Course on Two-dimensional Bose gases

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Introduction

The role of dimensionality in physics

Physics is qualitatively changed when dimension is reduced. Topology is not the same as in 3D.

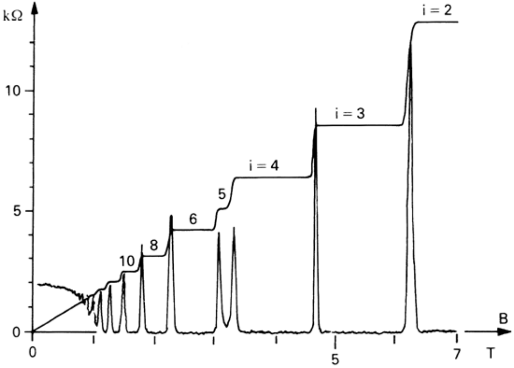
Examples include:

- in 1D: absence of thermalisation of a 1D gas, ‘fermionization’ of an interacting Bose gas, renormalization of the interactions, role of solitons...
- in 2D: absence of long-range order, (fractional) quantum Hall effect, Kosterlitz-Thouless transition, renormalization of the interactions, role of vortices...

Introduction

Example in 2D: the Quantum Hall Effect

- 2D electron gas a nction
- longitudinal curre
- measure the tran



- plateaux of Hall resistance $R = \frac{V_y}{I_x} = \frac{h}{\nu e^2}, \quad \nu \in \mathbb{N}^*$
- longitudinal resistance $R_x = \frac{V_x}{I_x} = 0$

2D: A marginal dimension

Scaling symmetry, topology, quasi long-range order… and lots of logs

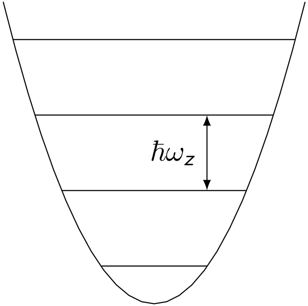
2D is a very special case!

- **Condensation and superfluidity**
    - No BEC at $T > 0$ for the homogeneous ideal gas
    - BEC recovered in a trap
    - Interactions induce a quasi long-range order…
    - …and enable a transition to a superfluid state
- **Topology**
    - Role of vortices in the superfluid transition
    - Analogy with Quantum Hall effect for the rotating gas
    - A KT(HNY) transition for the vortex lattice
- **Scaling invariance**
    - (almost) no length scale, dimensionless interaction strength
    - Equation of states depending only on $\alpha = \mu / k_B T$
    - Undamped monopole mode

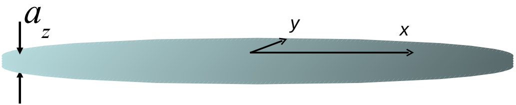
Production of 2D gases

General idea

Experimental realization of 2D gases: strongly confine the transverse direction z



$$k_B T \ll \hbar\omega_z \quad \mu \ll \hbar\omega_z$$



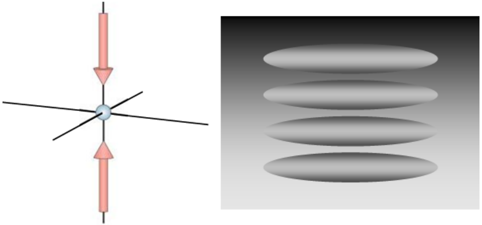
$$a_z = \sqrt{\frac{\hbar}{M\omega_z}}$$

Production of 2D gases

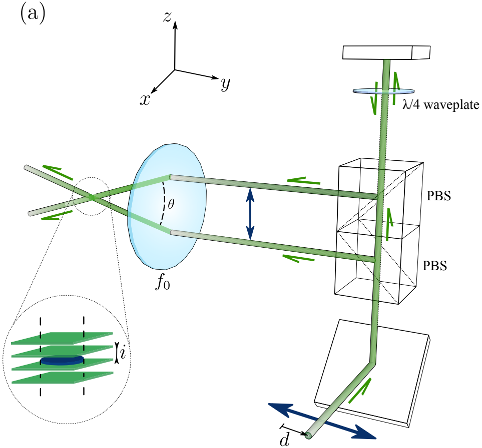
Optical lattices

Experimental realization of 2D gases: strongly confine the transverse direction ($k_B T, \mu \ll \hbar\omega_z$)

1. optical lattices along 1 axis



series of parallel 2D gases



Bloch, Nat. Phys. **1**, 23 (2005)

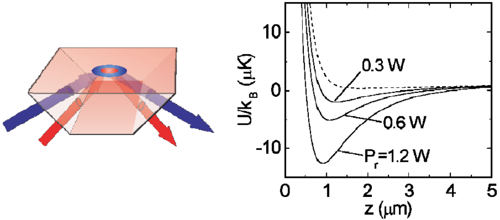
Ville et al., PRA **95**, 013632 (2017)

Production of 2D gases

Optical lattices

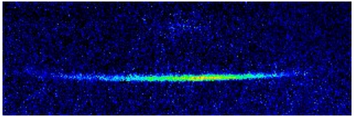
Experimental realization of 2D gases: strongly confine the transverse direction ($k_B T, \mu \ll \hbar\omega_z$)

1. optical lattices along 1 axis
2. 2D optical surface traps / rf-dressed magnetic traps



First 2D BEC in R. Grimm's group

Rychtarik et al., PRL **92**, 173003 (2004)



2D Bose gases in adiabatic potentials @ LPL

Colombe et al., EPL **67**, 593 (2004)

Merloti et al., NJP **15**, 033007 (2013)

See poster by Pedro Gaspar

Outline of the lecture

1. Introduction to 2D quantum gases
2. Quasi long-range order in 2D
    - From 3D to 2D
    - Hydrodynamic formulation and superfluidity
    - Phase fluctuactions in the degenerate Bose gas
3. The Berezinskii-Kosterlitz-Thouless mechanism
    - Vortex pairs from thermal fluctuations
    - A transition triggered by vortex pairs breaking
    - Experimental observations
4. Scaling symmetry in 2D: monopole mode and equation of state
    - Scaling symmetry
    - Scale invariance of the equation of states
    - Monopole mode

References

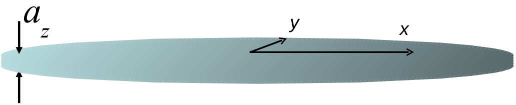
Remark: The lectures are based on a course given in São Paulo, see *Lecture notes*

General references:

- *Quantum Gases in Low Dimensions*, edited by L. Pricoupenko, H. Perrin and M. Olshanii, J. Phys IV **116** (2004)
  Les Houches lectures by Shlyapnikov, Castin, Olshanii, Stringari, Cirac and Douçot.
- *Many body physics with ultra cold gases*, I. Bloch, J. Dalibard and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008)
- Lectures at Collège de France by Jean Dalibard (in French), academic year 2016-2017
- Nobel lectures by J. Michael Kosterlitz and F. Duncan M. Haldane, Rev. Mod. Phys. **89**, 040501 & 040502 (2017)

... and (many) references therein.

1 Introduction to 2D quantum gases



1 A first glance at 2D quantum gases

- 1.1 Non-interacting 2D gas: BEC and log divergences
- 1.2 Contact interactions in 2D and scale invariance

1.1 Reminder on Bose-Einstein condensation

Bose-Einstein condensation is a saturation of the population in the excited states.

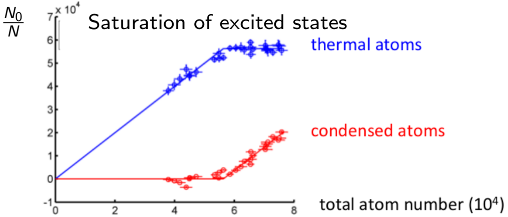
Assume $E_0 = 0$, use semi-classical approximation:

$$N_{\mathrm{exc}}(\mu, T) = \sum_{j>0} \frac{1}{e^{-\beta\mu} e^{\beta E_j} - 1} \simeq \int_0^{+\infty} d\varepsilon \frac{\rho(\varepsilon)}{e^{-\beta\mu} e^{\beta\varepsilon} - 1} \overset{\mu<0}{<} \int_0^{+\infty} d\varepsilon \frac{\rho(\varepsilon)}{e^{\beta\varepsilon} - 1} = N_{\mathrm{max}}(T).$$

$\Rightarrow$ BEC occurs only if $N_{\mathrm{max}}(T)$ is finite. IR behavior: $\sim \rho(\varepsilon)/\varepsilon$.

Example: in a **3D box** $L \times L \times L$,
$\varepsilon = p^2/2M$ and $\rho(\varepsilon) \propto \sqrt{\varepsilon} \Rightarrow \rho(\varepsilon)/\varepsilon \propto \varepsilon^{-1/2}$.
The integral converges $\Rightarrow$ BEC occurs

Hadzibabic group, 3D optical box potential
^{39}K, tunable interactions.



PRL **110**, 200406 (2013)

1.1 2D gas in a box: Bose-Einstein condensation?

