Simultaneous exact-null controllability and applications

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Outline

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Question

Two systems, one input

$$\begin{cases} \dot{z}_1(t) = A_1 z_1(t) + B_1 u(t), \\ \dot{z}_2(t) = A_2 z_2(t) + B_2 u(t). \end{cases}$$

Two different objectives

Given z_1^0, z_2^0 and a target z_1^τ, find one control u such that

$$z_1(\tau) = z_1^\tau, \qquad z_2(\tau) = 0.$$

$$u \in L^2(0, \tau; U)$$

exact target
$z_1(\tau) \;=\; z_1^\tau$

null target
$z_2(\tau) \;=\; 0$

Exact controllability for the first component + null controllability for the second component.

Why this is not just two separate controls

Control z_1 exactly $\rightarrow$ Control z_2 to zero $\rightarrow$ Same signal u

A necessary condition is immediate

Simultaneous exact-null controllability implies:

- (A_1, B_1) is exactly controllable;
- (A_2, B_2) is null controllable.

But it is not sufficient in general

Input-map formulation

Let Φ_τ^k be the input map

$$\Phi_\tau^k u = \int_0^\tau \mathbb{T}^k_{\tau-\sigma} B_k u(\sigma)\,\mathrm{d}\sigma.$$

Set

$$F_\tau = \begin{bmatrix} \Phi_\tau^1 \\ \Phi_\tau^2 \end{bmatrix}, \qquad G_\tau = \begin{bmatrix} I_{X_1} & 0 \\ 0 & \mathbb{T}^2_\tau \end{bmatrix}.$$

Equivalent condition

The simultaneous exact-null property is

$$\operatorname{Ran} G_\tau \subset \operatorname{Ran} F_\tau.$$

This is the point at which Hilbert-space range inclusion and duality enter.

Existing references: simultaneous control

Early and abstract simultaneous controllability

Russell's work on Maxwell equations is an early PDE example: after reduction, one is led to simultaneous control of two wave equations with different boundary conditions.

Tucsnak–Weiss

In [TW00], simultaneous exact controllability is studied for several systems driven by the same input. The emphasis is on exact controllability of all components and on abstract criteria, with applications to wave-type systems.

Difference with the present work

Here the objectives are mixed: one component is controlled exactly, whereas the second one is only required to vanish at the final time.

Existing references: hyperbolic–parabolic mechanisms

Thermoelasticity and related systems

Zuazua [Zua95] proved controllability results for the linear thermoelasticity system, where a hyperbolic elastic component is coupled with a parabolic temperature component. Lebeau–Zuazua later proved null controllability for linear thermoelasticity [LZ98].

Komornik–Tenenbaum

Komornik–Tenenbaum [KT15] use an Ingham–Müntz type theorem to obtain simultaneous observability results for string–heat and beam–heat systems in one space dimension.

Spectral vs. abstract approach

Those results rely heavily on one-dimensional spectral structures. The present argument replaces this by compactness–uniqueness and analyticity.

Existing references: robust observation and parabolic systems

Zuazua's stable observation of sums

Zuazua [Zua16] studies stable observation of additive superpositions of solutions of different PDEs. For wave–heat pairs generated by the same elliptic operator, observability of one component persists in the presence of the other.

Parabolic–parabolic simultaneous control

The purely parabolic case has a different nature: smoothing and Carleman estimates play a central role. Relevant works include Araruna–Chaves-Silva–de Teresa [ACdT26] and Burq–Moyano [BM24]. The controllability background is also connected with Carleman estimates such as those of Imanuvilov–Yamamoto [IY03].

Positioning

The contribution here is a hyperbolic–parabolic common-input theorem allowing boundary controls and a non-self-adjoint parabolic component.

Outline

Output maps

Let (A, C) be an observation system on a Hilbert space X, with output space Y. For $x \in D(A)$ and $t \in (0, \tau)$,

$$(\Psi_\tau x)(t) = C\mathbb{T}_t x.$$

If C is admissible, this map extends continuously to

$$\Psi_\tau : X \longrightarrow L^2(0, \tau; Y).$$

For the two observation systems used below,

$$(\Psi_\tau^k x_k)(t) = C_k \mathbb{T}_t^k x_k, \qquad k = 1, 2.$$

Restriction consistency

If $0 < \sigma < \tau$, then

$$(\Psi_\tau x)|_{(0,\sigma)} = \Psi_\sigma x.$$

This allows us to cut and translate observed trajectories.

