- 1/ Only the relative position of (external) indices has a relevant information
- 2/ Do not introduce a matrix representation for t^a 's or f 's \rightarrow Use algebraic relations.
- 3/ Built up your own lexicon of relations to simplify birdtracks

Structure constant definition

$$\text{Diagram 1} - \text{Diagram 2} = \text{Diagram 3} \quad (18)$$

Jacobi identity

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} = 0 \quad (19)$$

Color Conservation (*invariance condition*)

$$\text{Diagram 1} + \text{Diagram 2} + \dots + \text{Diagram 4} = 0 \quad (20)$$

Example of some useful relations in birdtracks

$$\begin{aligned} \text{Diagram 1} &= \frac{1}{2} \text{Diagram 2}, & \text{Diagram 3} &= N_c \text{Diagram 4}, & \text{Diagram 5} &= C_F \text{Diagram 6} \\ \text{Diagram 7} &= \frac{N_c}{2} \text{Diagram 8}, & \text{Diagram 9} &= \frac{-1}{2N_c} \text{Diagram 10} \end{aligned}$$

So on and so forth...

Projection operator symbolically noted as (for any $R_1 \otimes R_2$)

$$\mathcal{P}_\alpha = \begin{array}{c} R_1 \\ \diagdown \\ \quad \quad \alpha \\ \diagup \\ R_2 \end{array} \begin{array}{c} R_1 \\ \diagup \\ \quad \quad \alpha \\ \diagdown \\ R_2 \end{array} \quad (21)$$

Quadratic Casimir operator ($R_1 \otimes R_2$ case)

$$\hat{C}_2 = \left[\begin{array}{c} \text{---} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \quad (22)$$

And the two together

$$\hat{C}_2 \mathcal{P}_\alpha = C_\alpha \mathcal{P}_\alpha \quad (23)$$

By identification

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \mathcal{P}_\alpha = \frac{(C_{R_1} + C_{R_2} - C_\alpha)}{2} \mathcal{P}_\alpha \quad (24)$$

Introduction to Birdtracks - Few references

To get started

- Y. Dokshitzer. *Perturbative QCD (and Beyond)*. 1997
- P. Cvitanovic. *Group theory: Birdtracks, Lie's and exceptional groups*. 2008.
- S. Keppeler. *Birdtracks for $SU(N)$* . 2017
- S. Peigne. *Color in QCD: An Introduction Featuring the Birdtrack Pictorial Technique*. 2024.

Further reading on advanced topics

- M. Sjodahl and S. Keppeler. *Orthogonal multiplet bases in $SU(N_c)$ color space*. 2012; *Tools for calculations in color space*. 2013; *Hermitian Young Operators*. 2013
- J. Alcock-Zeilinger and H. Weigert. *Simplification Rules for Birdtrack Operators*. 2016; *Transition Operators*. 2016; *Compact Hermitian Young Projection Operators*. 2016
- B. Chargeishvili. *Automatic generation of orthogonal multiplet bases in $SU(N_c)$ color space*. 2024.
- J. Alcock-Zeilinger, S. Keppeler, S. Platzer, and M. Sjodahl. *Wigner δ_j symbols for $SU(N)$: Symbols with at least two quark-lines*. 2023; *Wigner δ_j symbols with gluon lines: completing the set of δ_j symbols required for color decomposition*. 2024

Part 3: $27 \otimes 27$

Setup and tensor basis

Consider the decomposition of the tensor product of two gluon:

$$8 \otimes 8 = 27 \oplus 0 \oplus 10 \oplus \overline{10} \oplus 8_+ \oplus 8_- \oplus 1 \quad (25)$$

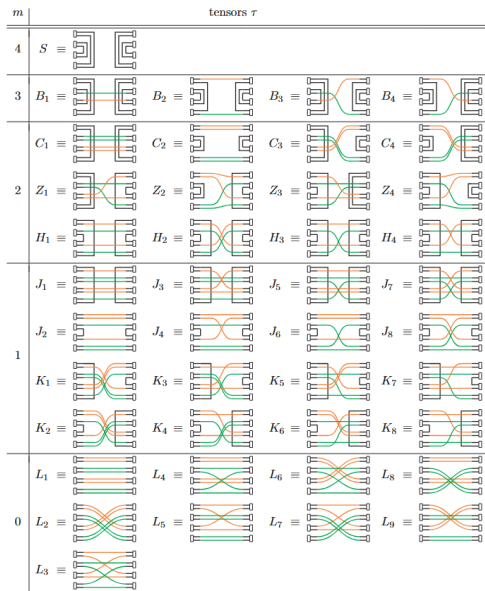
- Looking for **color enhancement** of the underlying process.
⇒ Irrep 27 is the highest dimension appearing in this decomposition.
- Imagine a process involving **four gluon jets**. There are kinematic regimes where gluon pairs can be considered as pointlike, and thus, are effectively higher dimensional color objects.
- What would be the corresponding **broadening** of such system?
What would be the associated **soft anomalous dimension matrix**?
- Requires efficient construction to the associated **color projectors**.

