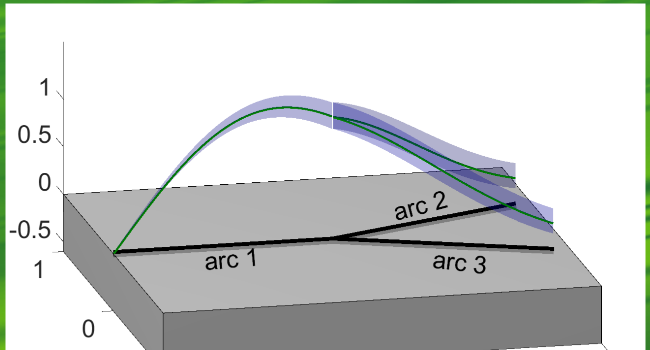
Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

On the stabilizability of hyperbolic systems by boundary control: How to deal with cyclic networks

Martin Gugat, FAU, Department Mathematik, DCN (AvH-Professorship)

Benasque



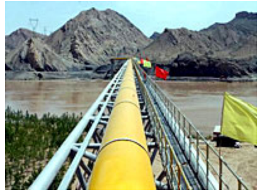
Application: Gas transport through pipelines

The system dynamics in a pipe is described by

the isothermal Euler equations

$$\begin{cases} \rho_t + q_x = 0 \\ \\ q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{f_g}{2\delta}\frac{q|q|}{\rho} - \rho\, g\, \sin(\alpha) \end{cases}$$

or a similar (linearized) model, see *A. Osiadacz, M. Chazykowski*, Comparison of isothermal and non-isothermal pipeline gas flow models, 2001.



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(12 years!)

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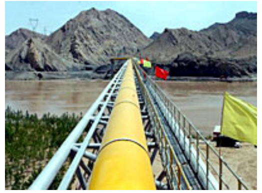

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The Hamiltonian form: With $v = \frac{q}{\rho}$ and

$$H(\rho,\, v) = \frac{1}{2}\rho\, v^2 + c^2\,[\rho \ln(\rho) - \rho]$$

is

$$\begin{cases} \rho_t + \partial_x\, H_v(\rho,\, v) = 0 \\ v_t + \partial_x\, H_\rho(\rho,\, v) = -\frac{f_g}{2\delta} v\,|v|, \quad (see\ CRC1701). \end{cases}$$

The model in diagonal form

For ideal gas, we have the *Riemann invariants*

$$R_{\pm}(\rho,\, q) = c\ln(\rho) \pm \frac{q}{\rho}.$$

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The *pressure* is given by exp$\left(\frac{1}{2\,c}\left(r_+ + r_- + \bar{R}_+ + \bar{R}_-\right)\right) > 0$ and the gas *velocity* is $\frac{r_+ - r_- + \bar{R}_+ - \bar{R}_-}{2}$.

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Travelling waves for the isothermal Euler equations. Let $\theta = \frac{f_g}{\delta}$.

Traveling waves solutions are (see *The isothermal Euler equations for ideal gas with source term: Product solutions, flow reversal and no blow up*; M GUGAT, S ULBRICH) (with a real parameter λ)

$$\begin{pmatrix} q(t,x) \\ \rho(t,x) \end{pmatrix} = \alpha(\lambda\, t - x) \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \quad \text{with} \quad \alpha(z) = C\, \exp\left(\frac{\lambda\,|\lambda|\,\theta + 2\,g\,\sin(\alpha)}{2\,c^2}\, z\right).$$

Here the *Riemann invariants* are $R_{\pm}(t,\, x) = \frac{\lambda\,|\lambda|\,\theta + 2\,g\,\sin(\alpha)}{2\,c}\,(\lambda\, t - x) \pm \lambda.$

- The characteristic curves are straight lines.

Another solution is $\rho = const,\; v = \frac{1}{const + \frac{1}{2}\theta\, t}$.

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$$\boxed{r_t + D(\bar{R})\, r_x = F'(\bar{R})\, r - [D'(\bar{R})(r)](\bar{R})_x}$$

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For the ideal gas we get

$$F'(\bar{R}) = -2\,\frac{f_g}{\delta}\,|\bar{R}_+(x) - \bar{R}_-(x)| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},\quad [D'(\bar{R})(r)]\ D^{-1}(\bar{R}) = \frac{1}{2}\begin{pmatrix} \frac{r_+ - r_-}{\bar{\lambda}_+} & 0 \\ 0 & \frac{r_+ - r_-}{\bar{\lambda}_-} \end{pmatrix}.$$

The matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive semidefinite.

Boundary control problems

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Yes!

The example by Bastin & Coron (interval-case)

In *Stability and Boundary Stabilization of 1-d Hyperbolic Systems*, Bastin & Coron consider

$$\begin{pmatrix} U \\ V \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_x + \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Together with the boundary conditions U(t, 0) = k V(t, 0), V(t, L) = U(t, L)

where $k \in (-1, 1)$ is a feedback parameter (and initial data $U(0, \cdot),\ V(0, \cdot))$) we have a **closed loop-system**.

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Bastin & Coron construct *product solutions* of the form

$$\begin{pmatrix} U(t,\, x) \\ V(t,\, x) \end{pmatrix} = \exp(\sigma\, t) \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}.$$

If such a solution can be found with $\sigma > 0$,
the system is *exponentially unstable* and cannot be stabilized.

The example by Bastin and Coron

TECHNICAL DETAILS - SKIP!

We consider the separation ansatz

$U(t,\,x) = \exp(\sigma\, t)\, f(x), \quad V(t,\,x) = \exp(\sigma\, t)\, g(x)$ for the pde $\begin{pmatrix} U \\ V \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_x + \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

For $\sigma \in (0,\,\beta)$ define $\omega = \sqrt{\beta^2 - \sigma^2} > 0$. The pde and $f(0) = k\, g(0)$ imply

$$\begin{aligned} f(x) &= \quad (\beta + k\,\sigma)\, \sin(\omega\, x) \quad -k\,\omega\, \cos(\omega\, x), \\ g(x) &= -(\sigma + k\,\beta)\, \sin(\omega\, x) \quad -\omega\, \cos(\omega\, x). \end{aligned}$$

For $k \neq -1$ we have $f(L) = g(L)$ if $\sigma \in (0,\,\beta)$ is such that

$$\boxed{(\sigma + \beta)\, \frac{\tan(\sqrt{\beta^2-\sigma^2}\, L)}{\sqrt{\beta^2-\sigma^2}} = \frac{k-1}{k+1}.}$$

For $k = -1$ we have $f(L) = g(L)$ if $\sigma \in (0,\,\beta)$ is such that $0 = \cos(\sqrt{\beta^2 - \sigma^2}\, L)$.
This is possible if $\beta\, L$ is sufficiently large in the sense that

$$\beta\, L > \pi/2.$$

The example by Bastin and Coron
TECHNICAL DETAILS - SKIP!

