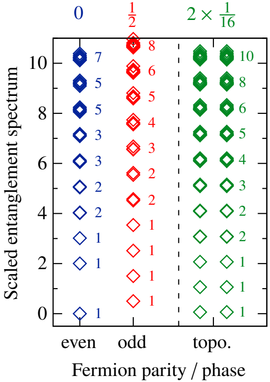
Rainbow chains and numerical RG for conformal spectra



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Entanglement in Strongly Correlated Systems
Benasque, 24 February 2025

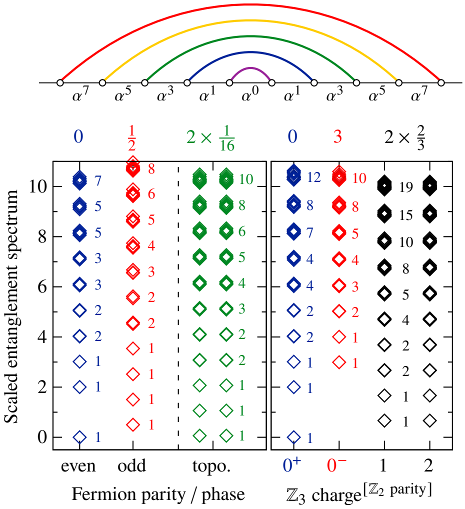
arXiv:2412.09685

Universität Zürich[UZH]

Swiss National Science Foundation

Outline

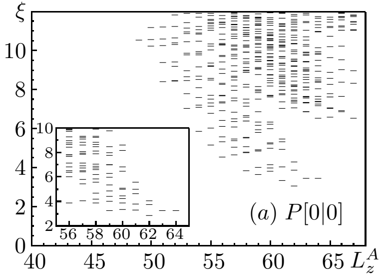
- Entanglement spectrum
  - in topological order
  - in CFTs
- Conformal transformations and rainbow chains
- Example 1: Transverse-field Ising model
- Numerical RG
- Example 2: Three-state Potts model
- Outlook



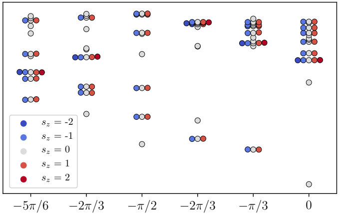
Entanglement spectrum

$$\rho_A = \sum_\alpha \lambda_\alpha |\alpha_A\rangle\langle\alpha_A| \implies \xi_\alpha := -\log \lambda_\alpha$$

- Direct probe of exotic physics
  - degeneracies in SPTs
  - edge CFTs in FQHE, chiral spin liquids,...
- Can simply read off of MPS ☺
- But in > 2D, area law is a problem ☹
  - need large bond dimensions
  - can only access narrow cylinders
  - edge CFT multiplets disperse



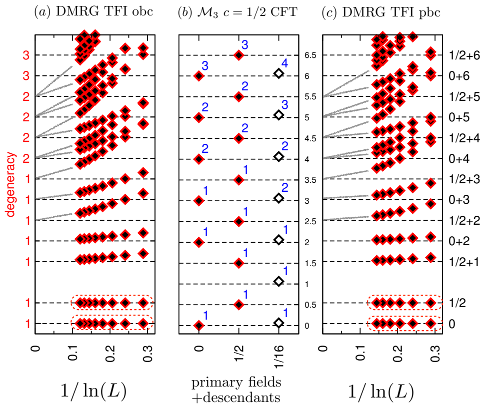
Li & Haldane, PRL **101**, 010504 (2008)



Szasz *et al.*, PRX **10**, 021042 (2020)

Entanglement spectrum of CFTs

- Matches energy spectrum of CFT with a boundary
  Läuchli, arXiv:1303.0741
  - same structure as chiral CFT!
- Conformal transformation from reduced to thermal d.m.
  Cardy & Tonni, J. Stat. Mech. **2016**,123103
  - "entanglement Hamiltonian" = CFT Hamiltonian
  - but system mapped to size $W \simeq \log L$
  - large finite-size effects ☹