The process under consideration here: $27 \otimes 27 \rightarrow 27 \otimes 27$

$$27 \otimes 27 = 125 \oplus \overline{81} \oplus 81 \oplus \overline{28} \oplus 28 \oplus \dots \oplus 8_+ \oplus 8_- \oplus 1 \quad (26)$$

- Grand total of 26 distinct irreps. Color space dimension is 729.
- The projector construction: still “pen-and-paper” tractable, easily apply to other cases.

List of tensors and their properties



τ	$\sigma_{12} \cdot \tau$	$\sigma_{34} \cdot \tau$	$\sigma_{14} \cdot \tau$	$\sigma_{23} \cdot \tau$	$\sigma_{13} \cdot \tau$	$\sigma_{24} \cdot \tau$	τ^*	τ^\dagger
S	S	S	L_2	L_2	L_1	L_1	S	S
L_1	L_2	L_2	L_1	L_1	S	S	L_1	L_1
L_2	L_1	L_1	S	S	L_2	L_2	L_2	L_2
B_1	B_4	B_3	L_7	L_6	J_2	J_2	B_2	B_2
B_2	B_3	B_4	L_6	L_7	J_1	J_1	B_1	B_2
B_3	B_2	B_1	K_2	K_2	L_5	L_4	B_4	B_4
B_4	B_1	B_2	K_1	K_1	L_4	L_5	B_3	B_3
L_4	L_6	L_6	J_1	J_2	B_4	B_3	L_5	L_5
L_5	L_7	L_7	J_2	J_1	B_3	B_4	L_4	L_4
L_6	L_4	L_4	B_2	B_1	K_1	K_2	L_7	L_6
L_7	L_5	L_5	B_1	B_2	K_2	K_1	L_6	L_7
J_1	K_2	K_1	L_4	L_5	B_2	B_2	J_2	J_1
J_2	K_1	K_2	L_5	L_4	B_1	B_1	J_1	J_2
K_1	J_2	J_1	B_4	B_4	L_6	L_7	K_2	K_2
K_2	J_1	J_2	B_3	B_3	L_7	L_6	K_1	K_1
C_1	C_4	C_3	L_8	L_9	C_2	C_2	C_2	C_1
C_2	C_3	C_4	L_9	L_8	C_1	C_1	C_1	C_2
C_3	C_2	C_1	C_4	C_4	L_9	L_8	C_4	C_4
C_4	C_1	C_2	C_3	C_3	L_8	L_9	C_3	C_3
L_8	L_9	L_9	C_1	C_2	C_4	C_3	L_9	L_9
L_9	L_8	L_8	C_2	C_1	C_3	C_4	L_8	L_8
H_1	H_2	H_2	L_3	L_3	H_1	H_1	H_1	H_1
H_2	H_1	H_1	H_2	H_2	L_3	L_3	H_2	H_2
L_3	L_3	L_3	H_1	H_1	H_2	H_2	L_3	L_3
H_3	H_4	H_4	J_7	J_6	K_6	K_7	H_4	H_3
H_4	H_3	H_3	J_8	J_7	K_7	K_8	H_3	H_4
J_7	K_8	K_7	H_3	H_4	J_8	J_8	J_7	J_7
J_8	K_7	K_8	H_4	H_3	J_7	J_7	J_8	J_8
K_7	J_8	J_7	K_8	K_8	H_4	H_3	K_8	K_8
K_8	J_7	J_8	K_7	K_7	H_3	H_4	K_7	K_7
Z_1	Z_2	Z_1	K_4	K_6	J_4	J_6	Z_2	Z_3
Z_2	Z_1	Z_2	K_5	K_3	J_5	J_3	Z_1	Z_1
Z_3	Z_3	Z_4	K_3	K_5	J_6	J_4	Z_4	Z_4
Z_4	Z_4	Z_3	K_6	K_4	J_3	J_5	Z_3	Z_3
J_3	K_4	K_3	J_6	J_3	Z_4	Z_2	J_6	J_2
J_4	K_3	K_4	J_4	J_5	Z_1	Z_3	J_5	J_4
J_5	K_6	K_5	J_5	J_4	Z_2	Z_4	J_4	J_5
J_6	K_5	K_6	J_3	J_6	Z_3	Z_1	J_3	J_6
K_3	J_4	J_3	Z_3	Z_2	K_6	K_3	K_6	K_4
K_4	J_3	J_4	Z_1	Z_4	K_4	K_5	K_5	K_3
K_5	J_6	J_5	Z_2	Z_3	K_5	K_4	K_4	K_4
K_6	J_5	J_6	Z_4	Z_1	K_3	K_6	K_3	K_5