Dual output maps

For a control system (A, B) with $B \in \mathcal{L}(U, X_{-1})$, the dual observation system is

$$\dot{z}^d(t) = A^* z^d(t), \qquad u^d(t) = B^* z^d(t).$$

Its output map is denoted by Ψ_τ^d and, for $x \in D(A^*)$,

$$(\Psi_\tau^d x)(t) = B^* \mathbb{T}_t^* x, \qquad t \in (0, \tau).$$

Thus

$$\Psi_\tau^d : X \longrightarrow L^2(0, \tau; U)$$

by admissibility duality.

Input–output duality

With $(R_\tau u)(t) = u(\tau - t)$,

$$\Phi_\tau^* = R_\tau \Psi_\tau^d.$$

This is the bridge between range inclusions and observability estimates.

Individual controllability and observability notions

For a control system (A, B) with input map Φ_τ:

Exact controllability

(A, B) is exactly controllable in time τ if

$$\operatorname{Ran}\Phi_\tau = X.$$

Equivalently, for the dual output map,

$$\|\Psi_\tau^d x\|_{L^2(0,\tau;U)} \geq c_\tau \|x\|_X.$$

Null controllability

(A, B) is null controllable in time τ if

$$\operatorname{Ran}\mathbb{T}_\tau \subset \operatorname{Ran}\Phi_\tau.$$

Equivalently,

$$\|\Psi_\tau^d x\|_{L^2(0,\tau;U)} \geq c_\tau \|\mathbb{T}_\tau^* x\|_X.$$

Simultaneous controllability notions

Consider two control systems with the same input space U.

Simultaneous exact–null controllability

They are simultaneously exact–null controllable in time τ if, for every z_1^0, z_2^0 and every target z_1^τ, there exists $u \in L^2(0,\tau;U)$ such that

$$z_1(\tau) = z_1^\tau, \qquad z_2(\tau) = 0.$$

Equivalently, for every $f_1 \in X_1$ and $z_2^0 \in X_2$,

$$\Phi_\tau^1 u = f_1, \qquad \mathbb{T}_\tau^2 z_2^0 + \Phi_\tau^2 u = 0.$$

Simultaneous approximate controllability

They are simultaneously approximately controllable in time τ if

$$\overline{\operatorname{Ran}\begin{bmatrix}\Phi_\tau^1 \\ \Phi_\tau^2\end{bmatrix}} = X_1 \times X_2.$$

Simultaneous observability notions

Let (A_1, C_1) and (A_2, C_2) have the same output space Y.

Simultaneous approximate observability

The two systems are simultaneously approximately observable in time τ if

$$\Psi_\tau^1 x_1 + \Psi_\tau^2 x_2 = 0 \quad \Longrightarrow \quad x_1 = x_2 = 0.$$

Equivalently, no nontrivial pair of initial states can produce cancelling outputs.

Simultaneous exact–final-state observability

They are simultaneously exact–final-state observable in time τ if there exists $c_\tau > 0$ such that

$$\|\Psi_\tau^1 x_1 + \Psi_\tau^2 x_2\|_{L^2(0,\tau;Y)} \geq c_\tau \left(\|x_1\|_{X_1}^2 + \|\mathbb{T}_\tau^2 x_2\|_{X_2}^2\right)^{1/2}.$$

Dual observability inequality

The range inclusion is equivalent to the estimate

$$\left\| \Psi_\tau^{1d} x_1 + \Psi_\tau^{2d} x_2 \right\|_{L^2(0,\tau;U)} \geq c_\tau \left(\|x_1\|_{X_1}^2 + \|\mathbb{T}_\tau^{2*} x_2\|_{X_2}^2 \right)^{1/2}.$$

First term

Exact observability of the first adjoint component.

Second term

Final-state observability of the second adjoint component.

The observed sum must determine both the exact state and the parabolic final state.

Outline

Main abstract theorem

Theorem (Main result)

Assume that

1. A_1 *generates a bounded strongly continuous group and* (A_1, B_1) *is exactly controllable in time* $\tau_1 > 0$*;*
2. A_2 *generates an analytic, exponentially stable semigroup and* $\mathbb{T}^2_t$ *is compact for every* $t > 0$*;*
3. (A_2, B_2) *is null controllable in every positive time.*

Then the two systems are simultaneously exact-null controllable for every

$$\tau > \tau_1.$$

No self-adjointness of the parabolic generator is required.

