Observers for gas network flow: Synchronization

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24.8.2026

Outline

Synchronization: What is it?

Why do we need observers?

Nodal stabilization and observers

Model for the flow in a natural gas pipe

What about mixtures (hydrogen)? The problem of embrittlement

The observer system and exponential synchronization:
Also here the exponential weights as used by JEAN-MICHEL CORON et al. are very useful!

More weight functions to deal more easily with the source terms and the **limits of observability**

Inhalt

Synchronization: What is it?

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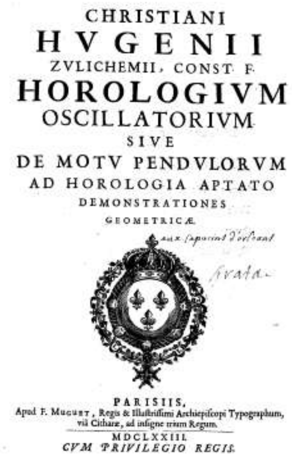

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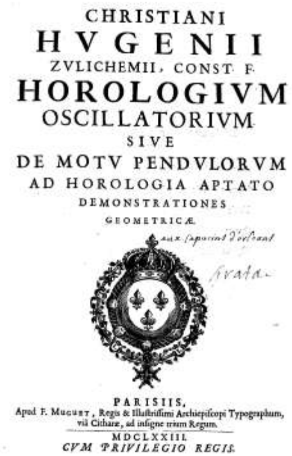
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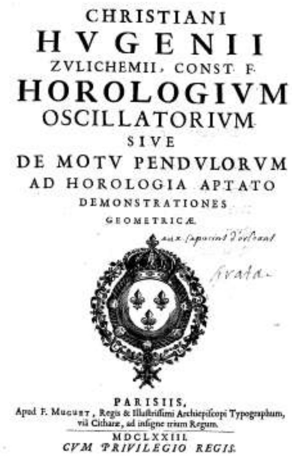
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- **Up to now, no networks! Synchronization of string networks ?**

Synchronization: What is it?

As an application, think of
Synchronization of velocities in pipeline flow of blended gas
`Journal of Mathematical Analysis and Applications` 2026.

- The flow of the gas mixtures is modelled following with conservation of mass and balance of momentum, where the effect of collisions is modelled by suitable coupling terms.
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In **observers**, there are $N = 2$ systems,

1. the original system and
2. the observer system.

The coupling only acts on the second system by information from the first system.

Inhalt

The control system

Control devices (=controllers)

See the results of DFG CRC 154-3

TRR 154 | Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks

In the operation of gas networks,

the control acts at

1. compressor stations,
2. valves,
3. regulators (pressure reducers).

Thus the control acts only in certain points (as in *boundary control*).

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Pointwise measurements: **Nodal Measurements**

Also measurements only provide **data** for the current state that is located in a finite number of points in space i.e. **nodal measurements**.

What is the use of observers?

Why do we need observers?

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- The **output** of the observer at the time t is an approximation of the **full system state** at t.

Optimal control problems and observers

Optimal Control

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The **compressor cost** is a generic objective functional in the mathematical literature:

$$\int_0^T \sum_{v\in V_c} A_v\, q^v(t) \left[\left(\frac{p_{out,v}(t)}{p_{in,v}(t)} \right)^{R_v} - 1 \right] dt$$

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In order to *approximate the full initial state* **observers** are needed!

The definition of the optimal boundary control problems is under scientific discussion: See preprint BREITKOPF, GUGAT, S. ULBRICH

The optimal control problems under discussion

The definition of the optimal boundary control problems is under scientific discussion:

- What is the right regularity?
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- **Desired regularity** can be enforced in
  - the objective function (regularization terms penalize violation)
  - or state constraints (bounds for derivatives).

