Reduced-model observers for isothermal gas flows on networks

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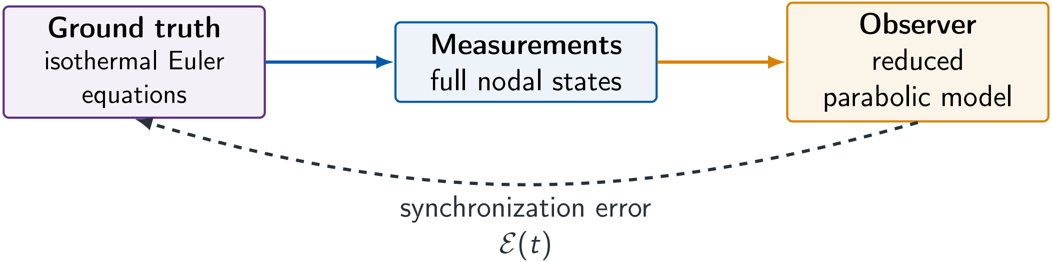
Can observations compensate for a reduced model?



Central question

The measurements drive the observer towards the true state - but the observer itself introduces a persistent modelling error. How do the effects interact?
Can we retain synchronization while avoiding to resolve sound waves?

The ground truth is hyperbolic

On every pipe $e \in E$:

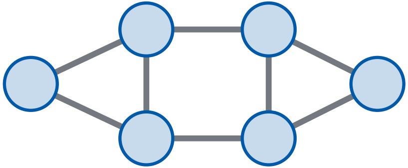
$$\partial_t \rho^e + \partial_x(\rho^e v^e) = 0,$$

$$\partial_t(\rho^e v^e) + \partial_x\left(\rho^e (v^e)^2\right) + \partial_x p(\rho^e) = -\gamma^e \rho^e v^e |v^e|,$$

with $p(\rho) = c^2 \rho$.

At every junction:

- conservation of mass,
- continuity of pressure.



The present result assumes full-state measurements at every node.

Model background: Dafermos (2016); Gugat–Giesselmann (2023).

Low Mach and large friction remove fast dynamics

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Set $t = \varepsilon^{-1}\tau$, $v = \varepsilon w$ and $\gamma = \varepsilon^{-2}\lambda$. Here ε is essentially the **Mach number**.

$$\begin{cases} \partial_\tau \rho + \partial_x(\rho w) = 0, \\ \varepsilon^2 \left[\partial_\tau(\rho w) + \partial_x(\rho w^2)\right] + \partial_x p(\rho) = -\lambda \rho w |w|. \end{cases}$$

In the limit $\varepsilon \to 0$:

$$\boxed{\partial_\tau \widehat{\rho} + \partial_x(\widehat{\rho}\widehat{w}) = 0, \qquad \partial_x p(\widehat{\rho}) = -\lambda \widehat{\rho}\widehat{w}|\widehat{w}|.}$$

We can write

$$\rho p'(\rho)\partial_x \rho = -\lambda |\rho w|(\rho w).$$

The momentum equation loses its evolutionary terms and becomes a nonlinear relation between pressure gradient and mass flux.

Low-Mach/high-friction scaling: Brouwer–Gasser–Herty (2011)

Rigorous low Mach-large friction limit on networks: Egger–Giesselmann (2023).

The observer is a doubly nonlinear parabolic problem

Eliminating the flux from the limiting system and defining a new variable by

$$\Phi'(\rho) = \rho p'(\rho), \qquad u := \Phi(\rho)$$

we can write

$$\partial_\tau \beta(u) - \partial_x \mu(\partial_x u) = 0, \qquad \widehat{\rho} = \beta(u), \qquad \widehat{\rho}\widehat{w} = -\mu(\partial_x u),$$

with nonlinear monotone functions β and μ.

Full model

- two evolutionary equations,
- finite-speed of propagation,
- fast acoustic time scale.

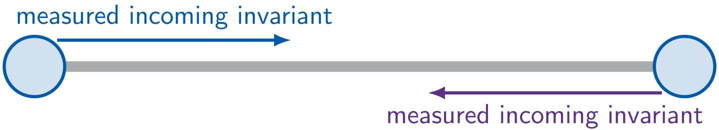
Reduced observer

- one evolutionary variable,
- nonlinear, degenerate diffusion.

Motivation: solve a structurally simpler observer without losing reliable approximation.

No model error $\rightarrow$ nodal data erase the initial error

The hyperbolic system has two Riemann invariants, R_+ and R_-, being transported to left and right, respectively. At a node, the measurements prescribe the invariant entering each pipe.



If ground truth and observer both use the full hyperbolic model, then

$$\boxed{\mathcal{E}(t) \leq e^{-Ct}\mathcal{E}(0).}$$

Here $\mathcal{E}$ is a weighted L^2-type squared error summed over all pipes.

Current measurement assumption

We require the full nodal state at every node, since there is no device for measuring Riemann invariants.

Does model error accumulate?