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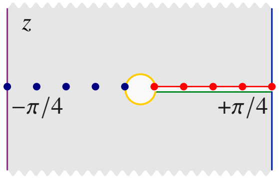
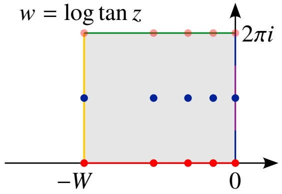
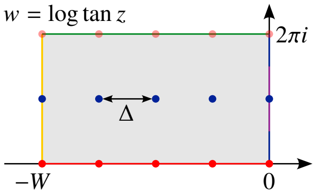


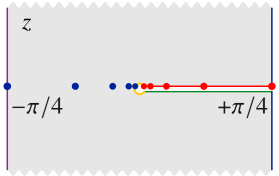


Upside down!

- Want sites distributed uniformly in w for better scaling
- Unequal distances in z
  - near the middle: exponential decay
- How to represent in a spin chain?
  - each term stands for $\int \mathrm{d}x\, \mathcal{H}(x) \approx \mathcal{H}_{\text{mid}}\ell$
  - upon conformal transformation
    - ℓ scales as $|\mathrm{d}z/\mathrm{d}w|$
    - $\mathcal{H}$ scales as $|\mathrm{d}z/\mathrm{d}w|^{-2}$

  $\Rightarrow$ Hamiltonian terms scale as $1/\ell$
- Largest Hamiltonian terms in the middle
- Exponential decay $\sim e^{-n\Delta}$ going outwards





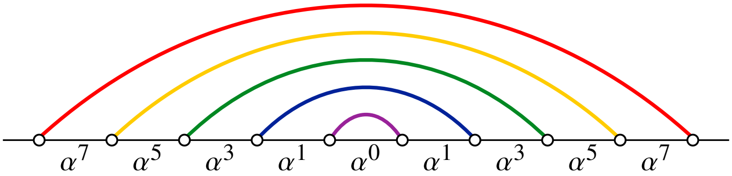
Rainbow chains

- Proposed before as example of volume-law ground states!

  Vitagliano, Riera, & Latorre, New J. Phys. **12**, 113049 (2010)

  Ramírez, Rodríguez-Laguna, & Sierra, J. Stat. Mech. **2014**, P10004

  - but only studied for free fermions (aka XX spin chain)
- If $\alpha = e^{-\Delta/2} \ll 1$ (Δ large)
  - separation of scales
  - GS ≈ "dimers" on rainbow arcs
- **(How much) will this volume-law entanglement limit us?**



Transverse-field Ising model

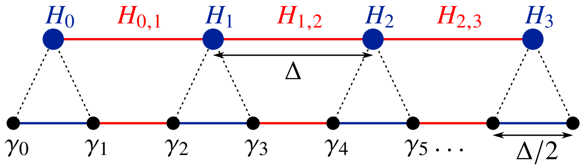
$$H = -\sum_i J_{i+1/2}\sigma_i^x\sigma_{i+1}^x + \sum_i h_i\sigma_i^z$$

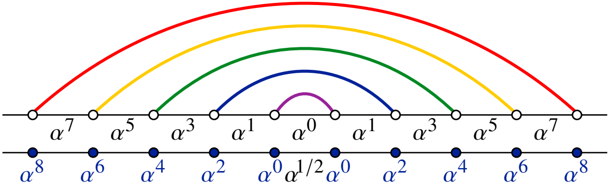
- critical point at $J = h$
- rewrite as Majorana tight-binding chain:

$$H = \sum_{i=0}^{2L-1} it_{i/2}\gamma_i\gamma_{i+1}$$

  - critical point at uniform t
  - rainbow chain prescription generalises
- Fermionic Gaussian state