Inhalt

The stabilization problem and observers

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feedback laws

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The stabilization problem as a special case of observers with exponential decay

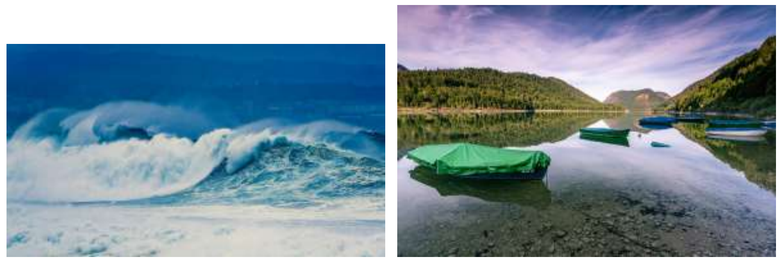

Figure: Dynamic versus static states

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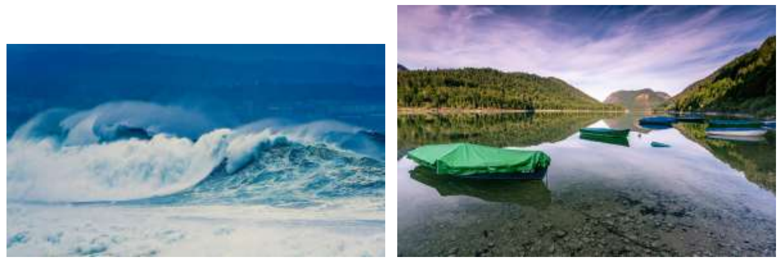

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- The time-independent constant yield the **feedback gains**.

Hence the *limits of stabilizability* imply *limits of observability*!

Inhalt

Model for the flow in a single pipe

The isothermal Euler equations for ideal gas (see e.g. BANDA, HERTY, KLAR(2006) NHM)

For a horizontal pipe, we have

$$\begin{cases} \rho_t + q_x = 0 \\ q_t + \left(c^2\,\rho + \frac{q^2}{\rho}\right)_x = -\frac{1}{2}\,\theta\,\frac{q\,|q|}{\rho} \end{cases}$$

- ρ: density
- q: flow rate
- c: sound speed
- $\theta = \frac{f_g}{D}$
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With sufficiently regular control actions,

no shocks

are created.

Riemann invariants - the system in diagonal form

For a general pressure law $p(\rho)$ the **Riemann invariants** are given by

$$R_{\pm}(\rho, q) = \tilde{R}(\rho) \pm \frac{q}{\rho} \quad \text{with} \quad \tilde{R}(\rho) = \int_1^{\rho} \frac{\sqrt{p'(r)}}{r}\, dr.$$

The **eigenvalues** are

$$\lambda_{\pm} = \frac{R_+ - R_-}{2} \pm \left(p'\left(\tilde{R}^{-1}\left(\frac{R_+ + R_-}{2}\right)\right)\right)^{1/2}.$$

In terms of the Riemann invariants, for an edge $e \in E$ with

$$\sigma^e(R_+^e,\, R_-^e) = \nu^e \left| R_+^e - R_-^e \right| \, (R_+^e - R_-^e)$$

the PDE has the **diagonal form**

$$\partial_t \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix} + \begin{pmatrix} \lambda_+^e & 0 \\ 0 & \lambda_-^e \end{pmatrix} \partial_x \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix} = \sigma^e(R_+^e,\, R_-^e) \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Model for the flow through pipe junctions: Node conditions

Node conditions

Our node conditions model

1. Conservation of mass

Model for the flow through pipe junctions: Node conditions

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Our node conditions model

1. Conservation of mass
2. Continuity of the pressure

Model for the flow through pipe junctions: Algebraic node conditions

Continuity of the pressure

For all $v \in V$ and $e, f \in E_0(v)$ we have

$$p^e(x^e(v)) = p^f(x^f(v)).$$

Also, *perfect mixing of the incoming entropies* has been suggested for the sake of thermodynamic consistency.

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Notation: Outer normal $\mathbf{n}(e, v)$

Let $G = (V, E)$. The pipes $e \in E$ correspond to intervals $[0, L^e]$. Define the '**outer normal**'

$$\mathbf{n}(e, v) = \begin{cases} 0 & \text{if } \; e \notin E_0(v), \\ -1 & \text{if } \; e \in E_0(v) \text{ and } \; x^e(v) = 0, \\ 1 & \text{if } \; e \in E_0(v) \text{ and } \; x^e(v) = L^e. \end{cases}$$

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Conservation of mass

Define the mass flow $m^e = \frac{\pi}{4}(D^e)^2 q^e$.
Define the **incidence matrix**

$$A = (\mathbf{n}(e, v))_{v \in V,\, e \in E}.$$

The balance of mass is given by

$$A\, m = 0.$$

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$A\, m = 0$ (Conservation of mass).