Bose-Einstein condensation is a saturation of the population in the excitated states.
Assume $E_0 = 0$, use semi-classical approximation:

$$N_{\mathrm{exc}}(\mu, T) = \sum_{j>0} \frac{1}{e^{-\beta\mu} e^{\beta E_j} - 1} \simeq \int_0^{+\infty} d\varepsilon \frac{\rho(\varepsilon)}{e^{-\beta\mu} e^{\beta\varepsilon} - 1} \overset{\mu<0}{<} \int_0^{+\infty} d\varepsilon \frac{\rho(\varepsilon)}{e^{\beta\varepsilon} - 1} = N_{\max}(T).$$

$\Rightarrow$ BEC occurs only if $N_{\max}(T)$ is finite. IR behavior: $\sim \rho(\varepsilon)/\varepsilon$.
In a **2D box** $L \times L$, $\rho(\varepsilon) = \mathrm{cst} \Rightarrow \rho(\varepsilon)/\varepsilon \propto \varepsilon^{-1}$ **Logarithmic divergence** at low ε

$$n_{\mathrm{exc}}(\mu, T)\lambda_{\mathrm{dB}}^2 = \mathcal{D}_{\mathrm{exc}}(\mu, T) = -\ln\left(1 - e^{\beta\mu}\right).$$

- As $\mu \to 0^-$, $\mathcal{D}_{\mathrm{exc}} \sim -\ln|\beta\mu|$ diverges logarithmically $\Rightarrow$ no BEC in a 2D box!
- Slow divergence: in a finite size system, some coherence subsists at the edge

1.1 2D gas in a harmonic trap: Bose-Einstein condensation?

In a **2D harmonic trap** ω_0, $\rho(\varepsilon) = \varepsilon/(\hbar\omega_0)^2 \Rightarrow \rho(\varepsilon)/\varepsilon = \text{cst} \Rightarrow$ integral **converges**

$$N_{\max}(T) = \left(\frac{k_B T}{\hbar\omega_0}\right)^2 \frac{\pi^2}{6} \quad \Rightarrow \quad \text{BEC occurs with } \boxed{k_B T_c = \frac{\sqrt{6N}}{\pi}\hbar\omega_0}$$

- **Max density?** Local chemical potential $\mu_{\text{loc}}(\mathbf{r}) = \mu - V(\mathbf{r})$

$$n_{\text{exc}}(\mu, T, \mathbf{r})\lambda_{\text{dB}}^2 = \mathcal{D}_{\text{exc}}(\mu, T, \mathbf{r}) = -\ln\left(1 - e^{\beta\mu}e^{-\beta V(\mathbf{r})}\right).$$

- At $\mu = 0$, $\mathcal{D}_{\text{exc}}(\mu = 0, T, \mathbf{r}) \sim -\ln|\beta V(\mathbf{r})|$ diverges logarithmically in the trap center $\mathbf{r} = 0$
- Interactions between particles will regularize this divergence.

1.2 Contact interactions in 2D and scale invariance

Contact interaction potential in dimension D:

$$V_{\rm int}(\mathbf{r}) = g_D \delta^{(D)}(\mathbf{r})$$

$V_{\rm int}$ has dimension of an energy, $\delta^{(D)}$ of an inverse volume in dimension D.

$$\text{3D: } g_{\rm 3D} = \frac{4\pi\hbar^2 a}{M} \qquad \text{2D: } g_{\rm 2D} = \tilde{g}\frac{\hbar^2}{M}$$

3D: scattering length a. 2D: $\tilde{g}$ dimensionless, **no length!**

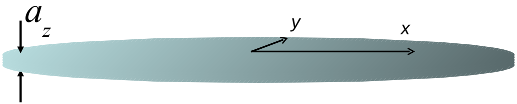
Consequences: **scaling symmetry** ($E_{\rm kin}$ and $E_{\rm int}$ scale alike)

- Scale invariant equation of state (EOS):

  $$\mathcal{D} = f(\alpha, \tilde{g}), \qquad \alpha = \frac{\mu}{k_B T}$$

  with f a universal function.
- Undamped monopole/breathing mode, sensitive to the EOS $\mu(n)$ at $T = 0$

2 Quasi long-range order in 2D

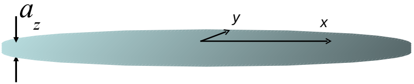


2 Quasi long-range order in 2D

- 2.1 From 3D to 2D
- 2.2 Reminder: hydrodynamic formulation of t-GPE & superfluidity
- 2.3 Phase fluctuations in the degenerate Bose gas

2.1 Reduce the Gross-Pitaevskii equation to 2D

A Bose gas confined to the ground state χ of a transverse harmonic trap.



$$a_z = \sqrt{\frac{\hbar}{M\omega_z}}$$

$$\chi(z) = \frac{1}{\pi^{1/4}\sqrt{a_z}} e^{-z^2/2a_z^2}$$ normalized solution of $$-\frac{\hbar^2}{2M}\partial_z^2\chi + \frac{1}{2}M\omega_z^2 z^2\chi = \frac{\hbar\omega_z}{2}\chi.$$

Ground state: assume the presence of a **condensate** or a **superfluid fraction** ⇒ GPE

Ansatz for the 3D order parameter: $\Psi_{3D}(\mathbf{r}, z) = \chi(z)\psi(\mathbf{r})$. Inject into 3D GPE:

$$\mu_{3\mathrm{D}}\chi\psi = -\frac{\hbar^2}{2M}\chi\nabla_\perp^2\psi - \frac{\hbar^2}{2M}\partial_z^2\chi\psi + V(\mathbf{r})\chi\psi + \frac{1}{2}M\omega_z^2 z^2\chi\psi + g_{3\mathrm{D}}|\psi|^2\psi|\chi|^2\chi.$$

Average over χ, i.e. multiply by χ^* and integrate:

$$\left(\mu_{3\mathrm{D}} - \frac{\hbar\omega_z}{2}\right)\psi = -\frac{\hbar^2}{2M}\nabla_\perp^2\psi + V(\mathbf{r})\psi + g_{3\mathrm{D}}\int dz|\chi(z)|^4|\psi|^2\psi.$$

2.1 Reduce the Gross-Pitaevskii equation to 2D

We get a 2D GPE:

$$\boxed{\mu\psi = -\frac{\hbar^2}{2M}\nabla_\perp^2\psi + V(\mathbf{r})\psi + g_{2\mathrm{D}}|\psi|^2\psi.}$$

with a 2D chemical potential:

$$\boxed{\mu = \mu_{2\mathrm{D}} = \mu_{3\mathrm{D}} - \frac{\hbar\omega_z}{2}}$$

and a 2D interaction constant:

$$g_{2\mathrm{D}} = g_{3\mathrm{D}}\int dz|\chi(z)|^4 = \frac{4\pi\hbar^2 a}{M}\frac{1}{\pi a_z^2}\int dze^{-2z^2/a_z^2} = \frac{4\pi\hbar^2 a}{M}\frac{1}{\sqrt{2\pi}a_z} = \frac{\hbar^2}{M}\tilde{g}$$

with $$\boxed{\tilde{g} = \sqrt{8\pi}\frac{a}{a_z}.}$$

2.1 Conditions for the 2D regime

2D regime: all energies $E < \hbar\omega_z$.

- Thermal energy: $k_B T < \hbar\omega_z$
- Interaction energy: $\frac{1}{2}g_{2\mathrm{D}}n < \hbar\omega_z$, or $\mu < 2\hbar\omega_z$, to prevent interactions to 'inflate' the gas transversally

$$\tilde{g}\frac{\hbar^2}{M}n < 2\hbar\omega_z \quad \Rightarrow \quad \tilde{g}na_z^2 < 2 \quad \text{or} \quad naa_z < \frac{1}{\sqrt{2\pi}}$$

Typical numbers: $\nu_z \sim 2\,\mathrm{kHz}$, $a_z \sim 250\,\mathrm{nm}$, $a \sim 5\,\mathrm{nm} \Rightarrow n < 330\,\mu\mathrm{m}^{-2}$

- A useful relation to compare μ and $k_B T$:

$$\frac{\mu}{k_B T} = \frac{\hbar^2}{Mk_B T}\tilde{g}n = \frac{\tilde{g}n\lambda_{\mathrm{dB}}^2}{2\pi} \quad \Rightarrow \quad \boxed{\frac{\mu}{k_B T} = \frac{\tilde{g}\mathcal{D}}{2\pi}.}$$

Typical numbers: $\tilde{g} \sim 0.1 \Rightarrow \tilde{g}/2\pi \sim 1/60$. Need very high $\mathcal{D}$ to have $\mu > k_B T$

2.2 Reminder: hydrodynamic formulation of t-GPE & superfluidity

Beyond the ground state: time-dependent 2D Gross-Pitaevskii equation for the superfluid:

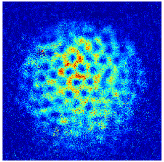
$$\boxed{i\hbar\partial_t\psi = -\frac{\hbar^2}{2M}\nabla^2\psi + V(\mathbf{r})\psi + \tilde{g}\frac{\hbar^2}{M}|\psi|^2\psi.}$$