Outline

PDE application

Let $\Omega \subset \mathbb{R}^n$ be smooth and bounded, and let $\Gamma \subset \partial\Omega$ be a nonempty relatively open subset satisfying the geometric control condition.

Wave equation

$$\begin{cases} w_{tt} - \Delta w = 0, \\ w = u \text{ on } \Gamma, \\ w = 0 \text{ on } \partial\Omega \setminus \Gamma. \end{cases}$$

Exactly controllable for $\tau > \tau_{GCC}$.

Advection-diffusion equation

$$\begin{cases} \theta_t - \Delta\theta - b \cdot \nabla\theta - c\theta = 0, \\ \theta = u \text{ on } \Gamma, \\ \theta = 0 \text{ on } \partial\Omega \setminus \Gamma. \end{cases}$$

Null controllable and exponentially stable.

Application: conclusion

For every $\tau > \tau_{GCC}$, every initial state

$$(w^0, w^1, \theta^0) \in L^2(\Omega) \times H^{-1}(\Omega) \times H^{-1}(\Omega)$$

and every wave target

$$(w^\tau, w_1^\tau) \in L^2(\Omega) \times H^{-1}(\Omega),$$

there exists one boundary control

$$u \in L^2(0, \tau; L^2(\Gamma))$$

such that

$$(w(\tau), w_t(\tau)) = (w^\tau, w_1^\tau), \qquad \theta(\tau) = 0.$$

Point of the example

The parabolic part may be non-self-adjoint, and the result is not intrinsically one-dimensional.

Outline

Architecture of the proof

Step 1: duality

Pass to the dual observation systems (A_1^*, B_1^*) and (A_2^*, B_2^*).

$$\Downarrow$$

Step 2: analyticity

Prove simultaneous approximate observability of the two dual systems.

$$\Downarrow$$

Step 3: compactness–uniqueness

Upgrade approximate observability to simultaneous exact–final-state observability.

$$\Downarrow$$

Step 4: duality again

Return to simultaneous exact–null controllability.

The compactness–uniqueness theorem (Theorem 4.3)

Let (A_k, C_k), $k = 1, 2$, be admissible observation systems with common output space Y, and output maps Ψ_τ^k.

Theorem (Theorem 4.3)

Assume that

1. A_1 *generates a strongly continuous group* $(\mathbb{T}_t^1)_{t\in\mathbb{R}}$ *and* (A_1, C_1) *is exactly observable in time* $\tau_1 > 0$*;*
2. (A_2, C_2) *is final-state observable in time* $\tau_2 > 0$*;*
3. $\mathbb{T}_t^2$ *is compact for every* $t > 0$*;*
4. *the two systems are simultaneously approximately observable in every time* $\sigma > \tau_1$*.*

Then, for every $\tau > \tau_1 + \tau_2$*,*

$$\|\Psi_\tau^1 x_1 + \Psi_\tau^2 x_2\|_{L^2(0,\tau;Y)} \geq c_\tau \big(\|x_1\|_{X_1} + \|\mathbb{T}_\tau^2 x_2\|_{X_2}\big).$$

If final-state observability holds in every positive time, the conclusion holds for every $\tau > \tau_1$*.*

Proof of Theorem 4.3: a bad sequence

Fix $\tau > \tau_1 + \tau_2$ and choose $\varepsilon > 0$ so that

$$\sigma := \tau - \tau_2 - \varepsilon > \tau_1.$$

If the estimate fails, there are $x_{1,n} \in X_1$, $x_{2,n} \in X_2$ such that

$$\|x_{1,n}\| + \|\mathbb{T}_\tau^2 x_{2,n}\| = 1, \qquad \Psi_\tau^1 x_{1,n} + \Psi_\tau^2 x_{2,n} \longrightarrow 0.$$

Write

$$w_n = \Psi_\tau^1 x_{1,n}, \qquad h_n = \Psi_\tau^2 x_{2,n}.$$

- $(x_{1,n})$ is bounded, hence (w_n) is bounded by admissibility.
- Since $h_n = -w_n + o(1)$, the sequence (h_n) is bounded as well.
- Final-state observability in time τ_2 gives boundedness of

$$q_n := \mathbb{T}_{\tau_2}^2 x_{2,n} \quad \text{in } X_2.$$

Proof of Theorem 4.3: compactness creates strong convergence

On the last interval $(\tau_2+\varepsilon,\tau)$, time invariance gives

$$h_n(\tau_2+\varepsilon+\cdot)=\Psi^2_\sigma(\mathbb{T}^2_\varepsilon q_n).$$

Since $\mathbb{T}^2_\varepsilon$ is compact and (q_n) is bounded,

$$\{h_n|_{(\tau_2+\varepsilon,\tau)}\}\quad\text{is relatively compact in } L^2.$$

Because $w_n+h_n\to 0$, the same holds for the restrictions of w_n.