See *On the Cauchy Problem for the p-System at a Junction*, COLOMBO GARAVELLO SIAM J. MATH. ANAL. 39 (2008), 1456-1471.

Inhalt

Hydrogen blending and Hydrogen embrittlement

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- The third eigenvalue is the *velocity* of the gas mixture.
- If the hydrogen has the same average velocity as the mixture, we can still obtain **Riemann invariants**. The new third Riemann invariant is the *concentration*.
- In general *Riemann invariants do not exist*.
  Instead we can use *characteristic variables*.

Hydrogen blending and Hydrogen embrittlement

We extended the observer to a model for a mixture of natural gas and hydrogen.

- We have chosen a 3×3 model with an *additional continuity equation* (see BANDA, HERTY, NGNOTCHOUYE, *Coupling Drift-Flux Models with Unequal Sonic Speeds*, Math. Comput. Appl. 2010).
- At the nodes *perfect mixing* is required, that is the outgoing concentrations are the same (see COLOMBO, MAURI 2008).
- The third eigenvalue is the *velocity* of the gas mixture.
- If the hydrogen has the same average velocity as the mixture, we can still obtain **Riemann invariants**. The new third Riemann invariant is the *concentration*.
- In general *Riemann invariants do not exist*.
  Instead we can use *characteristic variables*.
- There are *two time-scales*:
  Pressure waves are fast and hydrogen concentration travels much more slowly.

The system in diagonal form - (S)

$G = (V,\, E)$: graph of a pipeline network

- $E_0(v)$ denotes the set of edges in the graph that are incident to a vertex $v \in V$.
- $x^e(v) \in \{0, L^e\}$ denotes the end of the interval $[0, L^e]$ corresponding to $e \in E_0(v)$.
- For $v \in V$, let $\omega_v := 2\left(\sum_{f \in E_0(v)} |D^f|^2\right)^{-1}$.

Let the diagonal matrix $diag(S^e)$ contain $\lambda_\pm^e$; $\mu^v \in [-1,\, 1]$. The quasilinear system is

$$
(\mathbf{S}) \begin{cases}
S_+^e(0,x) = y_+^e(x),\ x \in (0, L^e),\ e \in E, \\
S_-^e(0,x) = y_-^e(x),\ x \in (0, L^e),\ e \in E, \\
\\
S_{out}^e(t, x^e(v)) = (1-\mu^v)\, u^e(t) + \mu^v\, S_{in}^e(t, L^e),\ t \in (0,T), \quad \text{if } \ |E_0(v)| = 1, \\
\\
S_{out}^e(t, x^e(v)) = -S_{in}^e(t,\, x^e(v)) + \omega_v \sum\limits_{g \in E_0(v)} (D^g)^2\, S_{in}^g(t,\, x^g(v)),\ t \in (0,T), \\
\hfill \text{if } \ |E_0(v)| \geq 2, \\
\partial_t \begin{pmatrix} S_+^e \\ S_-^e \end{pmatrix} + diag(S^e)\, \partial_x \begin{pmatrix} S_+^e \\ S_-^e \end{pmatrix} = \sigma^e(S_+^e,\, S_-^e) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \ \text{ on } \ [0,T] \times [0, L^e],\ e \in E.
\end{cases}
$$

Well–posedness of the system

Note that with **gas at rest**, we have a **steady state** with constant pressure and constant Riemann invariants

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Semi-global classical solutions (TA-TSIEN LI (Fudan university, Shanghai),...)

Let $T > 0$ and a real number J be given.
Then there exists $\varepsilon(T) > 0$ such that for all **initial data** $y_\pm^e \in C^1(0, L^e)$ such that

$$\tilde{I} := \|y_\pm^e - J\|_{C^1(0, L^e)} \leq \varepsilon(T)$$

and all **control functions** $u^e \in C^1(0, T)$ (e with $|E_0(v)| = 1$ for a $v \in V$) such that

$$\tilde{C} := \|u^e - J\|_{C^1(0,T)} \leq \varepsilon(T)$$

that satisfy the C^1*-compatibility conditions* for (**S**), there exists a

unique classical solution S of (**S**) on $[0, T]$.