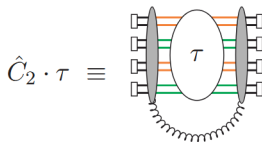
Using the Casimir to filter irreps

Strategy - First filtering

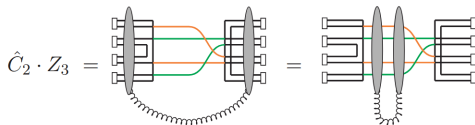
- All projectors and transition operators are linear combinations of $\tau \in \mathcal{T}$.
- All distinct irreps have a distinct eigenvalues (viewed as N_c dependent function).
- Action of the Casimir operator on all elements of $\mathcal{T} \implies$ split the vector space and construct the projection operators up to irrep degeneracy (simply by diagonalizing the matrix)

Step 2: Compute the action of \hat{C}_2 on all $\tau \in \mathcal{T}$.

Main trick: move the generators "inside" where there is the least amount of horizontal lines



Illustration



\implies Use Fierz identity, to remove the gluon.

Remaining problems to solve:

- The quadratic Casimir operator has the **same eigenvalue for irreps and their dual**: $C_R = C_{\bar{R}}$.
- The quadratic Casimir does not distinguish between **transition operators and projection operators**: $\hat{C}_2 \mathcal{P}_{8_+} = C_A \mathcal{P}_{8_+}$ and $\hat{C}_2 \mathcal{T}_{8_+ \rightarrow 8_-} = C_A \mathcal{T}_{8_+ \rightarrow 8_-}$

Strategy - Lifting of degeneracy

- Use the knowledge of the basis under hermitian conjugation to distinguish the irrep from their dual irrep.
- Use permutation operators on the left and on the right to remove the remaining degeneracy.

Step 3: Final results are obtained by looking at the kernel of:

$$\frac{\mathbb{1} + s_{12}\sigma_{12}}{2} \cdot \frac{\mathbb{1} + s_{34}\sigma_{34}}{2} \cdot \frac{\mathbb{1} + \lambda_{cc}\sigma_{cc}}{2} \cdot \hat{C}_2 - C_R \mathbb{1}$$

- s_{12} and s_{34} are eigenvalues of the permutation operators, and λ_{cc} the eigenvalue of the charge conjugation operator.

Some examples of projection operator:

Eigenspace of dimension 1:

$$\mathcal{P}_1 = \frac{4S}{(N_c - 1)N_c^2(N_c + 3)}$$
$$\mathcal{P}_{0_a} = \frac{4}{(N_c + 1)(N_c + 3)} \left\{ H_{12} - H_{34} - \frac{B_{12} + B_{34}}{N_c - 2} + \frac{2S}{(N_c - 2)(N_c - 1)} \right\}$$

Eigenspace of dimension 8:

$$\mathcal{P}_{35_+/\overline{35}_+} = \frac{2(N_c + 2)}{9(N_c + 1)(N_c + 3)} \left\{ 2(J_{12} + K_{12}) - 4(J_{78} + K_{78}) + J_{34} + J_{56} + K_{34} + K_{56} \right. \\ \left. - \frac{2(C_{12} + C_{34}) - 8(Z_{12} + Z_{34}) + 8(H_{12} + H_{34})}{N_c + 1} \pm 3(J_{34} - J_{56} + K_{34} - K_{56}) \right\}$$

