Chapter 2

Solution of the unitary 3-body problem

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- V. Efimov, Yad. Fiz. **12**, 1080 (1970) [Sov. J. Nucl. Phys. **12**, 589 (1971)]; Nucl. Phys. **A210**, 157 (1973)
- S. Tan, arXiv:cond-mat/0412764
- FW & Y. Castin, PRL **97**, 150401 (2006)

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$N_\uparrow = 2 \; , \; N_\downarrow = 1$

$$\Psi(\vec{r_1}, \vec{r_2}, \vec{r_3}) \; = \; -\Psi(\vec{r_2}, \vec{r_1}, \vec{r_3})$$

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ZRM

$r := r_{13} \to 0$

fixed $\vec{c} := \frac{\vec{r_1} + \vec{r_3}}{2}$

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$$\Psi(\vec{r_1}, \vec{r_2}, \vec{r_3}) \underset{r \to 0}{=} \left(\frac{1}{r} - \underbrace{\frac{1}{a}}_{=0} \right) \mathcal{A}(\vec{c}; \vec{r_2}) + O(r)$$

$$-\frac{\hbar^2}{2m} \sum_{i=1}^{3} \Delta_{\vec{r_i}} \Psi + \sum_{i=1}^{3} U(\vec{r_i}) \Psi = E\ \Psi$$

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$U(\vec{r}) = \dfrac{1}{2}\, m\, \omega^2\, r^2$

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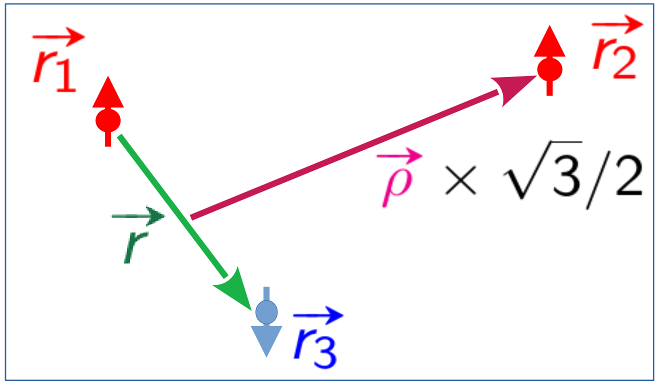

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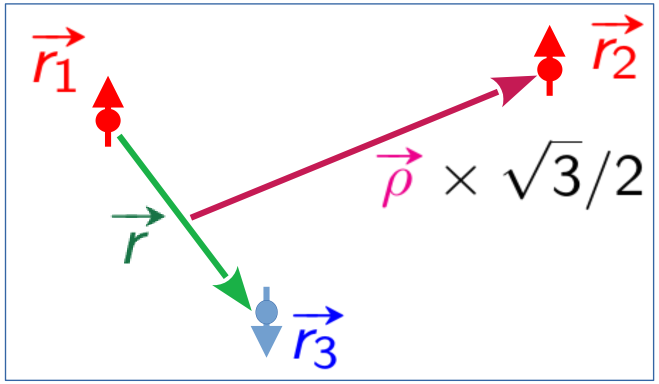

$$(\hat{P}_{12}\,\psi)(\vec{r}, \vec{\rho}) = -\psi(\vec{r}, \vec{\rho})$$

where $\hat{P}_{12}$ exchanges particles 1 and 2

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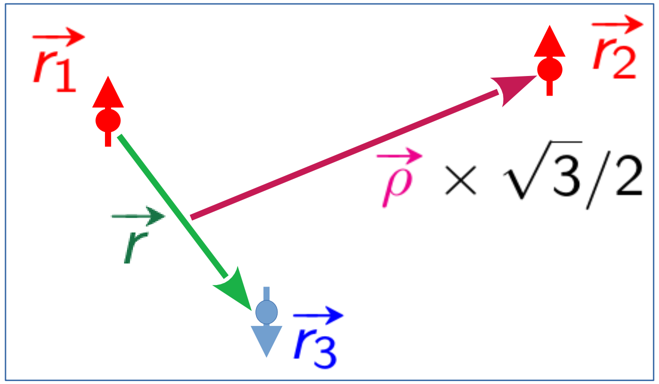

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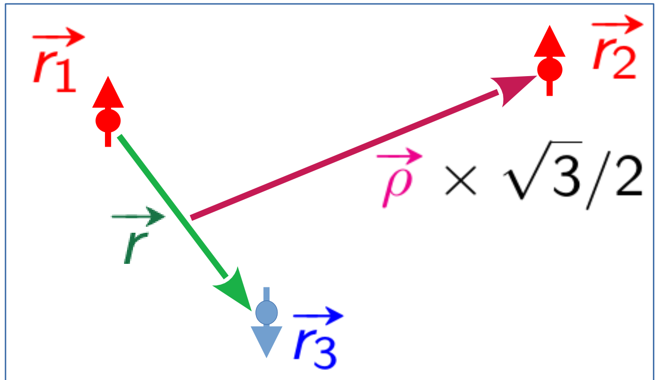

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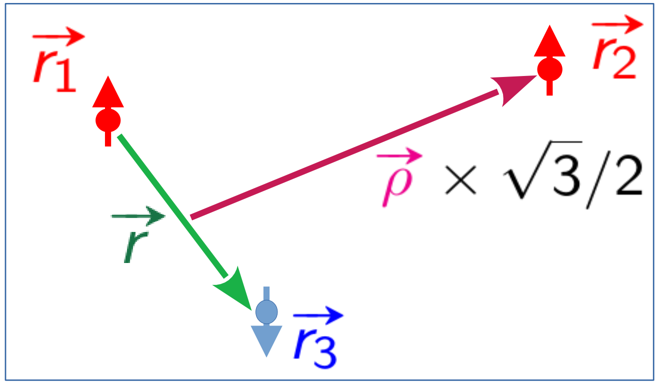