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that satisfy the C^1*-compatibility conditions* for (**S**), there exists a

unique classical solution S of (**S**) on $[0,\ T]$.

There exists a constant $\hat{C}_T > 0$ such that the solution satisfies the **a priori bound**

$$\max_{e\in E}\left\{\|S_+^e - J\|_{C^1((0,\,T)\times(0,\,L^e))},\ \|S_-^e - J\|_{C^1((0,\,T)\times(0,\,L^e))}\right\} \leq \hat{C}_T\ \max\{\tilde{I},\ \tilde{C}\}. \qquad (1)$$

Inhalt

The observer system (R)

The idea of the observer system is to replace the node conditions for (**S**) at $v \in V$ in the observer system (**R**) by a **linear combination of two terms**:

- the physical node conditions as in the original system (**S**): $S_{out} = F^v(S_{in})$;
- a nudging term of LUENBERGER type where the *difference*

$$\delta = F(R) - F(S)$$

  between the observer output and the value from the measurements is fed into the observer system (**R**).

$$\boxed{R_{out} = S_{out} + \mu^V \left[F^v(R_{in}) - F^v(S_{in})\right] = \mu^V F^v(R_{in}) + (1 - \mu^v)\, S_{out}.}$$

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The choice of the nodal parameter μ^v allows to control the influence of the measured values that are prone to measurement errors.

If $|1 - \mu^v|$ (i.e. the weight of the Luenberger term) is small, the influence of these errors is damped.
Then the synchronization has the tendency to be slower.

The observer system (R)

For $v \in V$ let $\mu^v \in [0,1]$ control the coupling of the original system to the observer system ($\mu^v = 1$: (**R**) decouples from (**S**); $\mu^v = 0$: full flow of information from (**S**)).

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The observer system has a similar structure as (**S**):

$$
(\mathbf{R}) \begin{cases}
R_+^e(0,x) = z_+^e(x),\ x \in (0,L^e),\ e \in E, \\
R_-^e(0,x) = z_-^e(x),\ x \in (0,L^e),\ e \in E, \\
\\
R_{out}^e(t, x^e(v)) = (1-\mu^v)\, u^e(t) + \mu^v\, R_{in}^e(t, x^e(v)),\ t \in (0,T),\ \text{if } \ |E_0(v)| = 1, \\
\\
R_{out}^e(t, x^e(v)) = S_{out}^e(t,\ x^e(v)) - \mu^v \left[R_{in}^e(t, x^e(v)) - S_{in}^e(t,\ x^e(v))\right] \\
\qquad\qquad + \mu^v\, \omega_v \sum_{g \in E_0(v)} (D^g)^2 \left[R_{in}^g(t,\ x^g(v)) - S_{in}^g(t,\ x^g(v))\right], \\
\qquad\qquad t \in (0,T),\ \text{if } \ |E_0(v)| \geq 2, \\
\\
\partial_t \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix} + diag(R^e)\, \partial_x \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix} = \sigma^e(R_+^e,\ R_-^e) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ on } \ [0,T] \times [0,L^e],\ e \in E.
\end{cases}
$$

The initial state (z_+^e, z_-^e) of (**R**) is an estimation of the initial state of the original system (**S**).

The difference system between (R) and (S)

For the proof of synchronization, we show that the solution of the difference system decays exponentially fast.