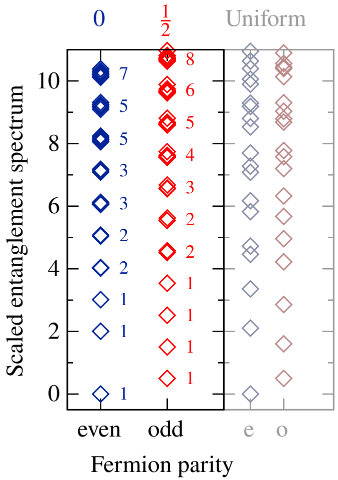
  $\Longrightarrow$ exact entanglement spectrum





TFIM on the rainbow chain

- Ising CFT has three conformal sectors:
  - identity ($h = 0$)
  - fermion ($h = 1/2$)
  - twist ($h = 1/16$)
- GS entanglement spectrum: $(0) \oplus (1/2)$
  - much narrower multiplets than for the uniform chain ☺
- Twist sector:
  - parity odd excited state
  - decouple first and last Majorana by hand
  - exact degeneracy $\Longleftrightarrow$ SPT phase



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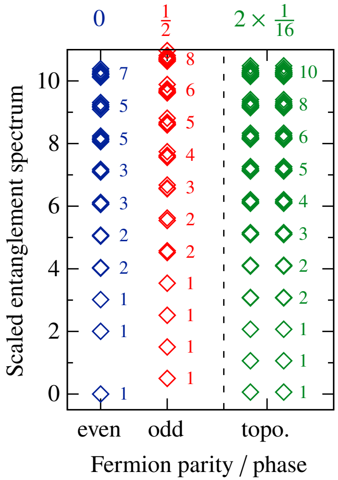
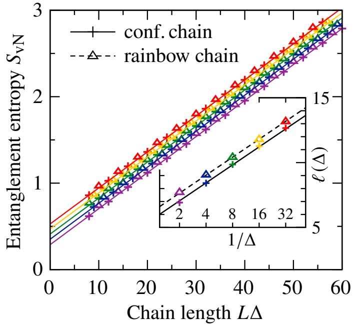


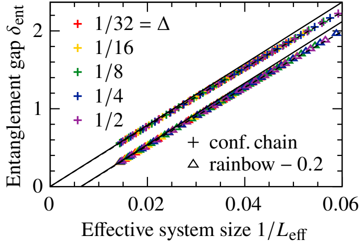
Finite-size scaling

- Entanglement entropy
  - $S_{\mathrm{vN}} \simeq \frac{c}{6} \log L$ for a uniform chain
    Calabrese & Cardy, J. Phys. A **42**, 504005 (2009)
  - effective length of rainbow chain $\sim e^{-L\Delta/2}$
    Ramírez, Rodríguez-Laguna, & Sierra, J. Stat. Mech. **2015**, P06002

$\Longrightarrow S_{\mathrm{vN}} \simeq \frac{c}{12}[L\Delta + \ell(\Delta)]$

  - define $L_{\mathrm{eff}} = \frac{12}{c} S_{\mathrm{vN}}$
- Entanglement gap: $\delta_{\mathrm{ent}} \simeq 4\pi^2/L_{\mathrm{eff}}$
  - mode spacing in k-space $\simeq 2\pi/L_{\mathrm{eff}}$
  - effective temperature $\beta = 2\pi$
- Entanglement spectrum
  - conformal multiplets converge as $\approx L_{\mathrm{eff}}^{-2}$
  - but at largest system sizes, $\sim L_{\mathrm{eff}}^{-3}$ (unexpectedly fast!)