Equivalent formulation of GPE with hydrodynamics equations: $\psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\,e^{i\theta(\mathbf{r},t)}$

(1) $\partial_t n + \nabla\cdot(n\mathbf{v}) = 0$ **continuity equation** $\mathbf{v} = \frac{\hbar}{m}\nabla\theta$ fluid velocity

(2) $M\partial_t\mathbf{v} = -\nabla\left(-\frac{\hbar^2}{2M}\frac{\nabla^2\left(\sqrt{n}\right)}{\sqrt{n}} + \frac{1}{2}Mv^2 + V(\mathbf{r}) + \tilde{g}\frac{\hbar^2}{M}n\right)$ **Euler equation**

- $\nabla\times\mathbf{v} = 0$ irrotational flow
- $\oint \mathbf{v}\cdot d\mathbf{s} = q_v h/M,\ q_v \in \mathbb{Z}$ quantized circulation
- First signature of superfluidity: quantum vortices, charge q_v



Homogeneous gas: the Bogolubov spectrum

$V(\mathbf{r}) = 0$. Small amplitude excitation spectrum around equilibrium $\psi_0(\mathbf{r},t) = \sqrt{n_0}$
Write $n(\mathbf{r},t) = n_0 + \delta n(\mathbf{r},t)$ and linearize for $\delta n(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)$

$$(1) \qquad \partial_t n + \nabla\cdot(n\mathbf{v}) \simeq \partial_t \delta n + n_0 \nabla\cdot\mathbf{v} = 0$$

$$(2) \quad M\partial_t \mathbf{v} = -\nabla\left(-\frac{\hbar^2}{2M}\frac{\nabla^2\left(\sqrt{n}\right)}{\sqrt{n}} + \frac{1}{2}Mv^2 + \tilde{g}\frac{\hbar^2}{M}n\right) \simeq -\nabla\left(-\frac{\hbar^2}{2M}\frac{\nabla^2\delta n}{2n_0} + \tilde{g}\frac{\hbar^2}{M}\delta n\right)$$

Look for a plane wave solution $\delta n(\mathbf{r},t) = \delta n_0 e^{i\mathbf{q}\cdot\mathbf{r}-i\omega t}$, same for $\mathbf{v}$:

$$(1) \qquad -\omega\delta n + n_0\,\mathbf{q}\cdot\mathbf{v} = 0 \qquad \times M\omega$$

$$(2) \qquad -\omega M n_0 \mathbf{v} = -\mathbf{q}\left(\frac{\hbar^2 q^2}{4M}\delta n + \tilde{g} n_0 \frac{\hbar^2}{M}\delta n\right) \qquad \cdot\mathbf{q}$$

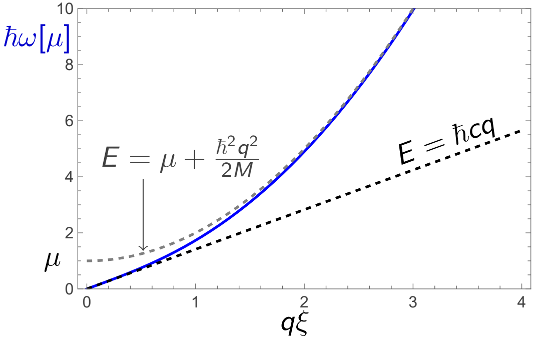
$$\Rightarrow \boxed{\omega^2 = \frac{\hbar^2 q^4}{4M^2} + \tilde{g} n_0 \frac{\hbar^2 q^2}{M^2} = \frac{\hbar^2 q^2}{4M^2}\left(q^2 + 4\tilde{g} n_0\right)}$$

Bogolubov spectrum

Sound and particles

$$\boxed{\omega = \frac{\hbar q}{2M}\sqrt{q^2 + 4\tilde{g}\,n_0}}$$

$$\boxed{\xi = \frac{1}{\sqrt{2\tilde{g}\,n_0}}}$$



Two relevant limits split by ξ^{-1}

ξ: healing length

- $q \ll \sqrt{2\tilde{g}\,n0} \equiv \xi^{-1}$: $\omega \simeq cq$
  sound waves with the speed of sound

$$\boxed{c = \frac{\hbar}{M}\sqrt{\tilde{g}\,n_0} = \sqrt{\frac{\mu}{M}}} \qquad \mu = \tilde{g}\,\frac{\hbar^2}{M}\,n_0$$

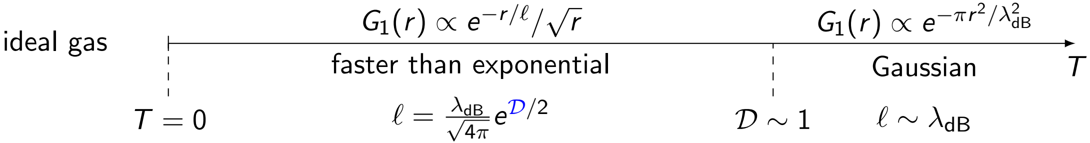
- $q \gg \xi^{-1}$: $\hbar\omega \simeq \mu + \frac{\hbar^2 q^2}{2M}$
  particle-like excitations on top of the condensate

Second signature of superfluidity: critical speed c for excitations

2.3 Characterize phase fluctuations

2D uniform gas: no BEC, Hohenberg-Mermin-Wagner theorem prevents long-range order, but how does the phase coherence decay for $r \to \infty$?

- Evaluate 1-body (phase) correlation function $\boxed{G_1(\mathbf{r},\mathbf{r}') = \langle \psi(\mathbf{r})\psi^\dagger(\mathbf{r}')\rangle}$
- Uniform gas: $G_1(\mathbf{r},\mathbf{r}') = G_1(\mathbf{r}-\mathbf{r}')$. $G_1(\mathbf{r})$ is Fourier transform of $\mathcal{N}(\mathbf{p})$
- Thermal gas, $\mathcal{N}(p)$ Gaussian $\Rightarrow$ G_1 Gaussian: $G_1(r) \sim e^{-\pi r^2/\lambda_{\rm dB}^2}$
- Degenerate **ideal gas**, $\mathcal{N}(p) \sim$ Lorentzian $\Rightarrow$ $G_1 \sim$ exponential: $G_1(r) \sim \sqrt{\ell/r}\,e^{-r/\ell}$
  Fast decay, but $\ell = \lambda_{\rm dB} e^{\mathcal{D}/2}/\sqrt{4\pi}$ very large for $\mathcal{D}$ large



ideal gas

$G_1(r) \propto e^{-r/\ell}/\sqrt{r}$

$G_1(r) \propto e^{-\pi r^2/\lambda_{\rm dB}^2}$

faster than exponential

Gaussian

T

$T = 0$

$\ell = \frac{\lambda_{\rm dB}}{\sqrt{4\pi}} e^{\mathcal{D}/2}$

$\mathcal{D} \sim 1$

$\ell \sim \lambda_{\rm dB}$

What about the **weakly interacting 2D Bose gas**?

G_1 for the weakly interacting Bose gas

- At high/medium T, interactions play a minor role ⇒ same behaviour as ideal gas
- At low T, density fluctuations δn are **suppressed** because of repulsive interaction
- **Classical field approach**: valid for large occupation $n_\varepsilon > 1$ of lower states

$$n_\varepsilon = \frac{1}{e^{\beta(\varepsilon-\mu)}-1} \simeq \frac{k_B T}{\varepsilon-\mu} > 1$$

- Describe the superfluid fraction by a mean field $\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}e^{i\theta(\mathbf{r})}$.
- ψ obeys GPE, $\mu = \tilde{g}n_0\hbar^2/M$. Excitation spectrum $\varepsilon-\mu =$ Bogolubov modes
- G_1 is dominated by **phase fluctuations**. $n(\mathbf{r}) \simeq n_0$.

$$G_1(\mathbf{r}) = \langle\psi(\mathbf{r})\psi^*(\mathbf{0})\rangle \simeq n_0\langle e^{i[\theta(\mathbf{r})-\theta(\mathbf{0})]}\rangle = n_0\, e^{-\frac{1}{2}\langle[\theta(\mathbf{r})-\theta(\mathbf{0})]^2\rangle} = n_0 e^{-\Delta\theta^2/2}$$

- Evaluate $\Delta\theta^2$ with the thermal population in the Bogolubov modes, up to some cut-off energy $E(q_T) = k_B T$ for the kinetic energy $\propto \hbar^2\nabla^2\theta/2M = \hbar^2 q^2/2M$
  NB Low-q cut-off given by $1/r$