For $k \neq -1$, in terms of $\omega\, L$, the condition for non–stabilizability is

$$\left(\beta\, L + \sqrt{(\beta\, L)^2 - (\omega\, L)^2}\right) \frac{\tan(\omega\, L)}{\omega\, L} = \frac{k-1}{k+1}. \tag{1}$$

So we analyze the *range* of the function on the left-hand side of (1)!

The *range* of a function
TECHNICAL DETAILS - SKIP!

We consider

$$\boxed{F(\beta\, L) = \sup_{s \in (0, \beta\, L)} \left[\beta\, L + \sqrt{(\beta\, L)^2 - s^2}\right] \frac{\tan(s)}{s} \geq 0.}$$

We substitute $y = \beta\, L$. For $y > 0$, define $F(y) = \sup_{s \in (0,y)} \left[y + \sqrt{y^2 - s^2}\right] \frac{\tan(s)}{s}$.
If $y \geq \pi$, then $F(y) \geq \lim_{s \to \pi-} \left[\pi + \sqrt{\pi^2 - s^2}\right] \frac{\tan(s)}{s} = 0$.

For $y > 0$, define

$$\boxed{G(y) = \inf_{s \in (0,y)} \left[y + \sqrt{y^2 - s^2}\right] \frac{\tan(s)}{s}.}$$

If $y > \frac{\pi}{2}$ then $G(y) \leq \lim_{s \to \frac{\pi}{2}+} \left[y + \sqrt{y^2 - s^2}\right] \frac{\tan(s)}{s} = -\infty$.
Thus for $y = \beta\, L \geq \pi$, the function on the left-hand side of (1) takes all values in $(-\infty, 0]$.

For all $k \in (-1,\ 1]$, this implies the instability of the system
because equation (1) has a solution $s = \omega\, L \in (\pi/2, \pi)$, where $\omega = \sqrt{\beta^2 - \sigma^2}$ correspond to $\sigma = \sqrt{\beta^2 - \omega^2} \in (0,\ \beta)$.

The example by Bastin and Coron

In fact, the following proposition is already proved by Bastin and Coron:

Proposition: If

$$\boxed{\beta\,\mathsf{L} \geq \pi,}$$

there is *no* real value of k such that the closed loop system with the pde

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and the boundary conditions $U(t,\, 0) = k\, V(t,\, 0), \quad V(t,\, L) = U(t,\, L)$
is exponentially stable. The bound is sharp!

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How is the situation on *networks*?

Now we consider a networks of strings.

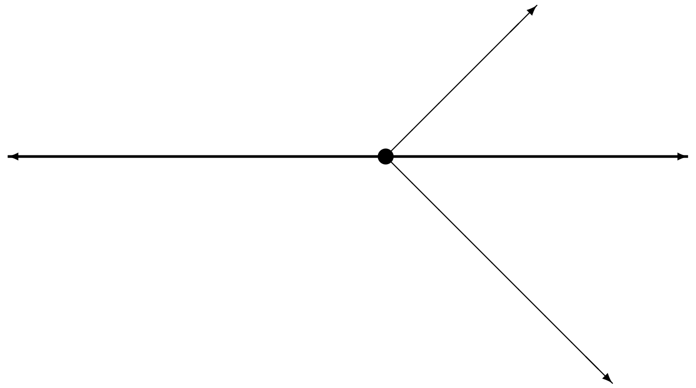

Figure: A star-shaped network with $N = 4$ edges.

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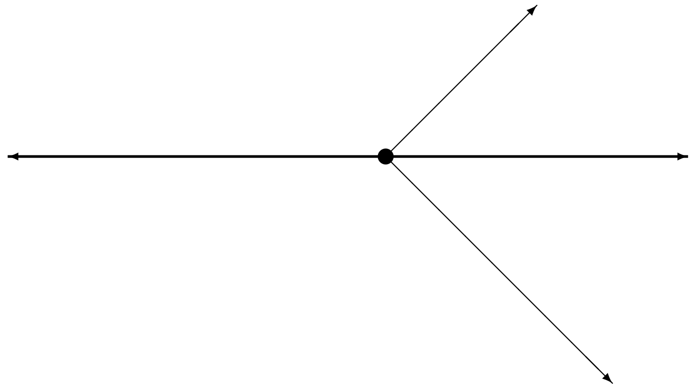

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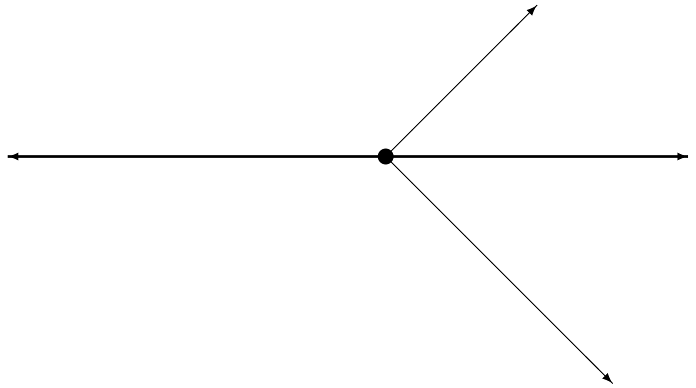

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$$\boxed{U^i_{tt} = U^i_{xx} - 2\,\varepsilon_i\,U^i_t - (\varepsilon_i^2 - \beta_i^2)\,U^i = 0.}$$

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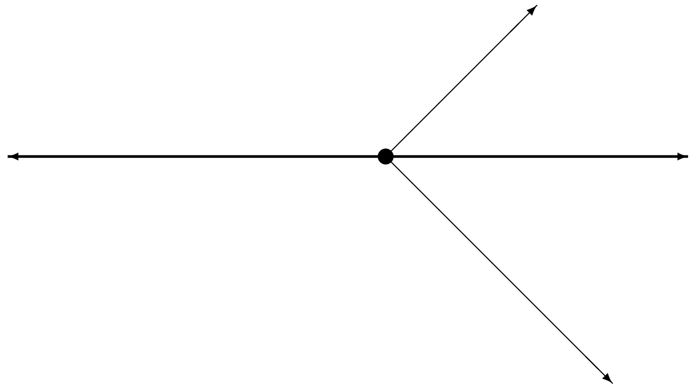