But

$$w_n(\tau_2+\varepsilon+\cdot)=\Psi^1_\sigma(\mathbb{T}^1_{\tau_2+\varepsilon}x_{1,n}).$$

Exact observability on the interval of length $\sigma>\tau_1$ therefore implies that

$$\mathbb{T}^1_{\tau_2+\varepsilon}x_{1,n}\quad\text{is Cauchy in } X_1.$$

Since $\mathbb{T}^1_{\tau_2+\varepsilon}$ is invertible, $(x_{1,n})$ converges strongly in X_1.

Proof of Theorem 4.3: uniqueness kills the limit

After extraction,

$$x_{1,n} \to x_1, \qquad q_n \rightharpoonup q.$$

Compactness gives

$$\mathbb{T}^2_\varepsilon q_n \to \mathbb{T}^2_\varepsilon q.$$

Passing to the limit in the observed cancellation on the last interval yields

$$\Psi^1_\sigma(\mathbb{T}^1_{\tau_2+\varepsilon} x_1) + \Psi^2_\sigma(\mathbb{T}^2_\varepsilon q) = 0.$$

By simultaneous approximate observability,

$$\mathbb{T}^1_{\tau_2+\varepsilon} x_1 = 0, \qquad \mathbb{T}^2_\varepsilon q = 0.$$

Thus $x_1 = 0$. Moreover,

$$\mathbb{T}^2_\tau x_{2,n} = \mathbb{T}^2_{\tau-\tau_2} q_n \longrightarrow \mathbb{T}^2_{\tau-\tau_2} q = 0$$

by compactness. This contradicts the normalization.

Mechanism

Compactness supplies convergence; approximate observability supplies uniqueness.

Three lemmas behind the analytic step

The analytic part of the proof rests on three elementary but crucial facts.

Lemma 5.3		Lemma 5.4		Lemma 5.5
output smoothness $\Rightarrow$ state regularity	$\rightarrow$	parabolic output is analytic	$\rightarrow$	analytic vector $\Rightarrow$ analytic group output

Role in Theorem 5.6

If two outputs cancel on $(0, \tau)$, analyticity of the parabolic output forces analyticity of the group output. Exact observability then converts this output analyticity into analyticity of the initial group state.

Lemma 5.3: smooth output forces state regularity

Assume that A_1 generates a bounded group and that (A_1, C_1) is exactly observable in time $\tau_1 > 0$:

$$\|\Psi^1_{\tau_1} x\|_{L^2(0,\tau_1;Y)} \geq k_{\tau_1} \|x\|_{X_1}.$$

Lemma (Lemma 5.3)

Let $\sigma > \tau_1$ and $m \in \mathbb{N}$. If

$$\Psi^1_\sigma x \in H^m(0, \sigma; Y),$$

then $x \in D(A_1^m)$ and, for every $j = 0, \ldots, m$,

$$\|A_1^j x\|_{X_1} \leq \frac{1}{k_{\tau_1}} \left\| \frac{d^j}{dt^j} \Psi^1_\sigma x \right\|_{L^2(0,\tau_1;Y)}.$$

Why it matters

The same observability constant works for every derivative order. Thus Cauchy estimates for the output become factorial bounds for $\|A_1^m x\|$.

Lemma 5.4: analytic semigroups give analytic outputs

Lemma (Lemma 5.4)

Assume that A_2 *generates an analytic semigroup* $(\mathbb{T}^2_t)_{t\geq 0}$ *and that*

$$C_2 \in \mathcal{L}(D(A_2), Y)$$

is an admissible observation operator. Then, for every $x_2 \in X_2$ *and every* $\tau > 0$*, the output* $\Psi^2_\tau x_2$ *has a representative on* $(0,\tau)$ *which extends holomorphically to a complex neighbourhood of every compact subinterval of* $(0,\tau)$.