G_1 for the weakly interacting Bose gas

- Population $n_{\mathbf{q}}$ of each phase mode: $N n_{\mathbf{q}} \dfrac{\hbar^2 q^2}{2M} \sim k_B T \Rightarrow n_{\mathbf{q}} \sim \dfrac{2 M k_B T}{N \hbar^2 q^2} = \dfrac{4\pi}{N q^2 \lambda_{\mathrm{dB}}^2}$
- Total phase fluctuations

$$\Delta\theta^2 \sim \sum_{\mathbf{q}, q<q_T} n_{\mathbf{q}} \simeq \frac{L^2}{4\pi^2} \int_{1/r}^{q_T} \mathrm{d}\mathbf{q} \frac{4\pi}{N q^2 \lambda_{\mathrm{dB}}^2} = \frac{2}{n_0 \lambda_{\mathrm{dB}}^2} \int_{1/r}^{q_T} \frac{q\,\mathrm{d}q}{q^2} = \frac{2}{\mathcal{D}_s} \int_1^{r q_T} \frac{\mathrm{d}u}{u}$$

Exact result (see lecture notes): $\Delta\theta^2 \simeq \dfrac{2}{\mathcal{D}_s} \ln(r q_T)$ for $r \to \infty$

- We finally get:

$$\boxed{G_1(r) \simeq n_0 \left(\frac{\ell_T}{r}\right)^{1/\mathcal{D}_s} \quad \text{for } r \to \infty} \qquad \ell_T = q_T^{-1}$$

Cut-off momentum from Bogolubov spectrum

$$E(q) = \hbar\omega(q) = \frac{\hbar^2 q}{2M}\sqrt{q^2 + 4\tilde{g} n_0} = \mu q\xi\sqrt{q^2\xi^2 + 2}$$

- Cut-off momentum $\hbar\omega(q_T) = k_B T$

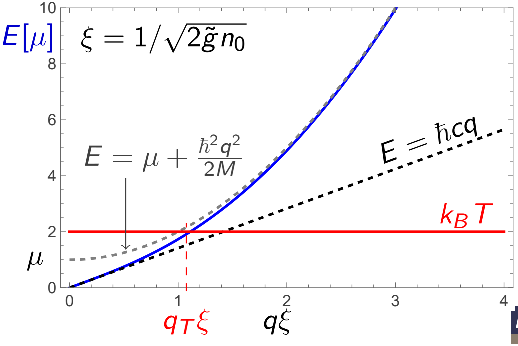
$$\Rightarrow \boxed{q_T\xi\sqrt{q_T^2\xi^2 + 2} = \frac{k_B T}{\mu} = \frac{2\pi}{\tilde{g}\mathcal{D}} = 4\pi\frac{\xi^2}{\lambda_{\mathrm{dB}}^2}.}$$

- $k_B T < \mu$ or $\tilde{g}\mathcal{D} > 2\pi$: **phonons**

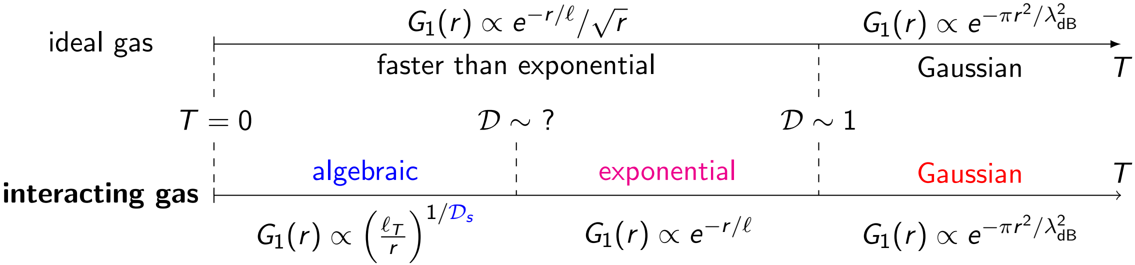
$$q_T \sim \xi/\lambda_{\mathrm{dB}}^2 < \xi^{-1}$$

- $k_B T > \mu$ or $\tilde{g}\mathcal{D} < 2\pi$: **free particles**

$$q_T \sim \lambda_{\mathrm{dB}}^{-1} > \xi^{-1}$$



Summary for the first order correlation function



ideal gas

$G_1(r) \propto e^{-r/\ell}/\sqrt{r}$

faster than exponential

$G_1(r) \propto e^{-\pi r^2/\lambda_{\rm dB}^2}$

Gaussian

T

$T = 0$ $\mathcal{D} \sim ?$ $\mathcal{D} \sim 1$

interacting gas

algebraic

exponential

Gaussian

T

$G_1(r) \propto \left(\frac{\ell_T}{r}\right)^{1/\mathcal{D}_s}$

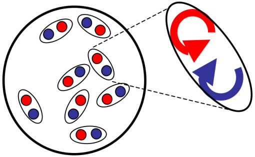
$G_1(r) \propto e^{-r/\ell}$

$G_1(r) \propto e^{-\pi r^2/\lambda_{\rm dB}^2}$

$$\ell = \lambda_{\rm dB} e^{\mathcal{D}/2}/\sqrt{4\pi} \qquad \ell_T \sim \min(\xi/\lambda_{\rm dB}^2, \lambda_{\rm dB}^{-1})$$

- Which value of $\mathcal{D}$ splits the two degenerate regimes of the weakly interacting gas?
- What is the nature of the transition?

3 The Berezinskii-Kosterlitz-Thouless mechanism



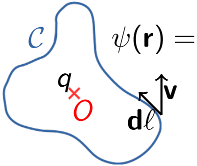
3 The Berezinskii-Kosterlitz-Thouless mechanism

3.1 Vortex pairs from thermal fluctuations

A few facts on vortices

- Superfluid to normal transition at some $\mathcal{D} = \mathcal{D}_c < 1$
- $G_1(r)$ transitions from algebraic to exponential
- Transition implies vortex pairs unbinding — but what is a vortex?

Reminder: quantization of circulation

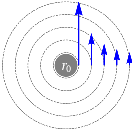


$$\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}e^{i\theta(\mathbf{r})}$$

$$\Gamma_{\mathcal{C}} = \oint_{\mathcal{C}} \mathbf{v}\cdot\mathbf{d}\ell = \frac{\hbar}{M}\oint_{\mathcal{C}} \nabla\theta(\mathbf{r})\cdot\mathbf{d}\ell$$

$$= \frac{\hbar}{M}(\theta_f - \theta_i) = 2\pi q\frac{\hbar}{M}, q \in \mathbb{Z}$$

Velocity field for a single vortex



$$\mathbf{v}(\mathbf{r}) = v(r)\,\mathbf{e}_\varphi$$

$$\Gamma_{\mathcal{C}} = \int_0^{2\pi} v(r)\,r\mathrm{d}\varphi = 2\pi r\,v(r)$$

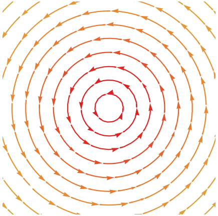
$$\Rightarrow v(r) = q\frac{\hbar}{Mr} \propto \frac{q}{r}$$

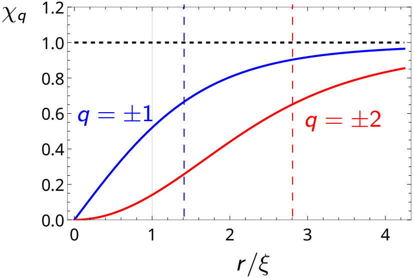
$$v(r_0) \simeq c = \frac{\hbar}{\sqrt{2}M\xi} \Rightarrow r_0 \simeq |q|\sqrt{2}\xi$$

Single vortex: wave function, velocity field, energy

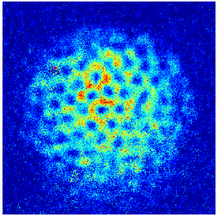
Wave function for a vortex of charge q: $\psi(r,\varphi) = \sqrt{n_0}\,\chi_q(r)\,e^{iq\varphi}$

$$\mathbf{v}(\mathbf{r}) = q\frac{\hbar}{Mr}\mathbf{e}_\varphi$$





vortex lattice



Energy of a vortex of charge q in a disk of radius R is dominated by kinetic energy:

$$E_q \simeq n_0 \int_{r_0}^{R} 2\pi r\, dr\, \frac{1}{2} M v(r)^2 = \pi n_0 \int_{r_0}^{R} dr\, \frac{q^2\hbar^2}{Mr} = q^2 \frac{\pi\hbar^2 n_0}{M} \ln\left(\frac{R}{r_0}\right) \simeq q^2 E_1$$

NB $|q|$ vortices of charge ± 1 have a lower energy $|q|E_1$

$\Rightarrow$ multiply charged vortex is unstable and decays into a vortex lattice

Two vortices: Coulomb-like interaction

Consider two vortices of charge q_1 and q_2 split by r, $r_0 < r \ll R$:

$$\begin{aligned} E_{q_1,q_2}(r) &\simeq \frac{\pi\hbar^2 n_0}{M}\left[(q_1+q_2)^2 \ln\frac{R}{r_0} - 2q_1 q_2 \ln\frac{r}{r_0}\right] \\ &\simeq \frac{\pi\hbar^2 n_0}{M}\left[(q_1^2+q_2^2)\ln\frac{R}{r_0} + 2q_1 q_2 \ln\frac{R}{r}\right] = E_{q_1} + E_{q_2} + V_{\mathrm{int}}(r) \end{aligned}$$