Let $\delta = R - S$. For the difference δ we obtain the system

$$
\text{(Diff)}\ \begin{cases}
\delta_+^e(0,x) = z_+^e(x) - y_+^e(x),\ x \in (0, L^e),\ e \in E,\\
\delta_-^e(0,x) = z_-^e(x) - y_-^e(x),\ x \in (0, L^e),\ e \in E,\\[1em]
\delta_{out}^e(t, x^e(v)) = \mu^v\, \delta_{in}^e(t, x^e(v)),\ t \in (0, T),\ \text{if }\ |E_0(v)| = 1,\\[1em]
\delta_{out}^e(t, x^e(v)) = \mu^v \left[-\delta_{in}^e(t,\ x^e(v)) + \omega_v \sum\limits_{g \in E_0(v)} (D^g)^2\, \delta_{in}^g(t,\ x^g(v)) \right],\\
\qquad\qquad\qquad\qquad\qquad\qquad t \in (0, T),\ \text{if }\ |E_0(v)| \geq 2,\\
\partial_t \begin{pmatrix} \delta_+^e \\ \delta_-^e \end{pmatrix} + diag(S^e + \delta^e)\, \partial_x \begin{pmatrix} \delta_+^e \\ \delta_-^e \end{pmatrix} + [diag(S^e + \delta^e) - diag(S^e)]\, \partial_x \begin{pmatrix} S_+^e \\ S_-^e \end{pmatrix}\\
= \left[\sigma^e(\delta_+^e + S_+^e,\ \delta_-^e + S_-^e) - \sigma^e(S_+^e,\ S_-^e)\right] \begin{pmatrix} -1 \\ 1 \end{pmatrix}\\
\quad \text{on }\ [0, T] \times [0, L^e],\ e \in E.
\end{cases}
$$

Result on Exponential Synchronization. Let $T > 0$ be given.

- Assume that for each node $v \in V$, $|\mu^v|$ is sufficiently small.
- Assume that on $[0, T]$ there exists a classical solution of (**Diff**) that satisfies a priori bounds for the C^1-norm and hence for the eigenvalues.
- Assume that the solution of (**S**) and the initial state of (**R**) satisfy appropriate smallness conditions in C^1.

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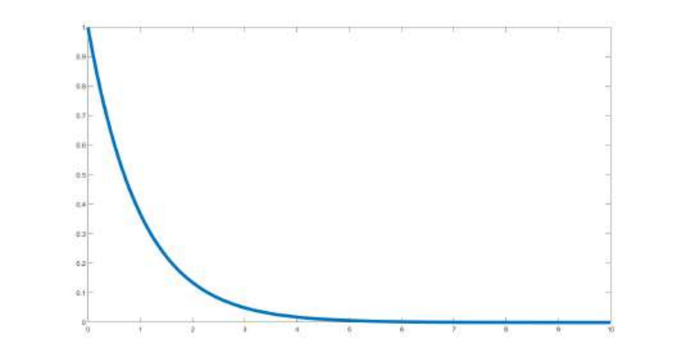
Then the solution of (**Diff**) decays **exponentially fast** in the sense that

there exist constants $C_1 > 0$ and $\chi > 0$ (indep. of T) such that for all $t \in [0,\ T]$ we have $\sum_{e\in E} \int_0^{L^e} |\delta_+^e(t,\ x)|^2 + \left|\delta_-^e(t,\ x)\right|^2\, dx \leq C_1\, \mathrm{e}^{-\chi\, t} \sum_{e\in E} \int_0^{L^e} |\delta_+^e(0,\ x)|^2 + \left|\delta_-^e(0,\ x)\right|^2\, dx.$



Discussion

- Due to the a priori bound, the smallness condition can be achieved for sufficiently small (in C^1) initial and boundary data for (**S**) and for initial data for (**R**) that are sufficiently close to the initial data of (**S**).

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For the proof we use **exponential weights**

$$h_\pm(x) = \exp(\mp\psi\, x)$$

(e.g. Coron, Bastin, d'Andrea-Novel, Hayat).
For boundary observers, see Castillo, Witrant, Prieur, Dugard)

in the **Lyapunov function**

$$\mathcal{E}_0^e(t) = \int_0^{L^e} h_+(x)\,|\delta_+^e(t,\,x)|^2 + h_-(x)\,|\delta_-^e(t,\,x)|^2\,dx.$$

We show that

$$\frac{d}{dt}\mathcal{E}_0(t) \le -\chi\,\mathcal{E}_0(t) - Q$$

with a positive real number $\chi > 0$ and

$$Q = \sum_{e\in E} h_+(x)\,\lambda_+^e(S+\delta)\,|\delta_+^e|^2(t,\,x) + h_-(x)\,\lambda_-^e(S+\delta)\,|\delta_-^e|^2(t,\,x)|_{x=0}^{L^e}.$$

