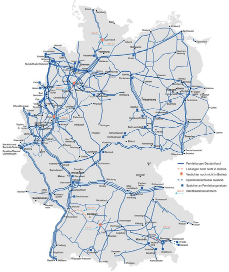
Asymptotic-Preserving Numerical Schemes for Gas Transport on Networks

Benasque XI: Differential Equations, Optimization, Numerics, and Machine Learning

Michael Thiele (joint work with Tabea Tscherpel)
TU Darmstadt



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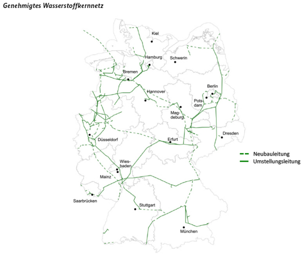

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1 https://www.bundeswirtschaftsministerium.de/Redaktion/DE/Bilder/Energie/gas-fernleitungsnetz-im-ueberblick.html

2 https://www.bundesnetzagentur.de/DE/Fachthemen/ElektrizitaetundGas/Wasserstoff/Kernnetz/start.html

1D compressible Euler equations with friction

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Find a **density** $\rho\colon (0,T)\times(0,\ell)\to\mathbb{R}_+$ and **velocity** $v\colon (0,T)\times(0,\ell)\to\mathbb{R}$ such that

$$
\begin{aligned}
\partial_t\rho + \partial_x(\rho v) &= 0 && \text{in} \quad (0,T)\times(0,\ell),\\
\varepsilon^2\partial_t v + \partial_x\left(\tfrac{\varepsilon^2}{2}v^2 + P'(\rho)\right) &= -\lambda|v|v && \text{in} \quad (0,T)\times(0,\ell),
\end{aligned}
$$

for given

- friction parameter $\lambda > 0$,
- smooth, convex pressure potential $P\colon \mathbb{R}_+\to\mathbb{R}$,
- Mach number $0 \le \varepsilon \ll 1$.

Limit $\varepsilon = 0$:

nonlinear parabolic equation for ρ.

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Goal: structure-preserving discretization

(i) high-order **energy-consistent** time-discretization

(ii) **asymptotic-preserving error estimates** (in $\varepsilon \geq 0$)

$$||\rho_\varepsilon - \rho_\varepsilon^{\tau h}|| + \varepsilon||v_\varepsilon - v_\varepsilon^{\tau h}|| \leq c(\tau + h),$$

with c depending on the regularity of u_ε, but is independent of ε.

Find (ρ, v) such that

$$\partial_t \rho + \partial_x(\rho v) = 0$$
$$\varepsilon^2 \partial_t v + \partial_x(\tfrac{\varepsilon^2}{2} v^2 + P'(\rho)) = -\lambda |v| v$$

Some schemes: (incomplete selection)

- Finite volume methods
  - **energy-consistent schemes** [Tadmor 1987], [Tadmor 2016], [Gassner, Winters, and Kopriva 2016]
  - **asymptotic-preserving schemes** [Jin 1999], [Redle and Herty 2025], [Busto and Dumbser 2022] [Anandan, Arun, Krishnamurthy, and Lukáčová-Medvid'ová 2026]
- Finite element methods
  - high-order **energy-consistent schemes** in time [Andrews and Farrell 2025] [Giesselmann, Karsai, and Tscherpel 2025]
  - first-order **asymptotic-preserving error estimates** [Egger, Giesselmann, Kunkel, and Philippi 2023]

Energy inequality

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$\varepsilon = 1$

Define the **Hamiltonian** as

$$\mathcal{H}(\rho, v) := \frac{\rho v^2}{2} + P(\rho) \qquad \text{and} \qquad \mathcal{H}'(\rho, v) = \begin{pmatrix} \frac{v^2}{2} + P'(\rho) \\ \rho v \end{pmatrix} := \begin{pmatrix} \eta \\ m \end{pmatrix}.$$