Why it matters

The second component is parabolic. Hence, away from $t = 0$, the map

$$t \longmapsto C_2 \mathbb{T}^2_t x_2$$

is analytic. If $\Psi^1_\tau x_1 + \Psi^2_\tau x_2 = 0$, then the first output inherits this analyticity on a slightly shorter interval.

Lemma 5.5: analytic vectors give analytic observed orbits

Lemma (Lemma 5.5)

Suppose that A_1 generates a strongly continuous semigroup and that

$$x \in \bigcap_{m\geq 0} D(A_1^m), \qquad \|A_1^m x\|_{X_1} \leq M_0 \frac{m!}{r^m} \quad (m \geq 0).$$

Then there exists a Y-valued real-analytic function g on $(0,\infty)$ such that, for every $\tau > 0$, $g|_{(0,\tau)}$ is a representative of $\Psi_\tau^1 x$. Moreover, g extends holomorphically to a complex neighbourhood of every compact subinterval of $(0,\infty)$.

Why it matters

Lemma 5.3 produces the factorial estimates; Lemma 5.5 turns them into real analyticity of the group output on all positive times.

How the three lemmas fit into Theorem 5.6

Start from a cancellation on $(0,\tau)$:

$$\Psi^1_\tau x_1 + \Psi^2_\tau x_2 = 0, \qquad \tau > \tau_1.$$

Choose $\varepsilon > 0$ and set $\sigma = \tau - 2\varepsilon > \tau_1$, $x = \mathbb{T}^1_\varepsilon x_1$. Then

$$\Psi^1_\sigma x = -\big(\Psi^2_{\tau-\varepsilon} x_2\big)(\varepsilon + \cdot).$$

1. Lemma 5.4 makes the right-hand side holomorphic near $[0,\sigma]$.
2. Cauchy's estimates give
$$\left\| \frac{d^m}{dt^m} \Psi^1_\sigma x \right\|_{L^2(0,\tau_1;Y)} \le C \frac{m!}{\rho^m}.$$
3. Lemma 5.3 gives $x \in D(A_1^m)$ and
$$\|A_1^m x\| \le C' \frac{m!}{\rho^m}.$$
4. Since $x_1 = \mathbb{T}^1_{-\varepsilon} x$, the same bounds hold for x_1; Lemma 5.5 gives analyticity of $\Psi^1_s x_1$ for all $s > 0$.

Analytic criterion for approximate observability (Theorem 5.6)

Theorem (Theorem 5.6)

Assume that

- (A_1, C_1) *is exactly observable in time* τ_1 *and* A_1 *generates a bounded group* $(\mathbb{T}_t^1)_{t\in\mathbb{R}}$*;*
- (A_2, C_2) *is final-state observable in some positive time;*
- A_2 *generates an analytic, strongly stable semigroup* $(\mathbb{T}_t^2)_{t\geq 0}$*.*

Then for every $\tau > \tau_1$*,*

$$\Psi_\tau^1 x_1 + \Psi_\tau^2 x_2 = 0 \quad \Longrightarrow \quad x_1 = x_2 = 0.$$

Theorem 5.6: transfer of analyticity

Assume

$$\Psi^1_\tau x_1 + \Psi^2_\tau x_2 = 0, \qquad \tau > \tau_1.$$

Choose $\varepsilon > 0$ with $\sigma = \tau - 2\varepsilon > \tau_1$ and set $x = \mathbb{T}^1_\varepsilon x_1$. Then

$$\Psi^1_\sigma x = -\big(\Psi^2_{\tau-\varepsilon} x_2\big)(\varepsilon + \cdot).$$

The right-hand side extends holomorphically to a complex neighbourhood of $[0, \sigma]$. Cauchy's estimates give

$$\left\| \frac{d^m}{dt^m} \Psi^1_\sigma x \right\|_{L^2(0,\tau_1;Y)} \leq M \sqrt{\tau_1}\, \frac{m!}{\rho^m}.$$

Exact observability applied to $A_1^m x$ yields, with the *same* observability constant,

$$\|A_1^m x\|_{X_1} \leq C \frac{m!}{\rho^m}, \qquad m \geq 0.$$

Hence x_1 is an analytic vector for the group and $t \mapsto C_1 \mathbb{T}^1_t x_1$ is real analytic on $(0, \infty)$.