The proofs

For the proof, it suffices to show that for $|\mu^v|$ sufficiently small, we have

$$Q = \sum_{v \in V} Q^v \geq 0,$$

where Q_v contains the terms in Q that correspond to a vertex $v \in V$.
Then **Gronwall's Lemma** implies

$$\mathcal{E}_0(t) \leq \exp(-\chi\, t)\, \mathcal{E}_0(0)$$

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- Similarly, an H^1-Lyapunov function can be found (ongoing research).
- For an H^2-Lyapunov function, we need additional assumptions since in the source term we have $r \mapsto r|r|$ which is C^1 but not C^2.
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See *An Observer for Pipeline Flow with Hydrogen Blending in Gas Networks*

M. Gugat, J. Giesselmann, *SIAM Journal on Control and Optimization 62 (2024)*, (contains the analysis for the 3×3 system that models the flow of the mixture).

Hydrogen embrittlement, "Photo by CEphoto, Uwe Aranas"

- We have also analyzed the effect of L^{∞}-errors (up to level η) in the measurements: It turns out that the observer synchronizes exponentially fast up to an error level η for the Lyapunov function!

Inhalt

Recent developments: Hyperbolic weights (Control and Cybernetics 53, 2024)

Since $\exp(\pm\psi x) = \cosh(\psi x) \pm \sinh(\psi x)$, a natural perturbation is the weight function

$$h_\pm(x) = \sqrt{\upsilon}\cosh(\psi\, x) \mp \sinh(\psi x)$$

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$$h_\pm(x) = \cosh(\psi\, x)\left[\sqrt{\upsilon} \mp \tanh(\psi\, x)\right].$$

Thus if $\sqrt{\upsilon} > |\tanh(\psi\, L)|$, we have $h_\pm(x) > 0$ for all $x \in [-L, L]$ and $h_+(x)/h_-(x) \leq 1$ and $h_-(x)/h_+(x) = (\sqrt{\upsilon} + 1)/(\sqrt{\upsilon} - \tanh(\psi\, L))$ for $x \geq 0$.

Figure 2 shows the graphs of h_+ and h_- for $\psi = L = 1$ and $\upsilon = \frac{3}{2}\tanh^2(\psi\, L)$.

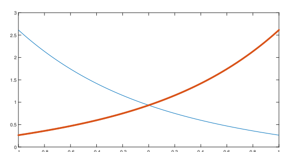

Figure: The weight functions h_+ and h_- for $\psi = L = 1$ and $\nu = \frac{3}{2}\tanh^2(\psi\, L)$.

We get better decay rates!

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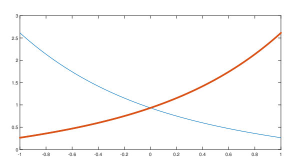

Figure: The weight functions h_+ and h_- for $\psi = L = 1$ and $\nu = \frac{3}{2}\tanh^2(\psi\, L)$.

We get better decay rates!

- The *derivatives* $h'_\pm$ can be represented as a *linear combination of the weight functions* h_+, h_-. Useful to estimate the *source terms* in the Lyapunov method!

Limits of observability: SKIP!

Results about the limits of stabilizability yield limits of observability!

- For quasilinear hyperbolic systems with non-zero source term it can happen that *if the interval is too long, the system is not observable!*
- The celebrated example by G. Bastin and J.-M. Coron:

$$\begin{pmatrix} \delta_+ \\ \delta_- \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \delta_+ \\ \delta_- \end{pmatrix}_x + \begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \delta_+ \\ \delta_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

with the boundary conditions

$$\delta_+(t,\,0) = k_0\,\delta_-(t,\,0), \quad \delta_-(t,\,L) = \delta_+(t,\,L)$$

where k_0 is a feedback parameter. It is shown that for parameters $(L,\,\mathcal{M})$ with $L > 0$, $\mathcal{M} > 0$ and

$$L\,\mathcal{M} \geq \pi$$

there is **no** feedback parameter $k_0 \in \mathbb{R}$ such that the system is exponentially stable.