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- The edges are coupled at $x = 0$ by node conditions:

A network of strings: Node conditions for coupling

At the central node: Continuity and Kirchhoff

For $i, j \in \{1, ..., N\}$

$$
\begin{aligned}
U^i(t, 0) - U^j(t, 0) &= 0, \\
\sum_{k=1}^{N} U_x^k(t, 0) &= 0.
\end{aligned}
$$

At the boundary node of edge number 1 at $x = L_1$ we have a homogeneous Dirichlet condition

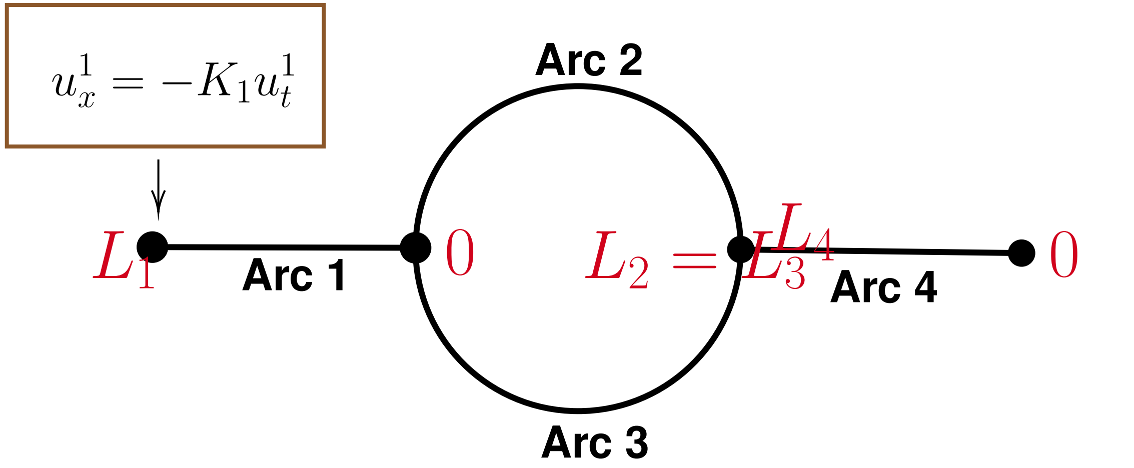
$$U_1(t, L_1) = 0$$

and **at the other boundary nodes** for $j \in \{2, ..., N\}$ at $x = L_j$ we have a Neumann *velocity feedback*

$$U_x^j(t, L_j) = K_j \, U_t^j(t, L_j)$$

with a real *feedback* parameter K_j.

A system with cyclic graph



A cyclic network with 4 edges: Boundary Feedback control at L_1,

- We consider a network with a circle and two additional edges.
- At one boundary node, feedback control takes place.
- At the other boundary node, a homogeneous Dirichlet condition is prescribed.

For $k \in \{1, 2, 3, 4\}$, let $\beta_k > 0, \varepsilon_k \geq 0$ be given.

For $I = \{1, 2, 3, 4\}$, consider the system:

$$\left\{ \begin{array}{l} U_{tt}^k = U_{xx}^k - 2\varepsilon_k \, U_t^k - (\varepsilon_k^2 - \beta_k^2) \, U^k, \quad t > 0, \, x \in (0, L_k), k \in I \\ U^1(t,0) = U^2(t,0) = U^3(t,0), \\ U^2(t, L_2) = U^3(t, L_3) = U^4(t, L_4), \\ \Sigma_{k=1,2,3} \, U_x^k(t,0) = 0, \\ \Sigma_{k=2,3,4} \, U_x^k(t, L_k) = 0, \\ U^4(t,0) = 0, \\ U_x^1(t, L_1) = -K_1 \, U_t^1(t, L_1). \end{array} \right. \tag{2}$$

A system with cyclic graph

The following result is from GUGAT, HUANG, WANG: *Limits of stabilization of a networked hyperbolic system with a circle*, `Control and Cybernetics` 52 (2023):

Theorem

Assume that $\beta_k = \beta > 0$, $\varepsilon_k = \varepsilon > 0$, $L_k = L$, that is the length of the arcs in the network and the parameters are the same for all arcs. Assume that for the initial state the C^0 compatibility conditions are satisfied and $\varepsilon \in (0,\, \beta)$.

- *If $L < L_{min} = \frac{\arctan \sqrt{\frac{2}{7}}}{\sqrt{\beta^2 - \varepsilon^2}}$, system (2) is stabilizable (with $|K_1|$ sufficiently small).*
- *If $L > L_{max} = \frac{\pi}{2\sqrt{\beta^2-\varepsilon^2}}$, the system is **not** stabilizable.*

- If ε is sufficiently close to $\beta > 0$, the value of L_{min} can become arbitrarily **large**.
  If β is sufficiently large, the value of L_{max} can become arbitrarily **small**.
- There is still a *gap* between L_{min} and L_{max}. (Open problem)

Now we state a result for the special case where $\varepsilon = 0$. In this case L_{max} is minimal as a function of ε.

Proposition Under the previous assumptions, for $\beta > 0$ and $\varepsilon = 0$, if

$$\boxed{L > L_{max} = \frac{\pi}{2\,\beta},}$$

the system is **not** stabilizable.