Applying chain rule in time and testing with (η, m) yields

$\langle f\,,\,g\rangle := \int_0^\ell f \cdot g \,\mathrm{d}x$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\ell \mathcal{H}(\rho, v) = \left\langle \partial_t \begin{pmatrix} \rho \\ v \end{pmatrix}, \mathcal{H}'(\rho, v) \right\rangle = \left\langle \partial_t \begin{pmatrix} \rho \\ v \end{pmatrix}, \begin{pmatrix} \eta \\ m \end{pmatrix} \right\rangle$$

$$\overset{\text{eq.}}{=} -\lambda \underbrace{\left\langle \frac{1}{\rho}|m|m\,,\, m \right\rangle}_{\geq 0} - \underbrace{\langle \partial_x m\,,\,\eta\rangle + \langle \eta\,,\,\partial_x m\rangle}_{=0} + \underbrace{\eta m\Big|_0^\ell}_{\text{bd. term}}.$$

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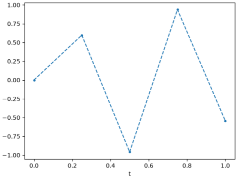
Time semi-discrete spaces for $s \in \{k-1, k\}$.

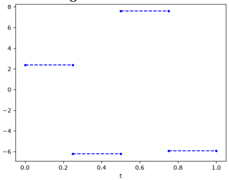
$\mathcal{L}^0_s(\mathcal{I}_\tau; H^1(\Omega)) :=$ piecewise polynomials of order s in $\mathcal{I}_\tau$

$\mathcal{L}^1_s(\mathcal{I}_\tau; H^1(\Omega)) := C([0,T]; H^1(\Omega)) \cap \mathcal{L}^0_s(\mathcal{I}_\tau; H^1(\Omega))$

$\mathcal{L}^1_1$-function



$\mathcal{L}^0_0$-function



Time-discrete modified cont. Petrov Galerkin (cPG) scheme [Giesselmann, Karsai, and Tscherpel 2025]

Find $\rho^\tau, v^\tau \in \mathcal{L}^1_k(\mathcal{I}_\tau; H^1(\Omega))$ such that

$$\langle \partial_t \rho^\tau , q \rangle + \langle \partial_x (m^\tau) , q \rangle = 0,$$

$$\varepsilon^2 \langle \partial_t v^\tau , g \rangle - \langle \eta^\tau , \partial_x g \rangle = \mathcal{B}(g) - \lambda \left\langle \frac{1}{\rho^\tau} |\Pi^0_\tau m^\tau| \Pi^0_\tau m^\tau , g \right\rangle ,$$

for all $q,\ g \in \mathcal{L}^0_{k-1}(\mathcal{I}_\tau; H^1(\Omega))$.

$$m^\tau := \rho^\tau v^\tau$$

$$\eta^\tau := \frac{\varepsilon^2}{2}(v^\tau)^2 + P'(\rho^\tau)$$

$$\langle f , a \rangle := \int_{t^i}^{t^{i+1}} \int_0^\ell f \cdot a \, \mathrm{d}x\mathrm{d}t$$

$\Pi^0_\tau m^\tau$ L^2-projection of m^τ onto $\mathcal{L}^0_{k-1}(\mathcal{I}_\tau; H^1(\Omega))$

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Discrete energy inequality

$$\int_0^\ell \mathcal{H}(\rho^\tau, v^\tau)\,\mathrm{d}x\Big|_{t^i}^{t^{i+1}} = \left\langle \partial_t \begin{pmatrix}\rho^\tau\\ v^\tau\end{pmatrix}, \mathcal{H}'(\rho, v)\right\rangle = \left\langle \partial_t \begin{pmatrix}\rho^\tau\\ v^\tau\end{pmatrix}, \begin{pmatrix}\eta^\tau\\ m^\tau\end{pmatrix}\right\rangle = \left\langle \partial_t \begin{pmatrix}\rho^\tau\\ v^\tau\end{pmatrix}, \Pi^0_\tau \begin{pmatrix}\eta^\tau\\ m^\tau\end{pmatrix}\right\rangle$$

with the temporal L^2-projection $\Pi^0_\tau : L^2((0,T); H^1(\Omega)) \to \mathcal{L}^0_{k-1}(\mathcal{I}_\tau; H^1(\Omega))$ defined by

$$\langle \Pi^0_\tau \xi\,,\, \psi\rangle = \langle \xi\,,\, \psi\rangle \quad \text{for all} \quad \psi \in \mathcal{L}^0_{k-1}(\mathcal{I}_\tau; H^1(\Omega)).$$