Theorem 5.6: propagate the cancellation and conclude

The two outputs are now real analytic on $(0, \infty)$. Since their sum vanishes on $(0, \tau)$, the identity theorem gives

$$\Psi_s^1 x_1 + \Psi_s^2 x_2 = 0 \qquad \text{for every } s > 0.$$

Take an exact-observability time $\tau_0 \geq \tau_1$. For every $s > 0$,

$$k_{\tau_0} \|\mathbb{T}_s^1 x_1\| \leq \|\Psi_{\tau_0}^1 (\mathbb{T}_s^1 x_1)\| = \|\Psi_{\tau_0}^2 (\mathbb{T}_s^2 x_2)\| \leq M_{\tau_0} \|\mathbb{T}_s^2 x_2\|.$$

The group is bounded, hence $\|\mathbb{T}_s^1 x_1\| \geq c\|x_1\|$. Strong stability of $\mathbb{T}^2$ gives, as $s \to \infty$,

$$x_1 = 0.$$

Then $\Psi_s^2 x_2 = 0$ for every $s > 0$. Final-state observability gives $\mathbb{T}_S^2 x_2 = 0$ for some $S > 0$; analyticity of the orbit gives $\mathbb{T}_t^2 x_2 = 0$ for every $t > 0$, and strong continuity at 0 yields $x_2 = 0$.

Proof of the main theorem: Step 1 – duality

Return to the control systems (A_k, B_k).

Component 1

Exact controllability of (A_1, B_1) in time τ_1 is equivalent to exact observability of

$$(A_1^*, B_1^*).$$

Since A_1 generates a bounded group, so does A_1^*.

Component 2

Null controllability of (A_2, B_2) in every positive time is equivalent to final-state observability of

$$(A_2^*, B_2^*)$$

in every positive time.

Moreover $(\mathbb{T}_t^{2*})$ is analytic and compact for $t > 0$, and exponential stability of $(\mathbb{T}_t^2)$ implies exponential, hence strong, stability of $(\mathbb{T}_t^{2*})$.

Proof of the main theorem: Step 2 – uniqueness by analyticity

Apply Theorem 5.6 to the two dual observation systems. For every $\sigma > \tau_1$,

$$\Psi_\sigma^{1d} x_1 + \Psi_\sigma^{2d} x_2 = 0 \quad \Longrightarrow \quad x_1 = x_2 = 0.$$

Thus the dual systems are simultaneously approximately observable in every time $\sigma > \tau_1$.

This is the nontrivial compatibility condition

Individual exact/null controllability alone does not guarantee simultaneous control. Analyticity supplies precisely the missing uniqueness property for the sum of the two observations.

Proof of the main theorem: Step 3 – compactness–uniqueness

For the dual systems we now have:

- exact observability of the first component in time τ_1;
- final-state observability of the second component in every positive time;
- compactness of $\mathbb{T}_t^{2*}$ for every $t > 0$;
- simultaneous approximate observability for every $\sigma > \tau_1$.

Therefore Theorem 4.3 yields, for every $\tau > \tau_1$,

$$\|\Psi_\tau^{1d} x_1 + \Psi_\tau^{2d} x_2\|_{L^2(0,\tau;U)} \geq c_\tau \left(\|x_1\|_{X_1}^2 + \|\mathbb{T}_\tau^{2*} x_2\|_{X_2}^2\right)^{1/2}.$$

This is simultaneous exact–final-state observability of the two dual systems.

Proof of the main theorem: Step 4 – back to control

By the range-inclusion/duality criterion,

$$\|\Psi_\tau^{1d} x_1 + \Psi_\tau^{2d} x_2\| \geq c_\tau \left(\|x_1\|^2 + \|\mathbb{T}_\tau^{2*} x_2\|^2 \right)^{1/2}$$

is equivalent to

$$\operatorname{Ran} \begin{bmatrix} I_{X_1} & 0 \\ 0 & \mathbb{T}_\tau^2 \end{bmatrix} \subset \operatorname{Ran} \begin{bmatrix} \Phi_\tau^1 \\ \Phi_\tau^2 \end{bmatrix}.$$

Equivalently, for every $f_1 \in X_1$ and $z_2^0 \in X_2$, there is one control u such that

$$\Phi_\tau^1 u = f_1, \qquad \mathbb{T}_\tau^2 z_2^0 + \Phi_\tau^2 u = 0.$$

Taking $f_1 = z_1^\tau - \mathbb{T}_\tau^1 z_1^0$ gives

$$z_1(\tau) = z_1^\tau, \qquad z_2(\tau) = 0.$$

Thus simultaneous exact–null controllability holds for every $\tau > \tau_1$.