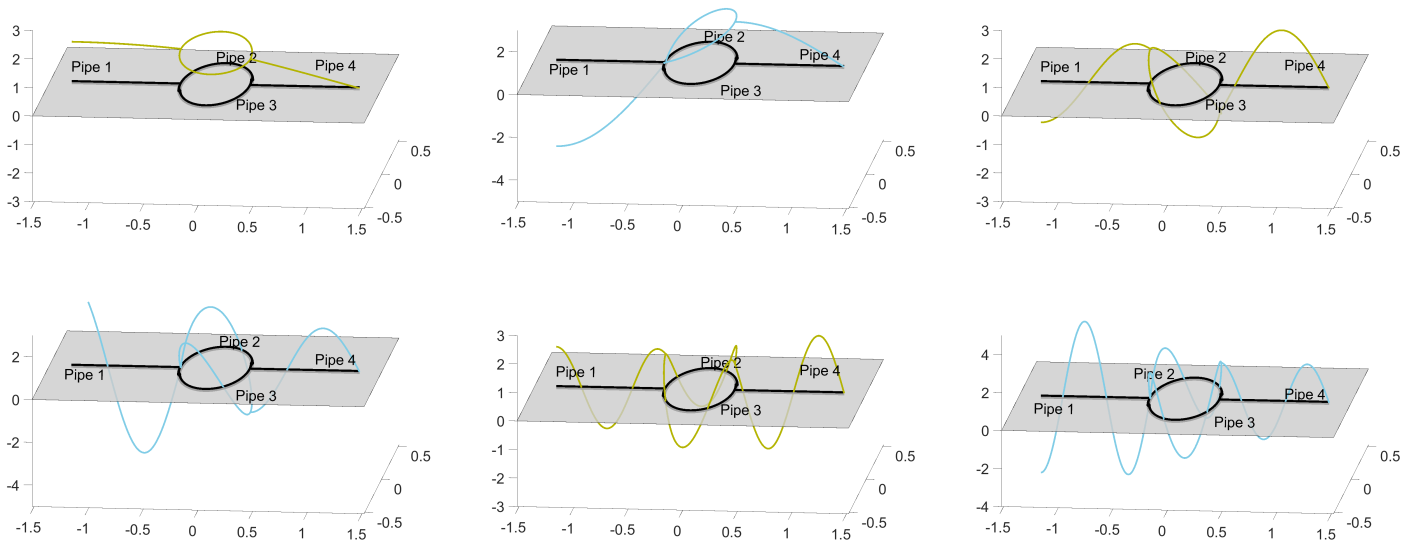
A system with cyclic graph

Illustrations of eigenfuncions

In this subsection, we present the figures of some of the eigenfunction in the system. We take $\varepsilon = \pi, c = \sqrt{1.01}\varepsilon, L = 1$. We obtain the eigenvalues and the corresponding eigenfuctions. The following figures shows the eigenfunction corresponding to the eigenvalues

$$\lambda_{0,1}^{-} = -\pi + \sqrt{1.01\pi^2 - \arctan^2\sqrt{\left(\tfrac{2}{7}\right)}},\ \lambda_{0,2}^{-} = (-1 + \sqrt{0.76})\pi,\ \lambda_{1,1}^{-} = -\pi + \sqrt{-1.01\pi^2 + (\arctan\sqrt{\tfrac{2}{7}} + \pi)^2}\mathrm{i},$$

$$\lambda_{1,2}^{-} = -\pi + \sqrt{1.24}\pi\mathrm{i},\ \lambda_{2,1}^{-} = -\pi + \sqrt{-1.01\pi^2 + (\arctan\sqrt{\tfrac{2}{7}} + 2\pi)^2}\mathrm{i},\ \lambda_{2,2}^{-} = -\pi + \sqrt{5.24}\pi\mathrm{i}.$$

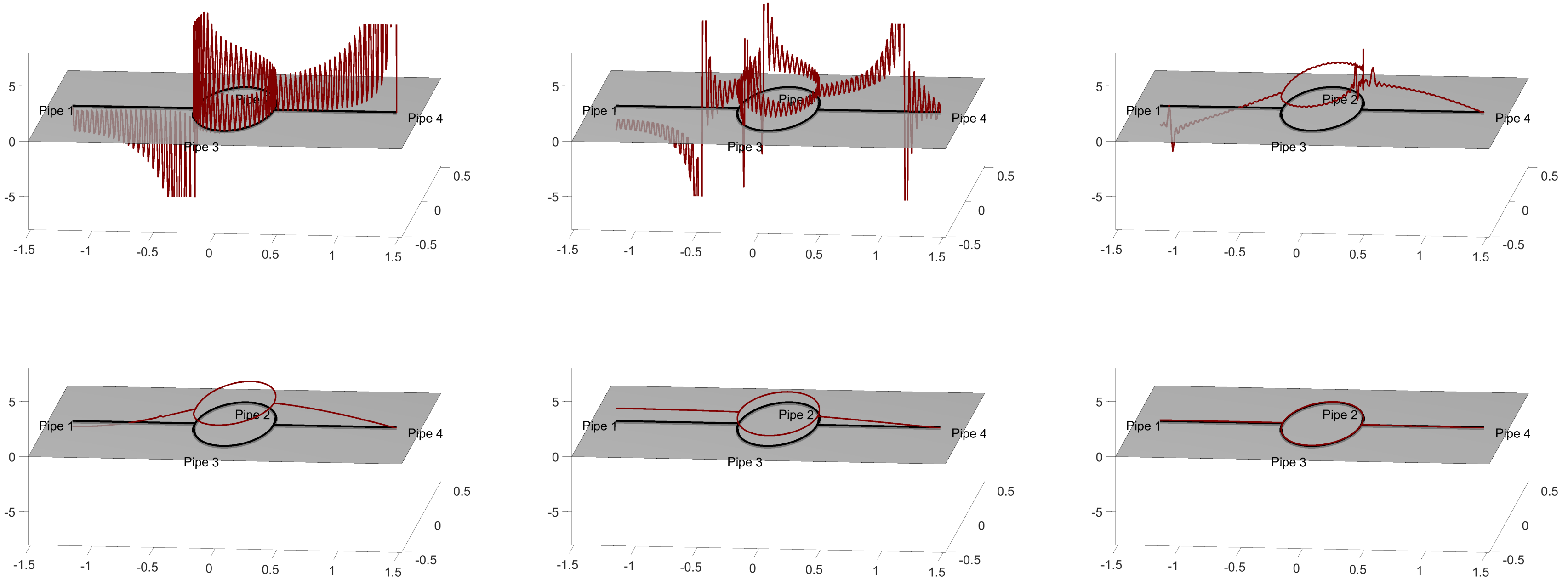


A system with cyclic graph

Illustrations of the evolution

We take $\varepsilon = \pi, \ \beta = \sqrt{1.01 \varepsilon}$, Then $L_{\min} = \frac{\arctan \sqrt{\frac{2}{7}}}{\sqrt{\beta^2 - \varepsilon^2}} \approx 1.5625.$

We take $L_1 = 1 < L_{\min}$. The time evolution of the network can be shown.
The initial data contains highly oscillatory parts that vanish rather quickly with time.