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Time-discrete modified cont. Petrov Galerkin (cPG) scheme [Giesselmann, Karsai, and Tscherpel 2025]

Find $\rho^\tau, v^\tau \in \mathcal{L}^1_k(\mathcal{I}_\tau; H^1(\Omega))$ such that

$$\langle \partial_t \rho^\tau , q\rangle + \langle \partial_x (m^\tau) , q\rangle = 0,$$

$$\varepsilon^2 \langle \partial_t v^\tau , g\rangle - \langle \eta^\tau , \partial_x g\rangle = \mathcal{B}(g) - \lambda \left\langle \frac{1}{\rho^\tau} |\Pi^0_\tau m^\tau| \Pi^0_\tau m^\tau , g \right\rangle,$$

for all $q,\ g \in \mathcal{L}^0_{k-1}(\mathcal{I}_\tau; H^1(\Omega))$.

$$m^\tau := \rho^\tau v^\tau$$

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Discrete energy inequality

$$\int_0^\ell \mathcal{H}(\rho^\tau, v^\tau)\,\mathrm{d}x \Big|_{t^i}^{t^{i+1}} = \left\langle \partial_t \begin{pmatrix} \rho^\tau \\ v^\tau \end{pmatrix}, \mathcal{H}'(\rho^\tau, v^\tau) \right\rangle = \left\langle \partial_t \begin{pmatrix} \rho^\tau \\ v^\tau \end{pmatrix}, \begin{pmatrix} \eta^\tau \\ m^\tau \end{pmatrix} \right\rangle = \left\langle \partial_t \begin{pmatrix} \rho^\tau \\ v^\tau \end{pmatrix}, \Pi^0_\tau \begin{pmatrix} \eta^\tau \\ m^\tau \end{pmatrix} \right\rangle$$

$$= \mathcal{B}(\Pi^0_\tau m^\tau) + \underbrace{\langle \partial_x m^\tau , \Pi^0_\tau \eta^\tau\rangle - \langle \eta^\tau , \partial_x \Pi^0_\tau m^\tau\rangle}_{=0} - \lambda \underbrace{\left\langle \frac{1}{\rho^\tau} |\Pi^0_\tau m^\tau| \Pi^0_\tau m^\tau , \Pi^0_\tau m^\tau \right\rangle}_{\geq 0}.$$

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Time discretization

- $\mathcal{I}_\tau$ time grid
- $\Pi^0_\tau : L^2((0,T)) \to \mathcal{L}^0_{k-1}(\mathcal{I}_\tau)$ a **time** L^2-proj.

Space discretization

- $\mathcal{T}_h$ quasi-uniform space grid
- $\Pi^1_h : L^2((0,\ell)) \to \mathcal{L}^1_r(\mathcal{T}_h)$ a **space** L^2-proj.