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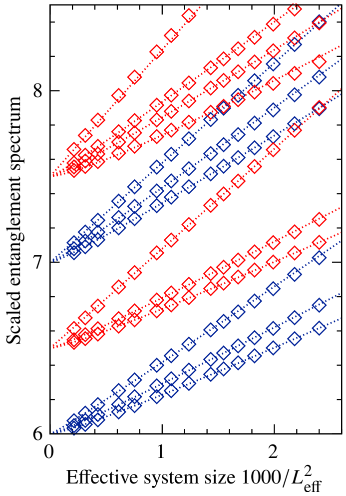
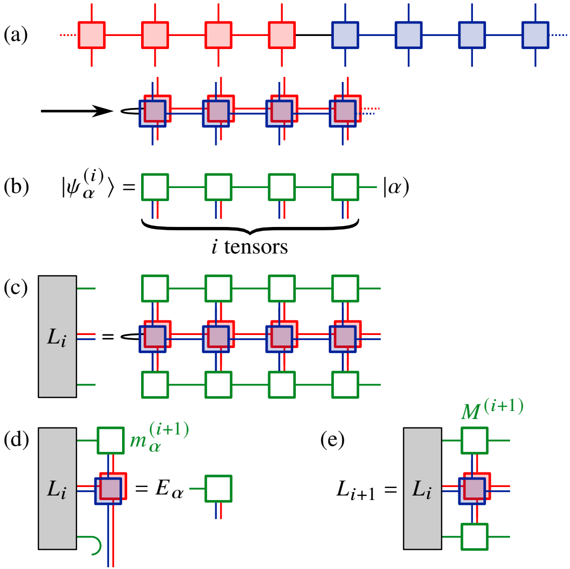


Numerical renormalisation group (NRG)

Want to solve interacting rainbow chains!

- DMRG struggles with wide range of terms ☹
- NRG: Originally for impurity problems
  Wilson, Rev. Mod. Phys. **47**, 773 (1975)
  - bath at different energy scales
    $\longrightarrow$ exponentially decaying chain
  - separation of scales (large Δ)
  - can iteratively diagonalise longer chains and discard high-energy states
- Naturally written in MPS form
  Weichselbaum, Phys. Rev. B **86**, 245124 (2012)
- Must handle all equal-energy terms at once
  $\Longrightarrow$ rainbow chain must be “folded”

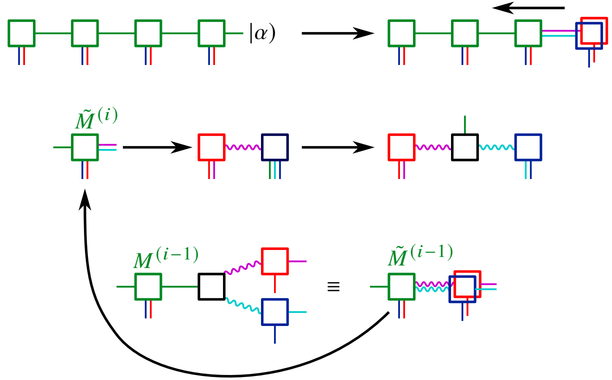


Numerical renormalisation group (NRG)

- GS wave function is "folded"
  - entanglement is spread along MPS

  $\Rightarrow$ can represent volume-law GS ☺
  - but no direct access to entanglement spectrum
- "Unzip" MPS into a regular one!
  - work from last tensor towards the middle
  - split off isometries for left and right side
  - fuse leftover to next tensor
- Result in mixed canonical form