Networks of strings

Relation to first order system

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Then due to the definition of V^i and (14), the function (U^i, V^i) solve

$$\begin{pmatrix} U^i \\ V^i \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U^i \\ V^i \end{pmatrix}_x + \begin{pmatrix} \varepsilon_i & \beta_i \\ \beta_i & \varepsilon_i \end{pmatrix} \begin{pmatrix} U^i \\ V^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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Also the node conditions and boundary conditions can be transformed similarly:
For example, the homogenous DIRICHLET conditions yield

$$V^1(t,\, L_1) = -\frac{1}{\beta_1}\left(U_x^1 + U_t^1\right).$$

The **feedback law** becomes

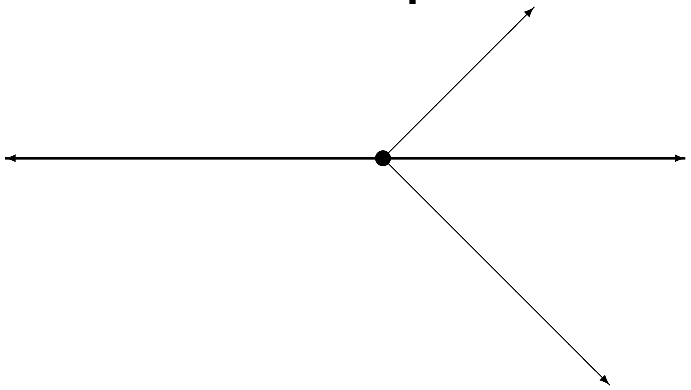
$$V^i(t,\, L_i) = -\frac{1}{\beta_i}\left(\varepsilon_i\, U^i + (K_i + 1)\, U_t^i\right).$$

Stabilization limits for networks

- Also for our **cyclic graph**, *boundary feedback stabilization* is *not always possible*!

Stabilization limits for networks

- Also for our **cyclic graph**, *boundary feedback stabilization* is *not always possible*!
- Now we consider a **star-shaped** networks of strings.



- We consider feedback control at all boundary nodes except one homogeneous Dirichlet node.

A star of strings

Gugat Gerster Syst. Control Lett. 131(2019)

Limits of stabilizability: Assume that for all $i \in \{1, ..., N\}$ we have $\beta_i > \varepsilon_i$

A star of strings

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- **Instability** if **ONE edge is sufficiently long**:
  If
  $$\beta_1^2 > \varepsilon_1^2 + \tfrac{9}{4}\,\frac{\pi^2}{L_1^2} \quad \text{and} \quad \beta_1 - \varepsilon_1 \leq \min_{i\in\{2,...,N\}} \{\beta_i - \varepsilon_i\}, \tag{3}$$
  there are **no values** of $K_2, \ldots, K_N \in (-\infty, 0]$ such that
  the closed loop system is **asymptotically stable.**

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  there are **no values** of $K_2, \ldots, K_N \in (-\infty, 0]$ such that
  the closed loop system is **asymptotically stable.**
- **Instability for a large number of short edges:**
  If
  $$\beta_i = \beta_1,\ \varepsilon_i = \varepsilon_1,\ L_i = L_1,\ K_i = K_2 \quad \text{and} \quad \sin^2(\sqrt{\beta_1^2 - \varepsilon_1^2}\, L_1) = \frac{1}{N},$$
  there are **no values** of $K_2 \in (-\infty,\, \infty)$ $(j \in \{2, ..., N\})$ such that the closed loop system is asymptotically stable.
  This implies
  $$L_1^2\ (\beta_1^2 - \varepsilon_1^2) = \left(\arcsin(\frac{1}{\sqrt{N}})\right)^2.$$
  Since we have $\lim_{N\to\infty} \arcsin(\frac{1}{\sqrt{N}}) = 0$, for N *sufficiently large* we obtain *arbitrarily small values of the lengths* $L_i > 0$, for which the system is not exponentially stable!

So here the *total length* of the strings $N\, L_1^2$ must be sufficiently large!

A sufficient condition for stabilizability

Assume that $\beta_i = \beta_1,\ \varepsilon_i = \varepsilon_1,\ L_i = L_1,\ K_i = K_2$. If $\varepsilon_1 > 0$ and

$$\varepsilon_1 \geq \beta_1 \geq 0$$

with $K_2 = 0$ the closed loop system is exponentially stable.

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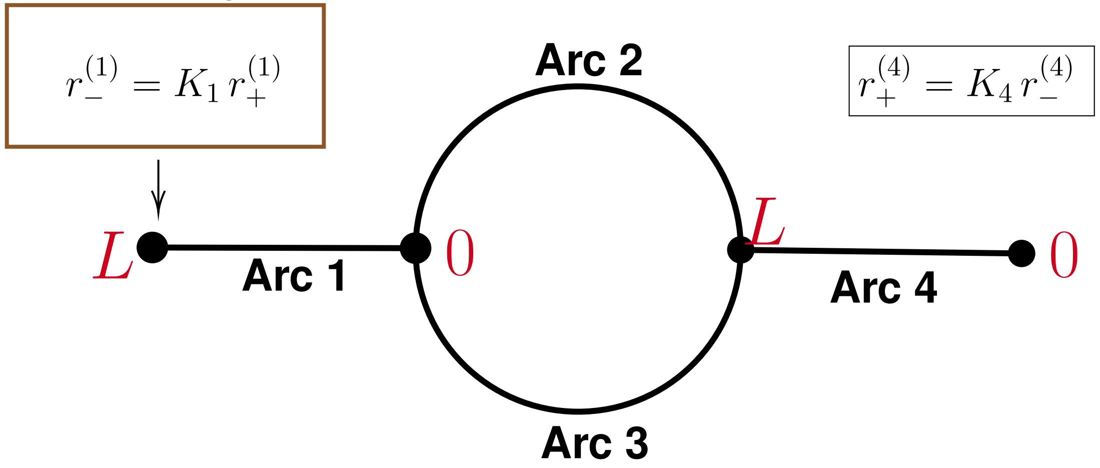
- This can be shown by the analysis of the *eigenfunctions* of the system.
- For $\varepsilon_1 \geq \beta_1 \geq 0$, the matrix of the source term is positive semi-definite.