Fully-discrete scheme

Find $\rho^{\tau h} \in \mathcal{L}^1_k(\mathcal{I}_\tau; \mathcal{L}^0_{r-1}(\mathcal{T}_h))$ and $v^{\tau h} \in \mathcal{L}^1_k(\mathcal{I}_\tau; \mathcal{L}^1_r(\mathcal{T}_h))$ such that

$$\left\langle \partial_t \rho^{\tau h}, q \right\rangle + \left\langle \partial_x \Pi^1_h \Pi^0_\tau m^{\tau h}, q \right\rangle = 0,$$

$$\left\langle \varepsilon^2 \partial_t v^{\tau h}, g \right\rangle - \left\langle \eta^{\tau h}, \partial_x g \right\rangle = \mathcal{B}(g) - \lambda \left\langle \frac{1}{\rho^{\tau h}} |\Pi^1_h \Pi^0_\tau m^{\tau h}| \Pi^1_h \Pi^0_\tau m^{\tau h}, g \right\rangle,$$

for all $q \in \mathcal{L}^0_{k-1}(\mathcal{I}_\tau; \mathcal{L}^0_{r-1}(\mathcal{T}_h))$ and all $g \in \mathcal{L}^0_{k-1}(\mathcal{I}_\tau; \mathcal{L}^1_r(\mathcal{T}_h))$.

$$m^{\tau h} := \rho^{\tau h} v^{\tau h}$$

$$\eta^{\tau h} := \frac{\varepsilon^2}{2}(v^{\tau h})^2 + P'(\rho^{\tau h})$$

$\mathcal{B} :=$ bd. term

✓ energy-consistency

Implementation: Add an auxiliary variable $\widetilde{m}^{\tau h} := \Pi^1_h \Pi^0_\tau m^{\tau h}$

A priori error estimate

Natural-distance: cf. [Liu and Barrett 1994]

$$\mathcal{N}(m,\hat m) = \| \, |m|^{\frac12} m - |\hat m|^{\frac12}\hat m \|_{L^2((0,T)\times(0,\ell))} \eqsim \Big(\int_0^T \int_0^\ell (|m|m - |\hat m|\hat m)\cdot(m-\hat m) \Big)^{\frac12}$$

Theorem [T., Tscherpel in prep.]

Let
- the flow regime be subsonic and the pressure potential P be strictly convex, smooth,
- (ρ, v) be smooth and $\rho \geq \underline{\rho}$,
- $(\rho^{\tau h}, v^{\tau h}) \in \mathcal{L}^1_k(\mathcal{I}_\tau; \mathcal{L}^0_{r-1}(\mathcal{T}_h) \times \mathcal{L}^1_r(\mathcal{T}_h))$ with $\rho^{\tau h} \geq \frac{2}{3}\underline{\rho}$.
- $k = 1$ and $r \geq 2$.

Then, (ρ, v) and $(\rho^{\tau h}, v^{\tau h})$ satisfy for all h, τ, ε sufficiently small uniformly in $\varepsilon \geq 0$.

$$\|\rho - \rho^{\tau h}\|_{L^\infty L^2} + \varepsilon \|v - v^{\tau h}\|_{L^\infty L^2} + \mathcal{N}(\rho v, \Pi^0_\tau \Pi^1_h(\rho^{\tau h} v^{\tau h})) \leq c(\tau + h^r).$$

Asymptotic preservation: uniform control of density error and $\mathcal{N}(\cdot,\cdot)$ for $\varepsilon \to 0$.

A priori error estimate (proof)

$$||\rho - \rho^{\tau h}||_{L^\infty L^2} + \varepsilon ||v - v^{\tau h}||_{L^\infty L^2} + \mathcal{N}(\rho v, \Pi_\tau^0 \Pi_h^1 (\rho^{\tau h} v^{\tau h})) \leq c(\tau + h^r).$$

Ingredients of the proof

1. error splitting

$$||\rho - \rho^{\tau h}||_{L^\infty L^2} \leq ||\rho - \hat{\rho}^{\tau h}||_{L^\infty L^2} + ||\hat{\rho}^{\tau h} - \rho^{\tau h}||_{L^\infty L^2}$$

2. choose discrete approx. $\hat{\rho}^{\tau h}$ of ρ and $\hat{v}^{\tau h}$ of v
3. stability via relative entropy method
4. estimate residual, e.g.,

$$||\partial_x[(\rho v) - \Pi_h^1(\hat{\rho}^{\tau h} \hat{v}^{\tau h})]||_{L^2 L^2} \leq \mathcal{O}(h^r)$$