$$\mathcal{P}_{35_-/\overline{35}_-} = \frac{2(N_c + 2)}{9(N_c - 1)(N_c + 5)} \left\{ 2(J_{12} - K_{12}) - 4(J_{78} - K_{78}) + J_{34} + J_{56} - K_{34} - K_{56} \right. \\ \left. - \frac{2(C_{12} - C_{34})}{N_c + 1} - \frac{24\overline{H}_{12}}{N_c + 3} + \frac{16(B_{12} - B_{34})}{(N_c + 1)(N_c + 3)} \pm 3 \left(J_{34} - J_{56} - K_{34} + K_{56} + \frac{8\overline{H}_{34}}{N_c + 3} \right) \right\}$$

And many more...

Soft anomalous dimension matrix

Let us apply the previously computed projectors to the soft anomalous dimension matrix: Consider the $27 \otimes 27 \rightarrow 27 \otimes 27$ process, and recall

$$\mathcal{Q}(b) = \frac{1+b}{2N_c} T_t^2 + \frac{1-b}{2N_c} T_u^2 \quad (27)$$

- $T_{t/u}^2$ are the Casimir operator acting on the t/u -channels.
- $b = (T - U)/(T + U)$ with $T = \ln(-s/t) - i\pi$ and $U = \ln(-s/u) - i\pi$

Easily obtained once the projectors are known, albeit in their own channel.

$$\mathcal{P}_\alpha = \sum_k a_k^\alpha \tau_k \Rightarrow \hat{C}_2 \cdot \mathcal{P}_\alpha = \sum_k a_k^\alpha \left(\hat{C}_2 \cdot \tau_k \right) = C_\alpha \mathcal{P}_\alpha \quad (28)$$

- **Known from the calculation of the projectors**
- **Computed from e.g. Young diagrams**

Remains to compute those projectors in a **common channel** (say s -channel) to be used in an observable

- All the **crossing relations** between tensors of the basis \mathcal{T} are known to construct the projectors
→ See crossing table

sADM 27x27 - Eigenvalues

Factoring invariant subspace according to $\lambda \sim \{\sigma_{cc}, \sigma_{14}\sigma_{23}, \sigma_{12}\sigma_{34}\}$.

Eigenvalues solve

$$\text{Det} \left(x I - \mathcal{Q}^{(s)}(b) \right) = \prod_{\lambda} f_{\lambda}(x, b), \quad (29)$$

with

$$f_{(+,+,+)}(x, b) = (x - r_1)(x - r_2)(x - r_3)p_1(x, b)p_2(x, b)p_2(x, -b)p_3(x, b) \quad (30a)$$

$$f_{(+,+,-)}(x, b) = x - r_1 \quad (30b)$$

$$f_{(+,-,+)}(x, b) = x - r_2 \quad (30c)$$

$$f_{(+,-,-)}(x, b) = x - r_3 \quad (30d)$$

$$f_{(-,+, -)}(x, b) = (x - r_2)(x - r_3)p_2(x, b) \quad (30e)$$

$$f_{(-,-,+)}(x, b) = (x - r_1)(x - r_3)p_1(x, b) \quad (30f)$$

$$f_{(-,-,-)}(x, b) = (x - r_1)(x - r_2)p_2(x, -b) \quad (30g)$$

where monomials are

$$r_1 = \frac{(5 - b)(N_c + 1)}{2N_c} ; \quad r_2 = 3 \frac{N_c + 1}{N_c} ; \quad r_3 = \frac{(5 + b)(N_c + 1)}{2N_c}. \quad (31)$$

Polynomials $p_1(x, b)$, $p_2(x, b)$, $p_3(x, b)$ of degree 4, 4, 6 respectively.

Concluding remarks

- Quickly introduced birdtracks, which can be used to compute **color factors in QCD**.
- Other aspect of QCD can be evaluated using birdtrack such as the **Clifford algebra for spinors**. e.g. Birdtracks methods for the spinor-helicity formalism are currently being worked-out in the community.
- Considered the color structure of high dimensional irreps, which can appear under specific kinematics, eg four gluon jet final state.
- Presented a method relying on elementary birdtrack which can be used to compute projection operators.
- Considered the **soft anomalous dimension matrix**, which relates to the broadening of quadrupoles for those color irreps.
- DM observed symmetry does not appear for the "next to simplest" case of self adjoint irrep $27 \otimes 27 \rightarrow 27 \otimes 27$.