The quasi-linear case

Consider again the network



but with the *isothermal* EULER *equations* on the edges. At the interior nodes $\bar{x} \in \{0,\ L\}$, the coupling is

$$\begin{pmatrix} r_+^{(1)}(t,\bar{x}) \\ r_+^{(2)}(t,\bar{x}) \\ r_+^{(3)}(t,\bar{x}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} r_-^{(1)}(t,\bar{x}) \\ r_-^{(2)}(t,\bar{x}) \\ r_-^{(3)}(t,\bar{x}) \end{pmatrix}$$

Boundary Feedback control at $x_1 = L$, $x_4 = 0$.
The matrix in the coupling conditions (conservation of mass and continuity of pressure) in terms of Riemann invariants is **orthogonal**!

- **For $\theta = 0$, stabilization is impossible** for a *cyclic* network also **if the length is small**!

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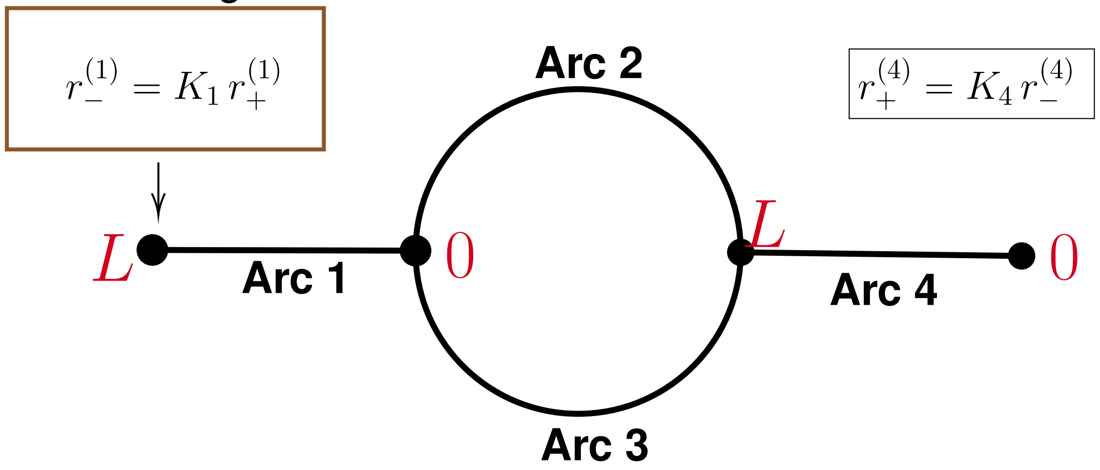

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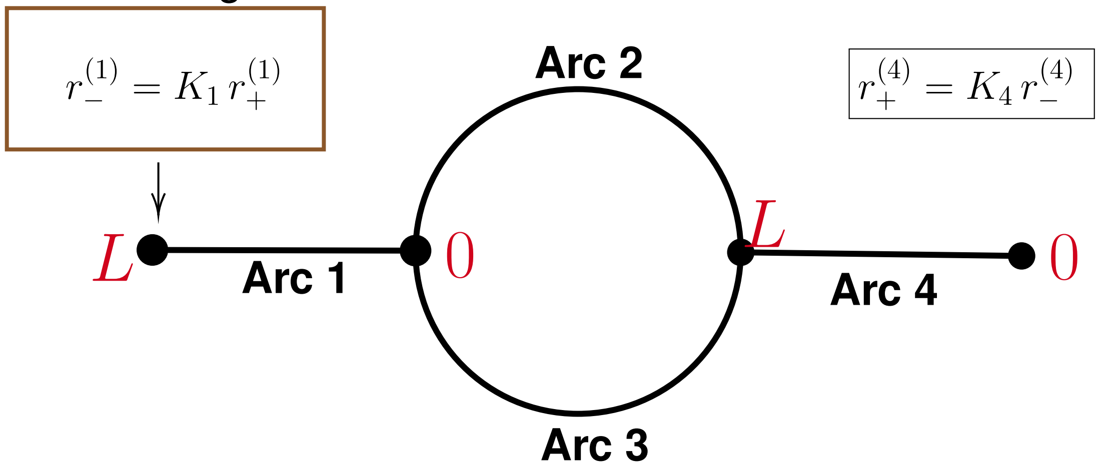

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  This can be seen as follows: Take a steady state with constant pressure, $v > 0$ in the cycle and $v = 0$ outside.
- A `remedy` is to include control action at an interior node, see *Stabilization of a cyclic network of strings by nodal control*, `J. Evol. Equ.` 2024, M. Gugat

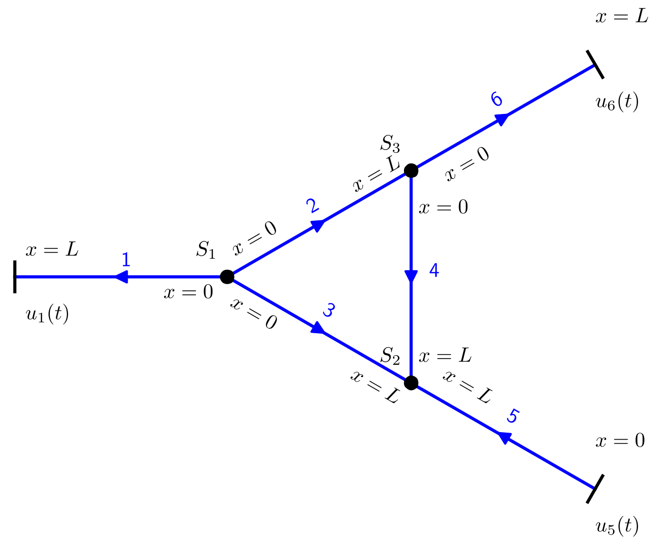


Figure: The graph of the string network

The quasi-linear case

Evolution of a **Lyapunov function**

We need an **additional damper** at a *cycle node*: Replace the coupling matrix Q at the **interior node** $x = L$ by $(1-\mu)Q$!

Consider the Lyapunov function

$$\boxed{\mathcal{E}^{(i)}(t) = \frac{1}{2} \int_{-L}^{L} w_+(x)\, |r_+^{(i)}(t,x)|^2 + w_-(x)\, |r_-^{(i)}(t,x)|^2 \, dx,}$$

$$\mathcal{E}(t) = \mathcal{E}^{(1)}(t) + \mathcal{E}^{(2)}(t) + \mathcal{E}^{(3)}(t) + \mathcal{E}^{(4)}(t)$$

with **hyperbolic weights**

$$\boxed{w_\pm(x) = \sqrt{v}\, \cosh(\psi\,(x+\eta)) \mp \sinh(\psi\,(x+\eta)),}$$

$\psi > 0$, $v > \tanh^2(\psi\,(L+|\eta|)$ see
New Lyapunov functions for systems with source terms, `Control Cybern.` 2024, *BOUNDARY STABILIZATION OF QUASI-LINEAR HYPERBOLIC SYSTEMS WITH VARYING TIME-DELAY*, `SICON` 2025, M. Gugat.