Coulomb-like interaction in 2D

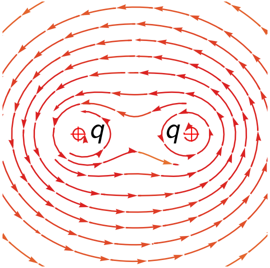
- $q_1 = q_2 = q$: large energy $\sim 4E_q$
- $q_2 = -q_1 = -q$: vortex dipole: $E_d = 2q^2 \dfrac{\pi\hbar^2 n_0}{M} \ln \dfrac{r}{r_0} \ll E_q = q^2 \dfrac{\pi\hbar^2 n_0}{M} \ln\dfrac{R}{r_0}$
  E_d independent of R, very small if $r \sim r_0 \sim \xi$

$\Rightarrow$ It costs little energy to create a vortex dipole

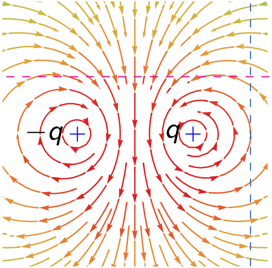
Two vortices: Coulomb-like interaction

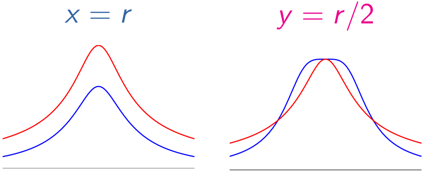
Consider two vortices of charge q_1 and q_2 split by r, $r_0 < r \ll R$:

$q_1 = q_2 = q$



dipole $q_1 = -q_2 = q$





1 vortex vs dipole

- Far field: twice the velocity
- The vortices rotate around each other

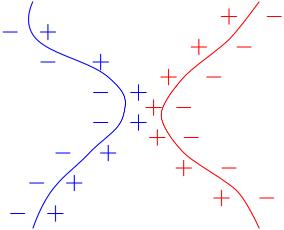
- Far field: vanishing velocity
- The two vortices drift along y

3.1 Vortex pairs from thermal fluctuations

$\psi(\mathbf{r}) = \mathcal{R}(\mathbf{r}) + i\mathcal{I}(\mathbf{r})$ fluctuates at $T > 0$

zero lines of $\mathcal{R}$ and $\mathcal{I}$ move in the plane...

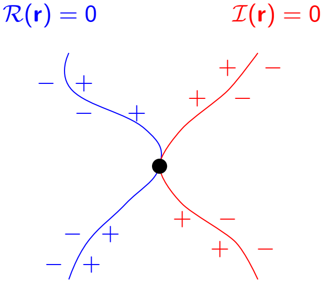
$\mathcal{R}(\mathbf{r}) = 0$ $\mathcal{I}(\mathbf{r}) = 0$



3.1 Vortex pairs from thermal fluctuations

$\psi(\mathbf{r}) = \mathcal{R}(\mathbf{r}) + i\mathcal{I}(\mathbf{r})$ fluctuates at $T > 0$

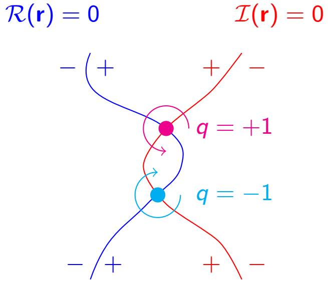
zero lines of $\mathcal{R}$ and $\mathcal{I}$ move in the plane... touch...



3.1 Vortex pairs from thermal fluctuations

$\psi(\mathbf{r}) = \mathcal{R}(\mathbf{r}) + i\mathcal{I}(\mathbf{r})$ fluctuates at $T > 0$

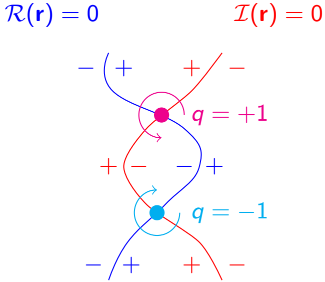
zero lines of $\mathcal{R}$ and $\mathcal{I}$ move in the plane... touch... yielding a dipole...



3.2 A transition triggered by vortex pairs unbinding

$\psi(\mathbf{r}) = \mathcal{R}(\mathbf{r}) + i\mathcal{I}(\mathbf{r})$ fluctuates at $T > 0$

zero lines of $\mathcal{R}$ and $\mathcal{I}$ move in the plane… touch… yielding a dipole… that breaks!



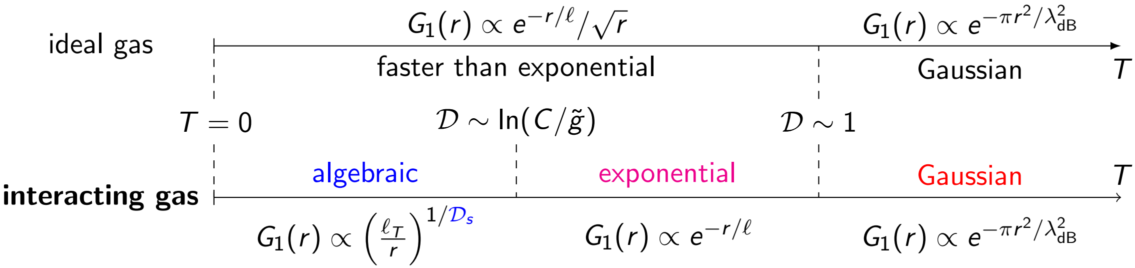
Condition for pair breaking and proliferation of free vortices?

Happens if **free energy** $F = E - TS < 0$

$$\begin{aligned} E &= \frac{\pi\hbar^2 n_0}{M} \ln\frac{R}{r_0} \\ S &= k_B \ln \mathcal{N}_s = 2k_B \ln\frac{R}{r_0} \end{aligned}$$

$$\boxed{F < 0 \Leftrightarrow \mathcal{D}_s < 4}$$

Final summary for the first order correlation function



ideal gas

$G_1(r) \propto e^{-r/\ell}/\sqrt{r}$ $\quad$ $G_1(r) \propto e^{-\pi r^2/\lambda_{\rm dB}^2}$

faster than exponential $\quad$ Gaussian $\quad$ T

$T = 0$ $\quad$ $\mathcal{D} \sim \ln(C/\tilde{g})$ $\quad$ $\mathcal{D} \sim 1$

interacting gas

algebraic $\quad$ exponential $\quad$ Gaussian $\quad$ T

$G_1(r) \propto \left(\frac{\ell_T}{r}\right)^{1/\mathcal{D}_s}$ $\quad$ $G_1(r) \propto e^{-r/\ell}$ $\quad$ $G_1(r) \propto e^{-\pi r^2/\lambda_{\rm dB}^2}$

$$\ell = \lambda_{\rm dB} e^{\mathcal{D}/2}/\sqrt{4\pi} \qquad \ell_T \sim \min(\xi/\lambda_{\rm dB}^2, \lambda_{\rm dB}^{-1})$$

$\mathcal{D}_s > 4$ corresponds to $\boxed{\mathcal{D} > \ln \frac{C}{\tilde{g}}}$ with $C \simeq 380$ $\qquad$ [Prokof'ev & Svistunov, PRL 2001]

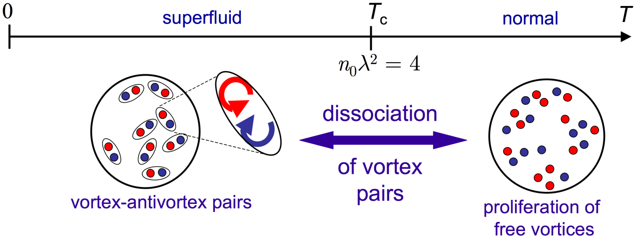
With $\tilde{g} \simeq 0.1$, $\mathcal{D}_c \simeq 8$.

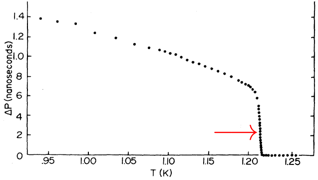
The Berezinskii-Kosterlitz-Thouless mechanism

KT transition in the homogeneous two-dimensional Bose fluid

2D is a very special case! Logs and topological phase transitions

- **2D homogeneous case** No long range order/BEC (Hohenberg–Mermin–Wagner theorem), but a Kosterlitz–Thouless transition to a superfluid state below T_{BKT}, relying on vortex-antivortex pairing. Universal jump of the superfluid density.





Bishop and Reppy, PRL **40**, 1727 (1978)

[ENS-CdF, NIST, Chicago, Seoul, Cambridge, Paris Nord, Oxford...]