Outline

What is new in the method?

- It treats a mixed hyperbolic/parabolic objective with a common control.
- It converts the problem into a observability estimate for a sum of two outputs.
- It combines a qualitative compactness-uniqueness argument with an analytic-continuation argument.
- It avoids relying on a common spectral basis or commuting differential operators.
- It applies to multidimensional boundary control and non-self-adjoint advection-diffusion dynamics.

Open question 1: stability of the parabolic component

Current assumption in Theorem 2.1

The second semigroup is assumed analytic, compact for $t > 0$, and exponentially stable. It gives final-state observability of the dual system and strong stability for the analytic argument.

Expected weakening

It should be enough to assume a spectral separation condition such as

$$\sigma(A_2) \cap i\mathbb{R} = \varnothing.$$

For analytic compact semigroups, the part of the spectrum in $\{\operatorname{Re}\lambda > 0\}$ is finite-dimensional.

Possible strategy

Split

$$X_2 = X_2^- \oplus X_2^+,$$

where X_2^- is exponentially stable and X_2^+ is finite-dimensional. Apply the theorem to X_2^-, then use finite-dimensional simultaneous controllability ideas, in the spirit of [TW00], for X_2^+.

Open question 2: weakening the group assumption

Where the group property is used

In Theorem 4.3, compactness gives convergence of

$$\mathbb{T}_s^1 x_{1,n}$$

for one fixed positive time s. To recover convergence of $x_{1,n}$ itself, one uses invertibility of $\mathbb{T}_s^1$.

Natural replacement

The argument should still work if, on the relevant time interval,

$$\|\mathbb{T}_s^1 x\|_{X_1} \geq m_s \|x\|_{X_1}, \qquad m_s > 0.$$

Thus a conservative group is not essential; a semigroup which is bounded below at the observation time may suffice.

Question

Can this cover damped or non-conservative hyperbolic systems where exact observability remains true but reversibility is lost?

Open question 3: simultaneous control cost

Qualitative nature of the present proof

The compactness–uniqueness step proves existence of a constant $c_\tau > 0$ in

$$\|\Psi_\tau^{1d} x_1 + \Psi_\tau^{2d} x_2\| \geq c_\tau \left(\|x_1\|^2 + \|\mathbb{T}_\tau^{2*} x_2\|^2\right)^{1/2},$$

but it gives no sharp information on c_τ.

Problem

Estimate the cost of simultaneous exact–null controllability as

$$\tau \downarrow \tau_1,$$

where τ_1 is the minimal exact controllability time of the group component.

Why it matters

For applications, the threshold is not enough: one also needs the blow-up rate of the norm of the control operator near the critical time.

Open question 4: cascade and transmission systems

Different but related models

The present setting is a *common-input* system: the two components are not dynamically coupled, but they must be controlled by the same signal.

Cascade and transmission problems

For genuinely coupled hyperbolic–parabolic systems, recent spectral works such as Lhachemi–Prieur–Trélat [LPT26] analyze wave–heat and heat–wave cascades and sharp reachable-space phenomena. Earlier, Zhang–Zuazua [ZZ04] studied a hyperbolic–parabolic transmission system arising in fluid–structure interaction.

Question

Can the compactness–uniqueness mechanism used here be adapted to such coupled systems, where the cancellation is not only in the observation but also in the dynamics?

Open question 5: several parabolic components

Expected extension

The method should extend to one exactly controlled group component and finitely many parabolic components driven to zero:

$$z_1(\tau) = z_1^\tau, \qquad z_2(\tau) = \cdots = z_N(\tau) = 0.$$

Dual inequality

One would need an estimate of the form

$$\left\| \Psi_\tau^1 x_1 + \sum_{j=2}^{N} \Psi_\tau^j x_j \right\| \geq c_\tau \left(\|x_1\|^2 + \sum_{j=2}^{N} \|\mathbb{T}_\tau^j x_j\|^2 \right)^{1/2}.$$

Main issue

The new point is not compactness itself, but the corresponding simultaneous approximate observability: no nontrivial finite sum of outputs should cancel identically.

Take-home message

One control can steer a group component exactly
and a parabolic component to zero.

The key is a dual observability inequality, obtained
from compactness-uniqueness plus analyticity.

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