- For $v = 1$, we obtain the exponential weights $w_+(x) = \exp(-\psi\,(x+\eta))$, $w_-(x) = \exp(\psi\,(x+\eta))$, (see Coron, Bastin, d'Andrea-Novel, `IEEE Trans. Autom. Control.` 2007).

Weights given by hyperbolic functions

- Let $\psi > 0$, η and $\upsilon > \tanh^2(\psi(L + |\eta|))$ be given.
  For $x \in [L, L]$, define

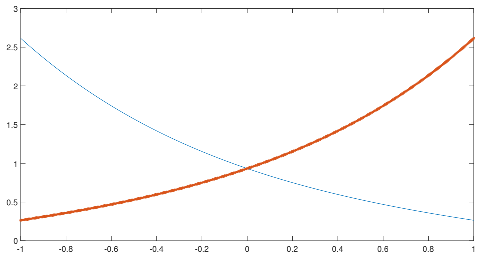
$$\boxed{w_\pm(x) = \sqrt{\upsilon}\,\cosh(\psi\,(x+\eta)) \mp \sinh(\psi\,(x+\eta)),}$$

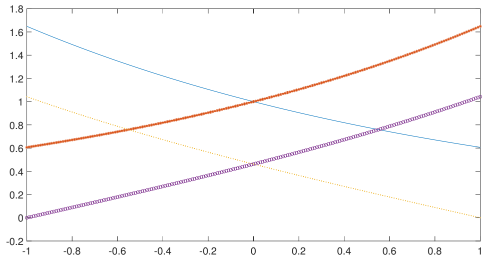
- (a) The functions $h_\pm(\cdot + \eta)$ are *strictly positive* and *strictly convex* on $[-L, L]$ (since $h_\pm'' = \psi^2 h_\pm$).
- (b) We have

$$h_+'(x+\eta) = -\psi\left(\frac{\upsilon+1}{2\sqrt{\upsilon}}\right) h_+(x+\eta) - \psi\left[\frac{1-\upsilon}{2\sqrt{\upsilon}}\right] h_-(x+\eta),$$

$$h_-'(x+\eta) = \psi\left(\frac{1-\upsilon}{2\sqrt{\upsilon}}\right) h_+(x+\eta) + \psi\left[\frac{\upsilon+1}{2\sqrt{\upsilon}}\right] h_-(x+\eta).$$

  If $\upsilon \in (\tanh^2(\psi(L+|\eta|)), 1]$, $h_+(\cdot + \eta)$ is decreasing and $h_-(\cdot + \eta)$ is increasing.





The quasi-linear case

Evolution of the Lyapunov function

For a steady state $\bar{R}$, let $r = R - \bar{R}$ and ε_0 be such that

$$|\partial_x \left(\lambda_\pm(x,\, \bar{R} - \bar{R}\right) | < \varepsilon_0 \tag{4}$$

Consider the system in diagonal form

$$\begin{pmatrix} r_+^{(i)} \\ r_-^{(i)} \end{pmatrix}_t + \begin{pmatrix} \lambda_+(x,\, r^{(i)}) & 0 \\ 0 & \lambda_-(x,\, r^{(i)}) \end{pmatrix} \begin{pmatrix} r_+^{(i)} \\ r_-^{(i)} \end{pmatrix}_x = \begin{pmatrix} G_+(x,\, r^{(i)}) \\ G_-(x,\, r^{(i)}) \end{pmatrix}. \tag{5}$$

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Theory of semi-global solutions (see Tatsien Li)

For given $\varepsilon_0 > 0$, $T > 0$, there exists $\varepsilon_T > 0$ such that for all initial states $r \in \mathcal{B}_{\varepsilon_T}^{C^1}(0)$ that are C^1-compatible with the feedback laws and the node conditions the system has a classical solution on $[0,\, T]$ that satisfies $|\partial_x \left(\lambda_\pm(x,\, \bar{R} - \bar{R}\right) | < \varepsilon_0$ and for some $\delta \geq \gamma > 0$ we have

$$-\delta \leq \lambda_-(\cdot) \leq -\gamma, \;\; \gamma \leq \lambda_+(\cdot) \leq \delta. \tag{6}$$

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Stability-Theorem: Assume that the closed loop system has a classical solution on $[0,\, T]$

such that (4) and (6) hold for all $t \in [0,\, T]$, that we have $(2\,\varepsilon_0 + 6\,\beta)\;\delta\, L < \gamma^2$ and the feedback gains $|K_1|$, $|K_4|$ and $|1 - \mu|$ are sufficiently small
Then the closed loop system (the L^2-norm of r) **decays exponentially fast** on $[0,\, T]$.

The quasi-linear case

Decay of the Lyapunov function: Choice of ψ, η, υ

Under the assumptions of the Stability-Theorem, there exist $\psi > 0$, $\eta > 0$, $\upsilon \in (\tanh^2(\psi\,(L+\eta)),\ 1]$

such that

$$\sqrt{\upsilon} \leq \tanh(\psi\eta)\frac{1+\frac{\gamma}{\delta}}{1-\frac{\gamma}{\delta}}, \tag{7}$$

and

$$3\beta + \varepsilon_0 \leq \frac{\gamma}{2}\,\psi\,\frac{1-\upsilon}{\sqrt{\upsilon}}. \tag{8}$$

For each such choice, $K_1 > 0$, $K_4 > 0$, $\mu \in (0, 1)$ can be chosen such that the **Lyapunov function** $\mathcal{E}(t)$ decays exponentially with the rate

$$\tilde{\psi} = \gamma\,\psi\,\sqrt{\upsilon}. \tag{9}$$

Channel networks

Our analysis also applies to networks of water-channels,

where the dynamics on each edge is governed by the

Saint-Venant equations.