2016 Nobel prize in physics to Haldane, Kosterlitz and Thouless

The two-dimensional Bose gas

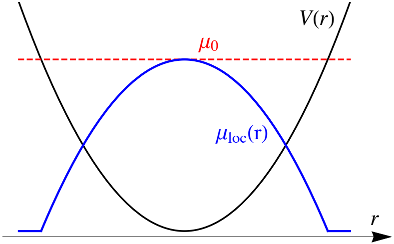
2D: A marginal dimension

- **trapped gas** $V(\mathbf{r})$:
  - **BEC** recovered in a harmonic trap (finite size helps)
  - **BKT** still relevant within local density approximation (LDA).

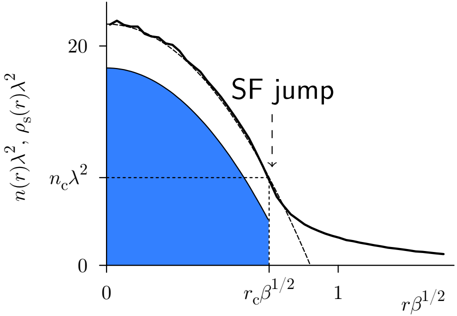
replace

μ by $\mu_{\rm loc}(\mathbf{r}) = \mu_0 - V(\mathbf{r})$,

$\alpha = \frac{\mu}{k_B T}$ by $\alpha_{\rm loc}(\mathbf{r}) = \alpha_0 - V(\mathbf{r})/k_B T$



BKT superfluid phase within LDA

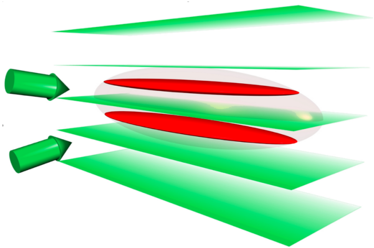


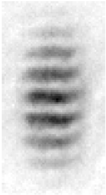
Holzmann & Krauth, PRL **100**, 190402 (2008)

The Berezinskii-Kosterlitz-Thouless mechanism

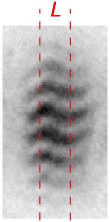
ENS experiment: observation of the transition and correlation function

ENS experiment: measure $G_1(x)$ decay by interferometry

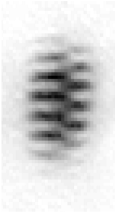




low T



phase fluc.



vortex

- correlation of the integrated contrast $\langle\psi_a(0)\psi_b^*(0)\psi_a^*(x)\psi_b(x)\rangle = |G_1(x)|^2$

$$\frac{1}{L}\int_{-L/2}^{L/2}|G_1(x)|^2\,dx \propto \frac{1}{L^{2\alpha}}$$

exponential decay: $\alpha = \frac{1}{2}$

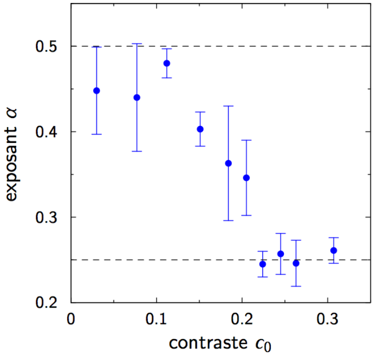
algebraic decay: $\alpha = \frac{1}{4}$

- statistics on phase defects (free vortices)

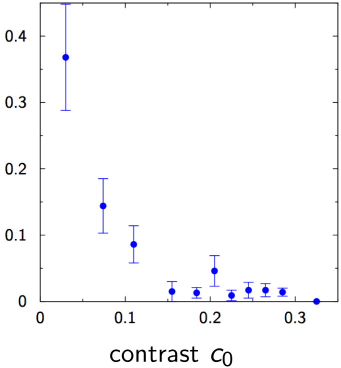
The Berezinskii-Kosterlitz-Thouless mechanism

ENS experiment: observation of the transition and correlation function

Results: Hadzibabic et al., Nature **441**, 1118 (2006)



Fraction of pictures with vortices

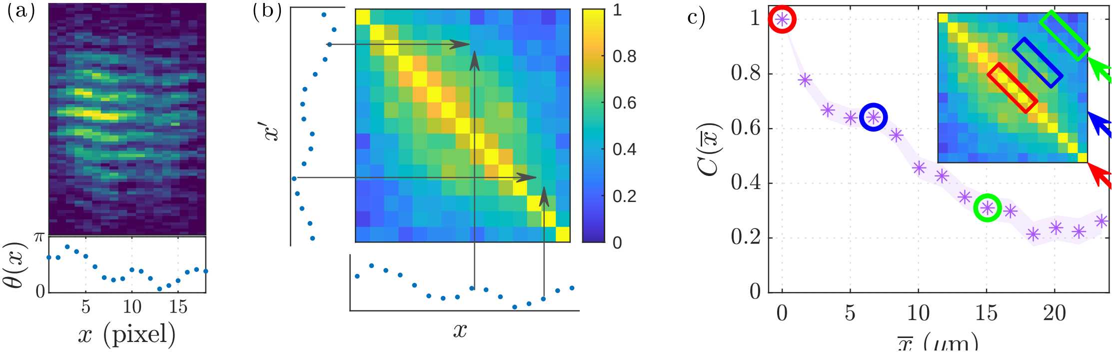


- contrast c_0 is a measure of temperature
- BKT transition evidenced by a step in exponent α and apparition of vortices

The Berezinskii-Kosterlitz-Thouless mechanism

Oxford experiment: decay of correlation function and vortices

Chris Foot's group, Sunami et al., PRL **128**, 250402 (2022)



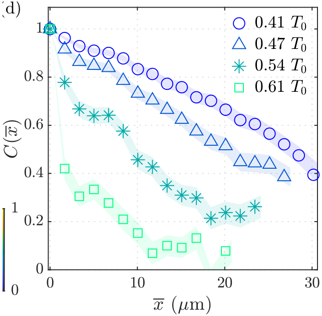
- (a) prepare two parallel 2D quantum gases, release and select a slice around $y = 0$
- (a) recover the local phase at each x from the fringes observed in time-of-flight
- (b) compute the (x, x') correlation, (c) plot as a function of distance $|x - x'|$

The Berezinskii-Kosterlitz-Thouless mechanism

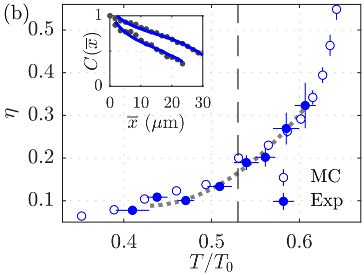
Oxford experiment: decay of correlation function and vortices

Chris Foot's group, Sunami et al., PRL **128**, 250402 (2022)

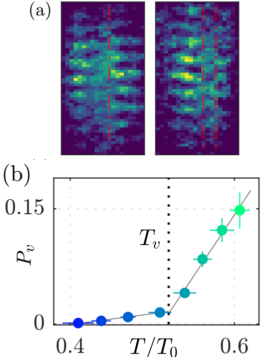
fit $C(x) \propto 1/L^{2\eta}$



exponent η



vortex probability P_v



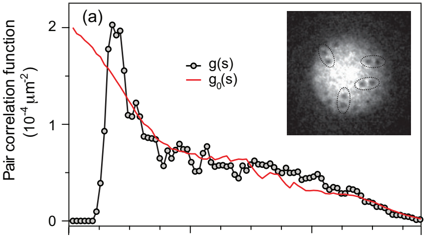
Recover the main features, more quantitative comparison to Monte-Carlo calculations.

The Berezinskii-Kosterlitz-Thouless mechanism

Seoul experiment: pairing of vortices

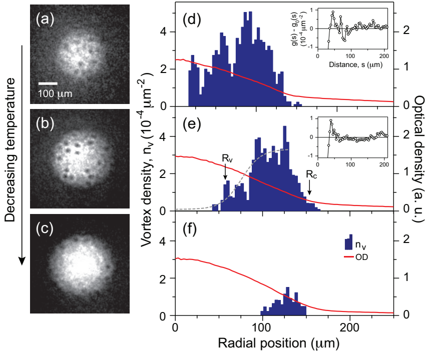
Yong-il Shin's group, Choi et al., PRL **110**, 175302 (2013)

pair correlation



distance between vortices

vortices pushed to the edges



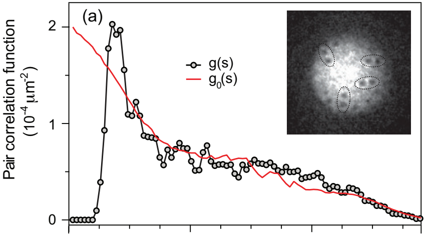
Most probable distance between vortices depends on T/T_{BKT}.