SKIP!

cycles without controls

- If the system graph contains a **cycle without a control node**, the *system cannot be stabilized*:
- Waves that travel around in the cycle cannot be stabilized from the boundary nodes!

See *Stabilization of a cyclic network of strings by nodal control*, Journal of Evolution Equations 25, 2024.

delay

- Time-delay can destabilize systems that are stable in the absence of delay!

The influence of **time delay** is studied in *Boundary stabilization of quasilinear hyperbolic systems with varying time-delay*, SICON 63, 2025.

SKIP!

Example: A system with constant time-delay $\tau \geq 0$ that is not stable for feedback gains $k_L > 0$ if L or $\mathcal{M}$ is sufficiently large.

Consider

$$\begin{pmatrix} \delta_+ \\ \delta_- \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \delta_+ \\ \delta_- \end{pmatrix}_x + \begin{pmatrix} 0 & \mathcal{M} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_+ \\ \delta_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}$$

with the boundary conditions $\delta_+(t,\,0) = \delta_-(t,\,0)$ and

$$\delta_-(t,\,L) = k_L\,\delta_+(t-\tau,\,L). \tag{4}$$

Consider the initial data

$$\delta_+(t,\,0) = \exp(\omega\,\mathcal{M}\,x) + (2\,\omega - 1)\exp(-\omega\,\mathcal{M}\,x),\;\; \delta_-(t,\,0) = 2\,\omega\;\exp(\omega\,\mathcal{M}\,x).$$

Then for all $t > 0$,

$$\delta_+(t,x) = \mathrm{e}^{\omega\,\mathcal{M}(t+x)} + (2\,\omega - 1)\,\mathrm{e}^{\omega\,\mathcal{M}\,(t-x)}, \quad \delta_-(t,x) = 2\,\omega\,\mathrm{e}^{\omega\,\mathcal{M}\,(t+x)}$$

satisfy (3) and the BC at $x = 0$. Condition (4) is equivalent to

$$\frac{1}{k_L} = \frac{1 + (2\omega - 1)\;\exp(-2\,\omega\,\mathcal{M}\,L)}{2\,\omega\;\exp(\omega\,\mathcal{M}\,\tau)}. \tag{5}$$

$$\frac{1}{k_L} = \frac{1 + (2\omega - 1)\, \exp(-2\,\omega\, \mathcal{M}\, L)}{2\,\omega\, \exp(\omega\, \mathcal{M}\, \tau)} \tag{6}$$

- For the right-hand side of (6) we have
$$\lim_{\omega\to\infty} \frac{1 + (2\omega - 1)\, \exp(-2\,\omega\, \mathcal{M} L)}{2\,\omega\, \exp(\omega\, \mathcal{M} \tau)} = 0.$$
Since
$$\lim_{L\to\infty} \frac{1 + (2\omega - 1)\, \exp(-2\,\omega\, \mathcal{M}\, L)}{2\,\omega\, \exp(\omega\, \mathcal{M}\, \tau)} = \frac{1}{2\,\omega\, \exp(\omega\, \mathcal{M} \tau)},$$
and
$$\lim_{\omega\to 0+} \frac{1}{2\,\omega\, \exp(\omega\, \mathcal{M}\, \tau)} = \infty.$$
- This implies that for all $k_L > 0$ there exist $L > 0$ and $\omega > 0$ such that (6) holds.
- Since $\lim_{t\to\infty} \delta_-(t,\, x) = \infty$, this implies that the system is unstable.
The construction requires that L **is chosen sufficiently large**.
- Moreover, $\lim_{\mathcal{M}\to\infty,\, \omega=\frac{1}{\mathcal{M}}} \frac{1+(2\omega-1)\, \exp(-2\,\omega\, \mathcal{M}\, L)}{2\,\omega\, \exp(\omega\, \mathcal{M}\, \tau)} = \infty$. This implies that for all $k_L > 0$ (for fixed $L > 0$) there exist $\mathcal{M} > 0$ and $\omega > 0$ such that (6) holds.
With this choice, the system is unstable.
The construction requires that $\mathcal{M}$ **is chosen sufficiently large**.

Thank you for your attention!