Solutions with shocks

Less regular solutions

Although in many applications these states are not desirable, also solutions with shocks can be considered, see e.g. *Feedback stabilization for entropy solutions of a 2×2 hyperbolic system of conservation laws at a junction*, by GIUSEPPE MARIA COCLITE, NICOLA DE NITTI, MAURO GARAVELLO , FRANCESCA MARCELLINI.

The analysis is only for pipes **without friction** (all steady states are constant).

- The graph is *star-shaped* with feedback at the boundary nods.

non-classical solutions

- For solutions in L^2 (Finite-Energy), the Lypunov analysis could be adapted.
  LIONS P.L., PERTHAME B., TADMOR E., *Kinetic formulation for the isentropic gas dynamics and p-system*, `Commun. Math. Phys.`, 163 (1994), pp. 415-431

The example of Bastin and Coron:

The influence of constant time-delay

Let $\tau > 0$ be a given time delay. For $t \geq 2\,\tau$, we replace the feedback law by

$$\boxed{\delta_+(t,\,0) = k\,\delta_-(t-\tau,\,0).}$$

To complete the system, a **compatible starting phase** on the time interval $[0, 2\,\tau)$ has to be added.

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For $t \in [0,\,2\tau)$ we define

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We choose $\zeta(t)$ as a smooth function with $\zeta(0) = 0$, $\zeta(2\,\tau) = \tau$, $\zeta'(0) = \zeta'(2\tau) = 0$ and $\zeta(t) \leq t$ for all $t \in [0, 2\tau]$.

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Let $\hat{k} > 0$ be given. If

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- Sufficiently large time-delay leads to a decrease of the critical length!
- Is this bound sharp? To be clarified (best in the workshop)!
- At this point, it is appropriate to mention DATKO's classical contributions to the study of time-delay, where it is shown that *arbitrarily small time-delay can destabilize a system that is otherwise stable.*

The influence of varying time-delay

The time-delay is described by a continuously differentiable function $\tau(t)$

with $\tau(t) \geq 0$ and $\tau'(t) < 1$. Thus we have $t - \tau(t) \geq -\tau(0)$. Assume that

$$\sup_t \tau(t) = \bar{\tau} < 1, \quad \sup_t |\tau'(t)| = \hat{\tau} < 1. \tag{11}$$

In order to **initialize the system with time-delay**, we proceed as follows:
Let $t_0 > \bar{\tau}$ be given such that $\tau(t_0) = \bar{\tau}$. For $t \in [0,\, t_0]$ define the function

$$\vartheta_0(t) = \left(1 - \frac{t}{t_0}\right)^2 \left(5 - 2\frac{t}{t_0}\right).$$

Then we have

$$\vartheta_0(t_0) = \vartheta_0'(t_0) = 0,\ \vartheta_0(2\,t_0) = 1,\ \vartheta_0'(2\,t_0) = 0. \tag{12}$$

Let $\sigma \in \{-1,\, 1\}$ and the initial state y_+, $y_- \in C^1([0,\, L])$ be given such that at $x = 0$ the C^1-*compatibility conditions*

$$y_+(0) = \sigma\, y_-(0), \tag{13}$$

$$-\lambda_+(0,\, y_+(0),\, y_-(0))\, y_+'(0) + G_+(0, y_+(0), y_-(0)) = \sigma\, \left[-\lambda_-(0,\, y_+(0),\, y_-(0))\, y_-'(0) + G_-(0, y_+(0), y_-(0))\right]$$

hold.

The influence of varying time-delay

Compatible starting phase

- We consider the initial condition

$$\delta_+(0,\, x) = y_+(x),\ \delta_-(0,\, x) = y_-(x). \tag{14}$$

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- Let a function $f_0 \in C^1([0, 2\,t_0])$ be given that satisfies the C^1 compatibility conditions with (14) at $x = L$, that is $f_0(0) = y_-(L)$,
$$f_0'(0) = G_-(L, y_+(L), y_-(L)) - \lambda_-(L, y_+(L), y_-(L))\, y_-'(L).$$
For $t \geq 0$ system (5) is completed with the boundary conditions
$$\delta_+(t,\, 0) = \sigma\, \delta_-(t,\, 0) \tag{15}$$
and (with a real number k_L that serves as the feedback gain)
$$\delta_-(t,\, L) = \begin{cases} f_0(t), & t \in [0,\, t_0) \\ (1 - \vartheta_0(t))\, f_0(t) + \vartheta_0(t) k_L\, \delta_+(t - \tau(t),\, L), & t \in [t_0,\, 2\,t_0) \\ k_L\, \delta_+(t - \tau(t),\, L) & t \in (2t_0,\, T]. \end{cases} \tag{16}$$
Due to (12) the boundary condition is continuously differentiable with respect to t.

The influence of time-delay

The quasi-linear case interval [0, L]

Theorem StabiD: Assume that the closed loop system has a classical solution on $[0, T]$

such that (4) and (6) hold for all $t \in [0, T]$, that we have $(2\,\varepsilon_0 + 6\,\beta)\;\delta\,L < \gamma^2$ and the feedback gains $|K_0|$, $|K_L|$ are sufficiently small.
Then the closed loop system (the L^2-norm of r) **decays exponentially fast** on $[0, T]$.

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- To be precise that is there exist constants $C_1 > 0$ and $\tilde{\psi} > 0$ that are independent of T such that for all $t \in [2\,t_0,\, T]$

$$\int_0^L \delta_+^2(t,\, x) + \delta_-^2(t,\, x)\, dx$$

$$\leq C_1\, \mathrm{e}^{-\tilde{\psi}\,(t-2t_0)} \left(\int_0^L \delta_+^2(2t_0,\, x) + \delta_-^2(2t_0,\, x)\, dx + \int_0^{\tau(2t_0)} \mathrm{e}^{-\tilde{\psi}\, s}\, \delta_+^2(2t_0 - s,\, L)\, ds \right). \tag{17}$$

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- In the proof, in the Lyapunov function a **delay term** $\mathcal{D}(t)$ is added to $\mathcal{E}(t)$:

$$\mathcal{D}(t) := \frac{K_L}{2} \int_0^{\tau(t)} \exp(-\tilde{\psi}\, s)\, |\delta_+(t-s,\, L)|^2\, ds. \tag{18}$$

See *BOUNDARY STABILIZATION OF QUASI-LINEAR HYPERBOLIC SYSTEMS WITH VARYING TIME-DELAY*, `SICON` 2025, M. Gugat.

Final remarks

Figure: Der Wanderer über der Welt

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