The Berezinskii-Kosterlitz-Thouless mechanism

Seoul experiment: pairing of vortices

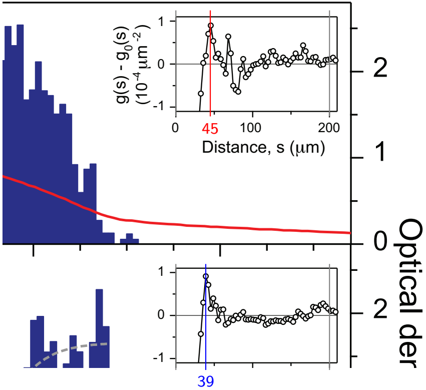
Yong-il Shin's group, Choi et al., PRL **110**, 175302 (2013)

pair correlation



distance between vortices

vortices pushed to the edges



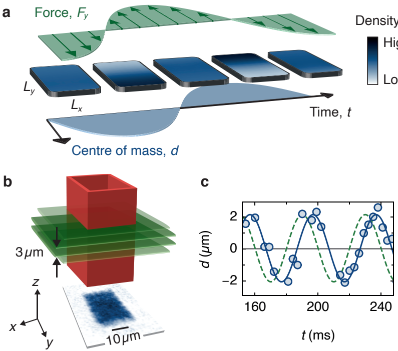
Most probable distance between vortices depends on T/T_{BKT}.

KT transition in the homogeneous 2D Bose gas

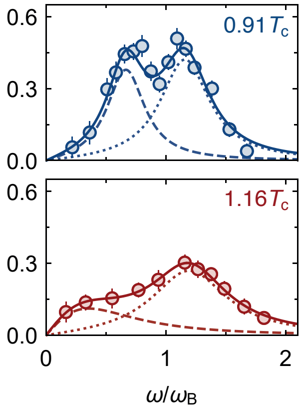
The Cambridge experiment: superfluid jump observed from first and second sound

2D homogeneous ^{39}K gas, $\tilde{g} = 0.64$

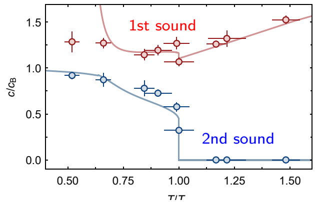
Shake the gas along y axis, record center-of-mass motion



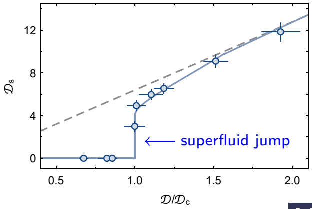
Spectroscopy of sound modes



Speed of sound



Superfluid phase-space density



⟵ superfluid jump

Recover $n_s\lambda_{\mathrm{dB}}^2 = 4$ jump at the transition [Christodoulou et al., Nature **594**, 191 (2021)]

Summary: BEC vs BKT in the two-dimensional Bose gas

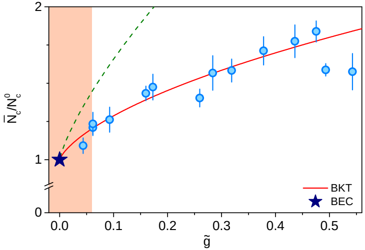
BEC or BKT depends on trapping and interactions

Summary:

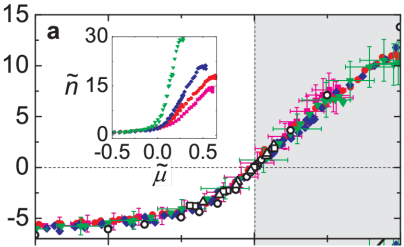
	ideal	interacting
homogeneous	no BEC, no SF	BKT SF [ENS-CdF]
trapped	BEC, no SF ⟷	BEC + BKT within LDA

BEC-BKT interplay in a harmonic trap

Fletcher et al., PRL **114**, 255302 (2015)



4 Scaling symmetry in 2D



4 Scaling symmetry in 2D

4.1 Scaling symmetry

Consider a dilation of space by a factor λ and of time by a factor λ^2. The scaling symmetry is verified if the energy scales as $1/\lambda^2$:

$$\left.\begin{array}{r}\mathbf{r} \longrightarrow \lambda\mathbf{r} \\ t \longrightarrow \lambda^2 t\end{array}\right\} \Rightarrow E \longrightarrow \frac{1}{\lambda^2}E, \qquad \text{or} \quad E(\lambda\mathbf{r}, \lambda^2 t) = \frac{1}{\lambda^2}E(\mathbf{r}, t).$$

Examples:

- Classical kinetic energy $\frac{1}{2}Mv^2$, velocity $\tilde{\mathbf{v}} = \frac{\mathrm{d}\tilde{\mathbf{r}}}{\mathrm{d}\tilde{t}} = \frac{\lambda\,\mathrm{d}\mathbf{r}}{\lambda^2\,\mathrm{d}t} = \frac{1}{\lambda}\mathbf{v}$
- Quantum kinetic energy $\frac{\hbar^2}{2M}\int \mathrm{d}^D\mathbf{r}|\nabla\psi|^2$, wavefunction $\widetilde{\psi}(\tilde{\mathbf{r}}) = \frac{1}{\lambda^{D/2}}\psi\left(\tilde{\mathbf{r}}/\lambda\right)$
- Interaction energy? Contact interactions $V_{\text{int}}(\mathbf{r}) = g\delta^{(\mathrm{D})}(\mathbf{r}) \propto \frac{1}{\text{length}^D}$

$\Rightarrow$ only true for $\mathrm{D} = 2$!

4.2 Scale invariance of the equation of states

Thermodynamic law for the occupation number of a configuration state $(N, \{\mathbf{r}_j, \mathbf{p}_j\})$ with N particles at positions and momenta $\{\mathbf{r}_j, \mathbf{p}_j\}$:

$$P\left(N, \{\mathbf{r}_j, \mathbf{p}_j\}\right) = \frac{1}{e^{\beta\left[E\left(N,\{\mathbf{r}_j,\mathbf{p}_j\}\right)-\mu\right]} - 1}$$

For a quantity $F\left(N, \{\mathbf{r}_j, \mathbf{p}_j\}\right)$ defined for a given configuration $(N, \{\mathbf{r}_j, \mathbf{p}_j\})$, the thermodynamic average reads:

$$\mathcal{F}(\mu, T) = \sum_N \int \prod_{j=1}^{N} \mathrm{d}^D\mathbf{r}_j \mathrm{d}^D\mathbf{p}_j P\left(N, \{\mathbf{r}_j, \mathbf{p}_j\}\right) F\left(N, \{\mathbf{r}_j, \mathbf{p}_j\}\right)$$

$$\boxed{\mathcal{F}(\mu, T) = \sum_N \int \prod_{j=1}^{N} \mathrm{d}^D\mathbf{r}_j \mathrm{d}^D\mathbf{p}_j \frac{F\left(N, \{\mathbf{r}_j, \mathbf{p}_j\}\right)}{e^{\beta\left[E\left(N,\{\mathbf{r}_j,\mathbf{p}_j\}\right)-\mu\right]} - 1}}$$

4.2 Scale invariance of the equation of states

- Assume scale invariance $E\left(N,\{\tilde{\mathbf{r}}_j,\tilde{\mathbf{p}}_j\}\right) = E\left(N,\{\lambda\mathbf{r}_j,\mathbf{p}_j/\lambda\}\right) = \frac{1}{\lambda^2}E\left(N,\{\mathbf{r}_j,\mathbf{p}_j\}\right)$
- Assume a scaling law for F:

$$F\left(N,\left\{\lambda\mathbf{r}_j,\frac{1}{\lambda}\mathbf{p}_j\right\}\right) = \frac{1}{\lambda^{2\nu}}F\left(N,\{\mathbf{r}_j,\mathbf{p}_j\}\right)$$

- Then for $\mathcal{F}$ we get

$$\mathcal{F}\left(\frac{1}{\lambda^2}\mu,\frac{1}{\lambda^2}T\right) = \sum_N\int\prod_{j=1}^{N}\mathrm{d}^D\tilde{\mathbf{r}}_j\mathrm{d}^D\tilde{\mathbf{p}}_j\frac{F\left(N,\{\tilde{\mathbf{r}}_j,\tilde{\mathbf{p}}_j\}\right)}{e^{\beta\left[\lambda^2E\left(N,\{\tilde{\mathbf{r}}_j,\tilde{\mathbf{p}}_j\}\right)-\mu\right]}-1} = \frac{1}{\lambda^{2\nu}}\mathcal{F}(\mu,T)$$

- Consequence:

$$\boxed{\mathcal{F}(\mu,T) = (k_BT)^{\nu}\ f\left(\frac{\mu}{k_BT}\right)}$$

4.2 Scale invariance of the equation of states

- Apply to density:

$$n_{(N,\{\tilde{\mathbf{r}}_j,\tilde{\mathbf{p}}_j\})}(\mathbf{r}) = \sum_{j=1}^{N} \delta^{(2)}(\mathbf{r}-\tilde{\mathbf{r}}_j) = \sum_{j=1}^{N} \delta^{(2)}(\mathbf{r}-\lambda\mathbf{r}_j) = \frac{1}{\lambda^2}\sum_{j=1}^{N} \delta^{(2)}(\mathbf{r}-\mathbf{r}_j)$$