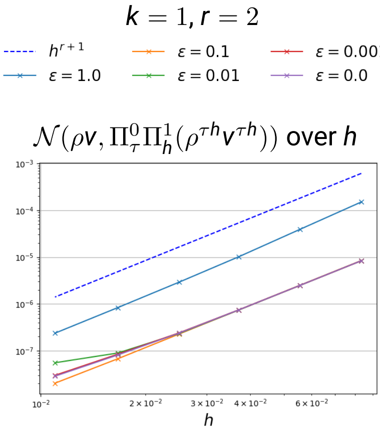
Crucial for optimal order in space

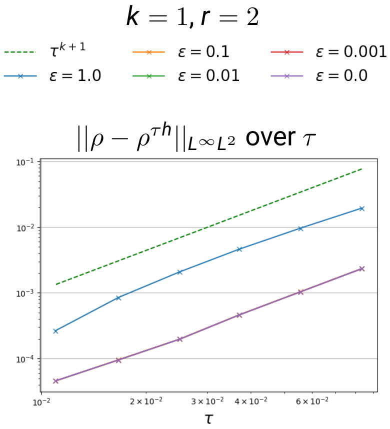
Density weighted interpolation operator for $\hat{v}^{\tau h}$.

Numerical experiments

Asymptotic preserving property in $\varepsilon \geq 0$

- $\ell = 1$
- $\gamma = 1$
- $(\rho^0, v^0) = (1, 0)$
- smooth boundary data
- ideal gas law $P(\rho) = \rho \log(\rho)$

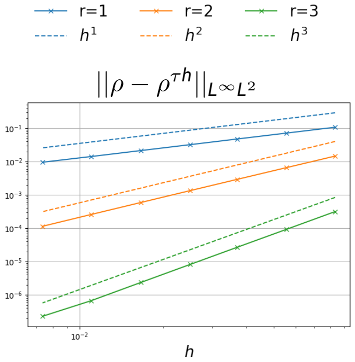




Numerical experiments
Higher order

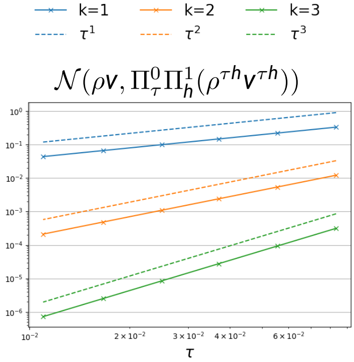
- $\ell = 1$
- $\varepsilon = 0.01$
- $\gamma = 1$
- periodic bc.
- pressure law $p(\rho) = \rho$
- manufactured solution

order in space



h^r (as expected)

order in time



τ^k (observed)

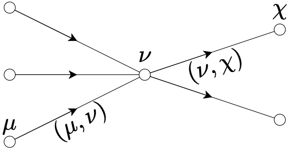
Extension to Networks

Continuous setting

- $(\mathcal{V}, \mathcal{E})$ nodes/edges

$$\eta_e = \frac{\epsilon^2 v_e^2}{2} + P'(\rho_e)$$

$$m_e = \rho_e v_e$$



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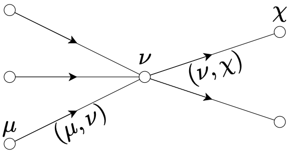

Goal: energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{e \in \mathcal{E}} \int_0^{\ell_e} \mathcal{H}(\rho_e, v_e) \leq \sum_{e=(\nu,\mu) \in \mathcal{E}} \eta_e m_e \Big|_{\nu}^{\mu}$$

Extension to Networks
Continuous setting

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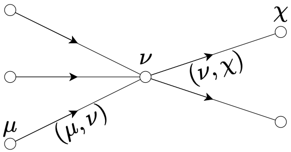
$$\eta_e = \frac{\epsilon^2 v_e^2}{2} + P'(\rho_e)$$
$$m_e = \rho_e v_e$$