Inserting the reduced solution $(\widehat{\rho}, \widehat{w})$ into the scaled momentum equation gives the residual

$$r_\varepsilon = \varepsilon^2 \left[\partial_\tau(\widehat{\rho}\widehat{w}) + \partial_x(\widehat{\rho}\widehat{w}^2)\right].$$

Let $\mathcal{R}_\varepsilon(\tau)$ denote its contribution to the error estimate.

Accumulation

Without a stabilizing mechanism, one expects only

$$\mathcal{E}(\tau) \lesssim \mathcal{E}(0) + \int_0^\tau \mathcal{R}_\varepsilon(s)\,\mathrm{d}s.$$

Even a uniformly small residual may then produce an error growing like τ.

Damped memory?

Can synchronization give

$$\mathcal{E}(\tau) \lesssim e^{-C\tau}\mathcal{E}(0) + \int_0^\tau e^{-C(\tau-s)}\mathcal{R}_\varepsilon(s)\,\mathrm{d}s?$$

Then

$$\sup_{\tau\geq 0} \mathcal{E}(\tau) \lesssim \mathcal{E}(0) + \frac{1}{C}\left\|\mathcal{R}_\varepsilon\right\|_{L^\infty}.$$

Central question: Do the measurements turn accumulation into fading memory?

Persistent forcing is filtered

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Consider a hyperbolic observer with a perturbation $g = (g_+, g_-)$ and exact nodal data. Set

$$\mathcal{G}(\tau) := \|g_+(\tau)\|^2_{L^2(E)} + \|g_-(\tau)\|^2_{L^2(E)} \,.$$

The weighted energy estimate gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{E}(\tau) + C\mathcal{E}(\tau) \le C_0\mathcal{G}(\tau).$$

Consequently,

$$\mathcal{E}(\tau) \le e^{-C\tau}\mathcal{E}(0) + C_0 \int_0^\tau e^{-C(\tau-s)}\mathcal{G}(s)\,\mathrm{d}s.$$

In particular,

$$\boxed{\mathcal{E}(\tau) \le e^{-C\tau}\mathcal{E}(0) + \frac{C_0}{C}\,\|\mathcal{G}\|_{L^\infty(0,\tau)}}$$

A persistent perturbation produces an error floor, not accumulation.

Reduced boundary data

Let $n^e(\nu) \in \{-1, 1\}$ denote the outward normal. The measurement prescribes the incoming Riemann invariant:

$$c \log \widehat{\rho}^e(\tau, \nu) - n^e(\nu) \widehat{w}^e(\tau, \nu) =: b^\nu(\tau). \tag{BC}$$

Recall the pressure potential

$$u = \Phi(\widehat{\rho}), \quad \Leftrightarrow \quad \widehat{\rho} = \beta(u) := \Phi^{-1}(u),$$

while the reduced momentum equation gives $\widehat{\rho}\,\widehat{w} = -\mu(\partial_x u)$. Consequently, (BC) becomes

$$n^e(\nu)\mu(\partial_x u^e) + \underbrace{\beta(u^e)\left[c \log \beta(u^e) - b^\nu(\tau)\right]}_{=:B^\nu(u^e)} = 0. \tag{$\widehat{\mathrm{BC}}$}$$

Difficulty
The boundary operator is not globally monotone. It may decrease for small densities.

Resolution
On the relevant density range, a suitable lower bound makes the operator monotone. A monotone continuation below this range allows us to apply monotone-operator theory.

Result: existence and uniqueness of a weak solution.

Reduced-model observer synchronization

The reduced observer has the defect

$$r_\varepsilon = \varepsilon^2 \left[\partial_\tau(\widehat{\rho}\widehat{w}) + \partial_x(\widehat{\rho}\widehat{w}^2)\right],$$

it defines a perturbation to which the previous stability estimate applies.

Theorem (Kumar–JG–Gugat 2026)

If the density is suibaly bounded from below and under some regularity assumptions,

$$\boxed{\mathcal{E}(\tau) \le e^{-C\tau}\mathcal{E}(0) + C_K\varepsilon^2}$$

for the observer driven by exact nodal measurements.

- The error caused by the unknown initial state decays exponentially.
- The continuously generated model error does not accumulate.
- The squared synchronization error remains of order ε^2.

Oberservation prevents model errors from accumulating.

What we know and what remains open

Present result

- valid on general networks,
- full-state measurements at every node,
- exponential deacay of the initial error,
- synchronization up to a low-Mach error floor.

Next steps

- retain measurements at boundary nodes, but reduce measurements at interior junctions,
- understand the role of cycles and topology,
- add discretization and measurement errors.

Take-home message

A reduced parabolic observer approximates hyperbolic gas dynamics well because nodal observations suppresses initial uncertainty and keeps the low-Mach model error from accumulating.

Selected references

M. Gugat and J. Giesselmann,

An observer for pipeline flow with hydrogen blending in gas networks: exponential synchronization, SIAM J. Control Optim., 2023.

H. Egger and J. Giesselmann,

Regularity and long time behavior of a doubly nonlinear parabolic problem and its discretization, Num. Math., 2023.

J. Giesselmann and T. Kunkel,

Identification of minimal number of measurements allowing synchronization of a nodal observer for the wave equation, Comm. Math. Sci., 2025.