  $\Longrightarrow$ read off entanglement spectrum directly ☺

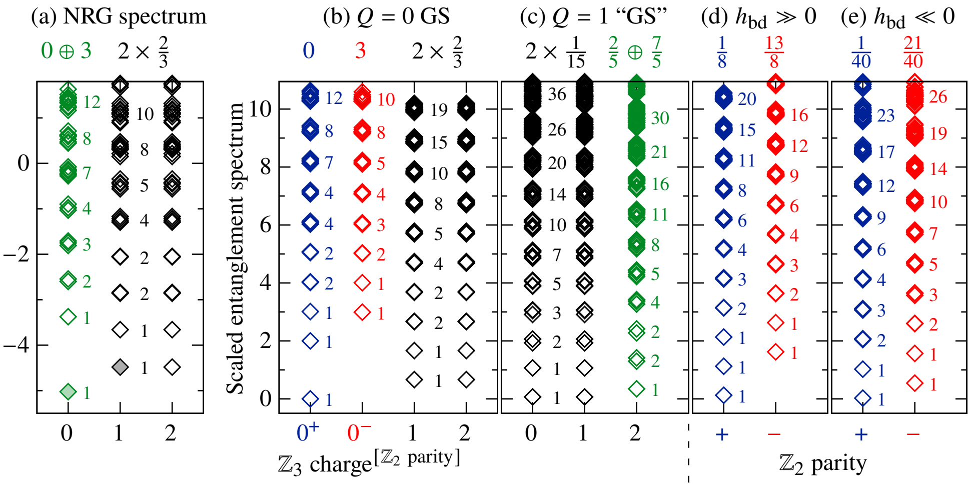


Three-state Potts model

$$H = -\sum_i J_{i+1/2}(X_i X_{i+1}^\dagger + X_i^\dagger X_{i+1}) - \sum_i h_i (Z_i + Z_i^\dagger) \qquad X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad Z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- S_3 symmetry (permutation of X eigenstates)
- For uniform couplings: critical point at $J = h$
- Described by $\mathcal{M}_5$ conformal minimal model
  Dotsenko, Nucl. Phys. B **235**, 54 (1984)
  - 10 primary fields, not all represented in Potts partition functions
  - free boundaries: $(0) \oplus (3) \oplus 2 \times (\frac{2}{3})$
  - can also apply $X + X^\dagger$ boundary fields (symmetry: $S_3 \to \mathbb{Z}_2$)
    Cardy, Nucl. Phys. B **275**, 200 (1986); **324,** 581 (1989)

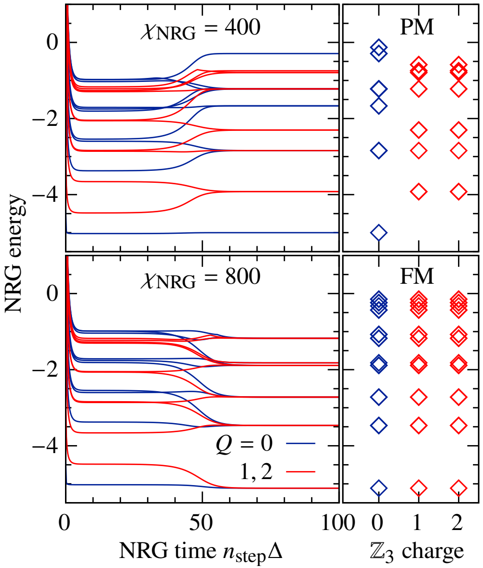
NRG results



- Get all 10 conformal towers $\Longrightarrow$ NRG fixed point carries full information on (chiral) CFT
- $Q = 1$ doesn't match any boundary CFT spectrum (!)

NRG stability

- CFT fixed point is unstable
  - eigenstate cutoff χ_{NRG} is an irrelevant perturbation
  - but it moves critical $g \equiv h/J$ away from 1
  - Δg amount and direction depends on Δ, χ_{NRG} randomly
- Can automatically search g_c
  - stabilises NRG for $\sim$ 500 steps ☺
  - excellent scaling collapse for $g \approx g_c$
  - implied $\nu = 16/3$ far higher than 3-state Potts $\nu = 5/6$!
    $\Longrightarrow$ NRG much more stable ☺