- $\mathcal{V}_\partial$ leaf nodes
- $\mathcal{V}_{\text{in}} := \mathcal{V} \setminus \mathcal{V}_\partial$ inner nodes
- $\mathcal{E}(\nu)$ edges adjacent to ν

$$n^e(\nu) = \begin{cases} 1 \text{ if } e = (\cdot, \nu) \\ -1 \text{ if } e = (\nu, \cdot) \end{cases}$$



Goal: energy conservation

$$\frac{\mathrm{d}}{\mathrm{dt}} \sum_{e \in \mathcal{E}} \int_0^{\ell_e} \mathcal{H}(\rho_e, v_e) \leq \sum_{v \in \mathcal{V}_\partial} \text{bd. data} + \sum_{\nu \in \mathcal{V}_{\text{in}}} \sum_{e \in \mathcal{E}(\nu)} \eta_e(\nu) m_e(\nu) n^e(\nu)$$

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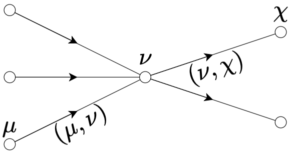

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Coupling conditions:

cf. [Egger and Giesselmann 2023]

- **Mass conservation:** $\displaystyle\sum_{e \in \mathcal{E}(\nu)} m_e(\nu) n^e(\nu) = 0$
- **Enthalpy continuity:** $\eta_e(\nu) = \text{const.}$ for all $e \in \mathcal{E}(\nu)$

- $(\mathcal{V}, \mathcal{E})$ nodes/edges

$$\eta_e^{\tau h} = \frac{\epsilon^2 (v_e^{\tau h})^2}{2} + P'(\rho_e^{\tau h})$$

$$m_e^{\tau h} = \rho_e^{\tau h} v_e^{\tau h}$$

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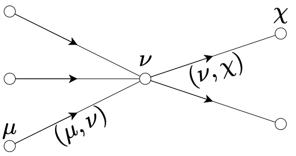


Goal: energy conservation

$$\sum_{e \in \mathcal{E}} \int_0^{\ell_e} \mathcal{H}(\rho_e^{\tau h}, v_e^{\tau h}) \Big|_{t^i}^{t^{i+1}} \leq \int_{t^i}^{t^{i+1}} \sum_{v \in \mathcal{V}_\partial} \text{bd. data}$$

$$\cancel{+ \sum_{\nu \in \mathcal{V}_{\text{in}}} \sum_{e \in \mathcal{E}(\nu)} \eta_e^{\tau h}(\nu) \Pi_\tau^0 \Pi_h^1 m_e^{\tau h}(\nu) n^e(\nu) \, dt}$$

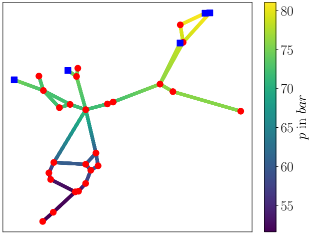
Discrete coupling conditions:

- **Mass conservation:** $\sum_{e \in \mathcal{E}(\nu)} \Pi_\tau^0 \Pi_h^1 m_e^{\tau h}(\nu) n^e(\nu) = 0$
- **Enthalpy continuity:** $\eta_e^{\tau h}(\nu) = \text{const. for all } e \in \mathcal{E}(\nu)$

Conclusion

Summary

- Galerkin scheme is of high-order in space and time (in experiments)
- asymptotic-preserving a priori error estimates uniform in $\varepsilon \geq 0$ for $k = 1$ (time) and $r \geq 2$ (space)
- extension to networks



Outlook

- improvement of convergence rate of the density error in time
- simulations on networks
- extension to mixture models

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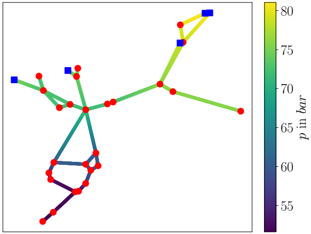

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Thank you!

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