- Density obeys a scaling law with $\nu = 1$ (can still depend on $\tilde{g}$)

$$n(\mu, T, \tilde{g}) = k_B T \ f\left(\frac{\mu}{k_B T}, \tilde{g}\right) = k_B T\, f(\alpha, \tilde{g}), \qquad \alpha = \frac{\mu}{k_B T}.$$

- Result for 2D phase space density $n\lambda_{\mathrm{dB}}^2$: $\lambda_{\mathrm{dB}}^2 \propto 1/T$

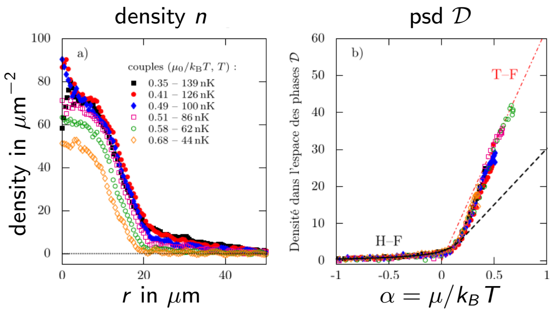
$$\Rightarrow \boxed{\mathcal{D}(\mu, T, \tilde{g}) = \phi(\alpha, \tilde{g})}$$

Scaling symmetry and universality

Measuring the Equation of state in a harmonic trap using LDA

Use local density approximation $\alpha(\mathbf{r}) = \alpha_0 - \beta V(\mathbf{r})$ to measure the EOS in a trap

- dimensionless interaction $g = \frac{\hbar^2}{M}\tilde{g}$
- EOS depends only on $\alpha = \mu/k_B T$: $\mathcal{D} = \phi(\alpha, \tilde{g})$ [ENS,Chicago]
- $\tilde{g} \simeq 0.1$: critical psd for BKT $\mathcal{D}_c \simeq 8$, critical $\alpha_c \simeq 0.16$
- Low $\mathcal{D}$, $\alpha < 0$ limit: $\mathcal{D} \simeq -\ln(1 - e^{\alpha})$
- Hartree correction: $\mu \to \mu - 2g_{2D}n$, $\alpha \to \alpha - \tilde{g}\mathcal{D}/\pi$
- High $\mathcal{D}$, $\alpha > 0$ limit: $\mathcal{D} \simeq \dfrac{2\pi\alpha}{\tilde{g}}$



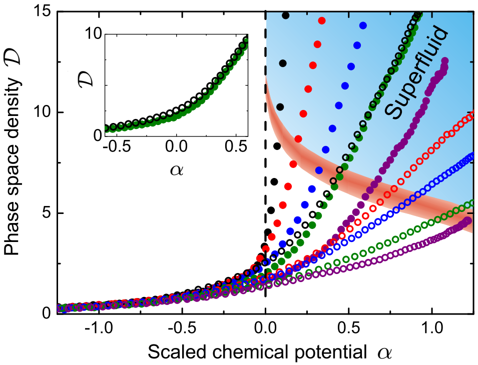
picture from T. Yefsah's PhD thesis

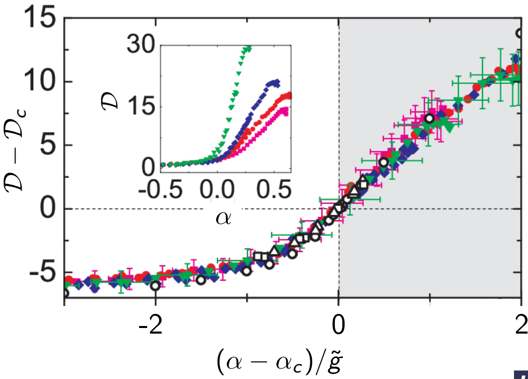
[Yefsah et al., PRL **107**, 130401 (2011)]

Scaling symmetry and universality

Measuring the Equation of state in a harmonic trap using LDA

universal law near $\mathcal{D}_c$: $\mathcal{D}-\mathcal{D}_c = f\left(\dfrac{\alpha-\alpha_c}{\tilde{g}}\right)$ a single parameter!



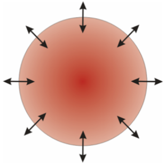


[Chin group, Nature **470**, 236 (2011) & PRL **110**, 145302 (2013)]

4.3 Monopole mode in 2D

- Weakly interacting Bose gas in a harmonic trap at ω_0, Thomas-Fermi regime
- Look for a scaling solution for the monopole mode

$$n(\mathbf{r},t) = \frac{1}{\lambda^2(t)} n_{\text{eq}}\left(\frac{\mathbf{r}}{\lambda(t)}\right)$$



- Thomas-Fermi: $n_{\text{eq}}(r) = n_0\left(1-\frac{r^2}{R^2}\right)$ with $g_{2\text{D}}\, n_0 = M\omega_0^2 R^2/2$
- Inject n and velocity field $\mathbf{v}(\mathbf{r},t) = \frac{\dot\lambda}{\lambda}\mathbf{r}$ into Euler's equation:

$$\ddot\lambda + \omega_0^2\left(\lambda - \frac{1}{\lambda^3}\right) = 0$$

- Scaling solution at $\Omega_M = 2\omega_0$ valid at any amplitude $\lambda_{\max}$:

$$\lambda(t) = \sqrt{\frac{1}{2}\left(\lambda_{\max}^2 + \frac{1}{\lambda_{\max}^2}\right) + \frac{1}{2}\left(\lambda_{\max}^2 - \frac{1}{\lambda_{\max}^2}\right)\cos 2\omega_0 t}.$$

Scaling symmetry and monopole mode of the 2D Bose gas

Pitaevskii-Rosch monopole mode **in an isotropic 2D harmonic trap**

- no damping, see also [Chevy et al., PRL **88**, 250402 (2002)], in a cigar
- strict 2D: $\mu(n) = g_{2D}n \Rightarrow \Omega_M = 2\omega_0$ for all amplitudes, linked to **scaling symmetry**
- beyond scaling: small amplitude monopole probes the compressibility $\Rightarrow \Omega_M$ is related to the **2D EOS** $\mu(n)$:

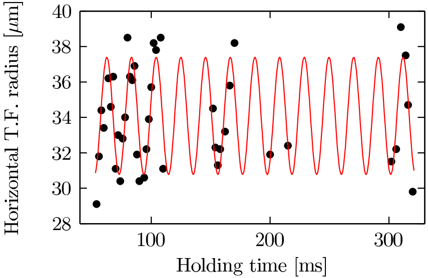
$$\Omega_M = \sqrt{2(2+\epsilon)}\,\omega_0 \qquad \text{with} \qquad \epsilon = \frac{n\mu''(n)}{\mu'(n)}$$

- **very strong transverse confinement**: Quantum anomaly: expected positive shift of Ω_M at the 0.5% level [Olshanii et al., PRL **105**, 095302 (2010)]
- 3D oblate: $\mu(n_{2D}) \propto n_{2D}^{2/3} \Rightarrow \Omega_M = \sqrt{\frac{10}{3}}\omega_0$ for small amplitudes

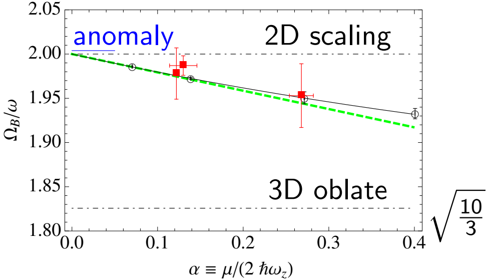
Scaling symmetry and monopole mode of the 2D Bose gas

Experiment: Karina Merloti (LPL) [thesis, NPJ **15**, 033007 & PRA **88**, 061603(R) (2013)]

no visible damping



Frequency sensitive to the EOS.

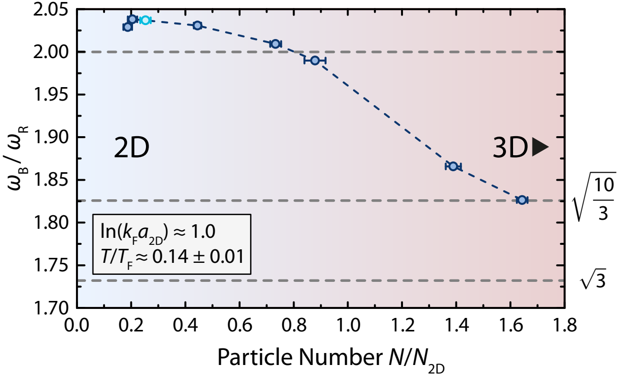


Observe a shift due to the interactions and progressive occupation of transverse modes as μ increases $\Rightarrow$ cross-over from $\Omega_B = 2\omega_0$ (2D) to $\Omega_B = \sqrt{\frac{10}{3}}\omega_0$ (3D oblate).

Quantum anomaly in the 2D Fermi gas

Related experiment with 2D Fermi gases: Selim Jochim (Heidelberg), Chris Vale (Melbourne) [PRL **121**, 120401 & 120402 (2018)]

Heidelberg data at small excitation:



Quantum anomaly visible deep in the 2D regime, then cross-over to 3D oblate.