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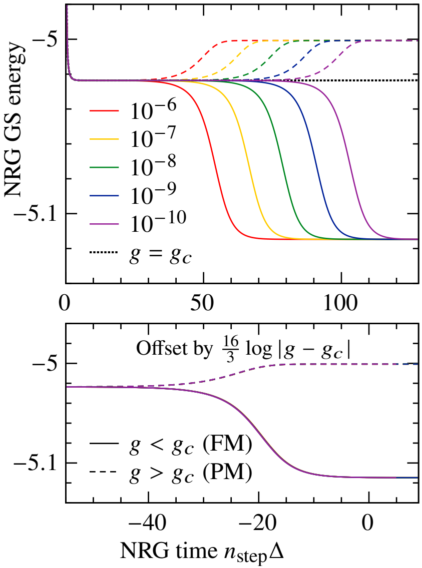
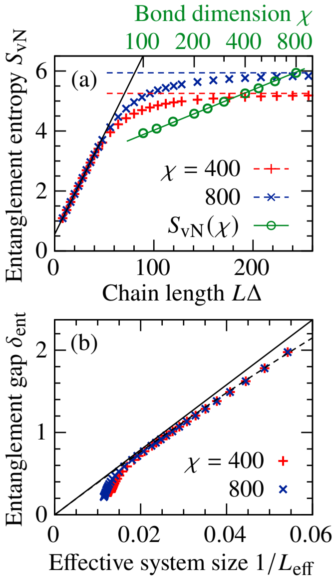


Unzip scaling

- Entanglement entropy
  - follows volume law for small systems
  - saturates at $S = \log \chi - \text{const.}$
  - can still handle $L \approx 250$ chains accurately
- Entanglement gap
  - follows $\delta_{\text{ent}} = 4\pi^2/L_{\text{eff}}$ until S saturates
  - then quickly shrinks
- Entanglement spectrum
  - converges to thermo. limit until saturation
  - then starts to diverge, faster for smaller χ
  - optimum around saturation ($L = 256$)



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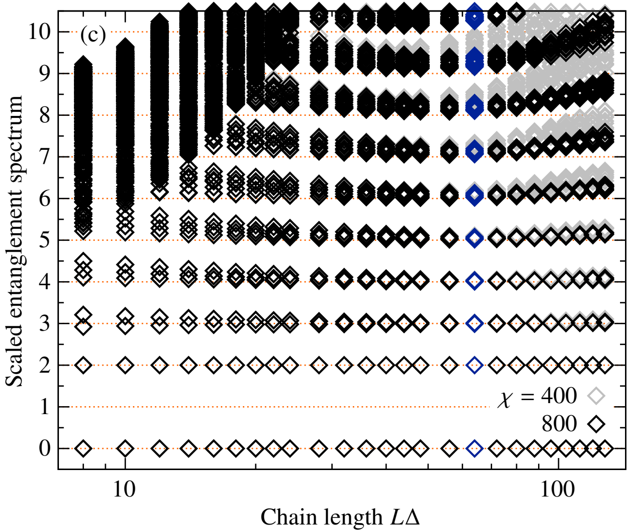


Conclusion & Outlook

Methods to compute (boundary/chiral) CFT spectra from ground state wave functions

- Rainbow chain: finite-size corrections scale with L, not $\log L$
  $\Longrightarrow$ can clearly resolve ~ 10 conformal levels ☺
- Interacting rainbow chains: low-energy states from NRG
  - fixed-point tensor describes **all** conformal towers: How to extract more CFT data?
  - not symmetric under chain reversal $\Longrightarrow$ chiral??

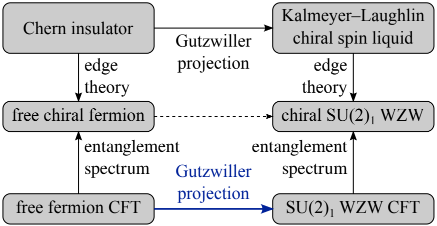
Technical improvements for NRG

- Compute fixed-point tensor directly (iDMRG/vumps?)
- More robust alternative to unzipping
  - MPS representation of Schmidt states
  - DMRG/iDMRG/vumps to compute them
  - how to obtain full spectrum?

Outlook: Parton constructions for CFTs

- Only need wave functions, not parent Hamiltonians!

  $\Longrightarrow$ applicable to any CFT wave function!
- Parton constructions successful for (chiral) topological order
  - fractional quantum Hall
  - chiral spin liquids…
- Equivalent construction on rainbow chains $\longrightarrow$ edge CFT?
- Two copies of free fermions + Gutzwiller projection
  - as edge theory: KL chiral spin liquid
  - get the expected spectrum ☺



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