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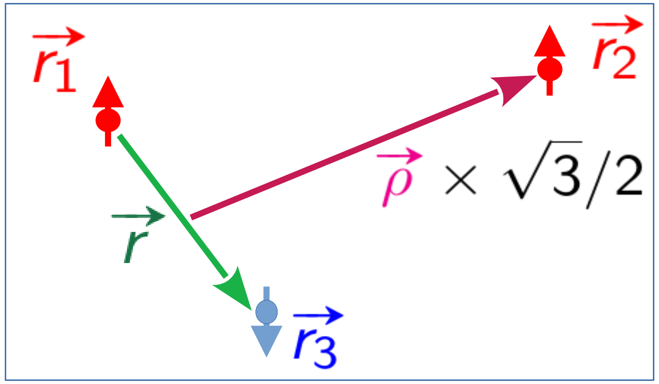

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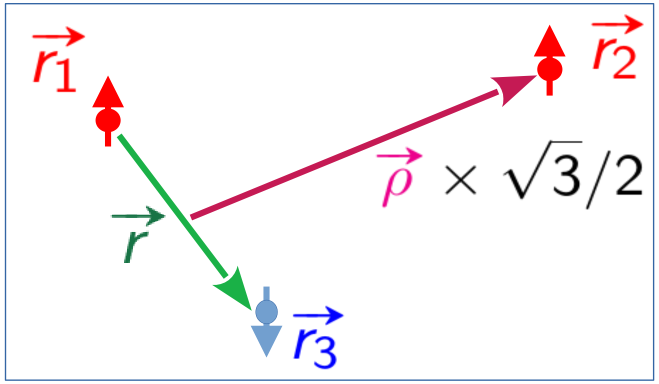

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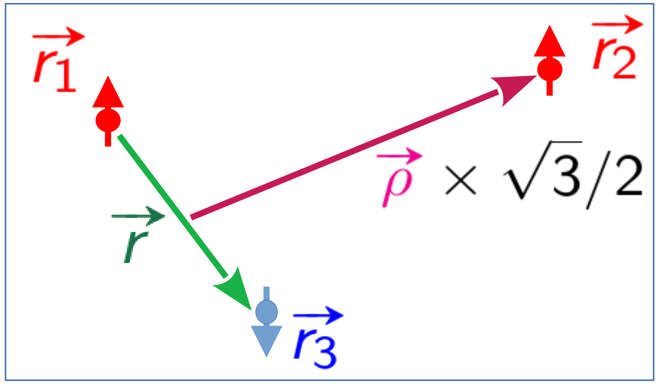

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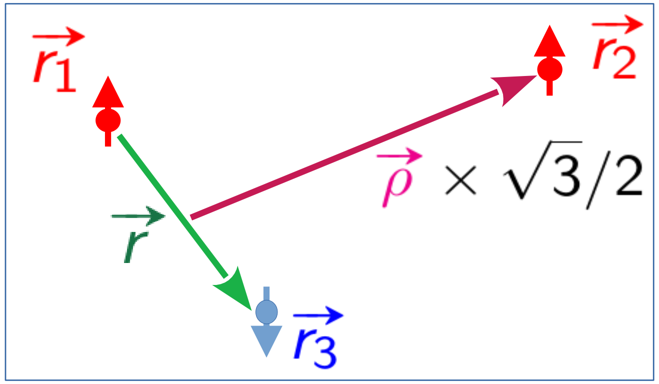

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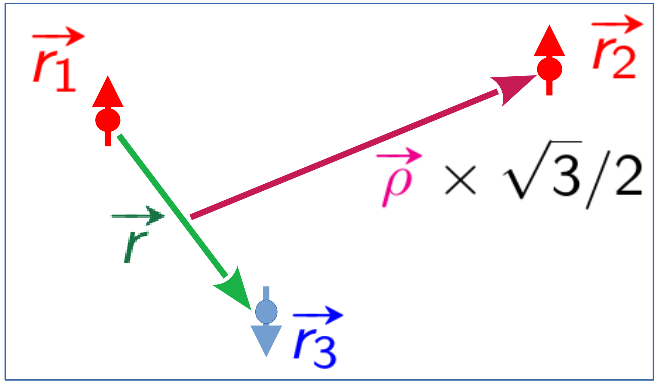

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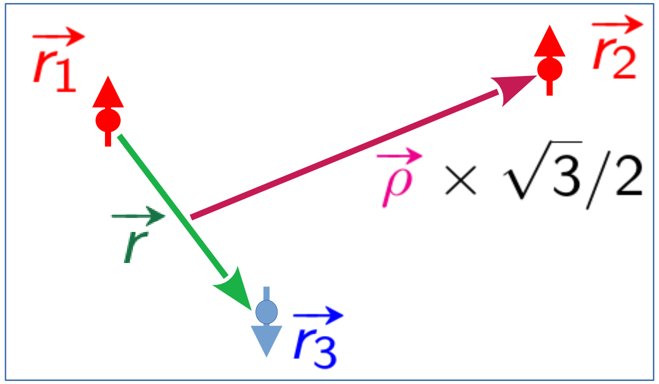

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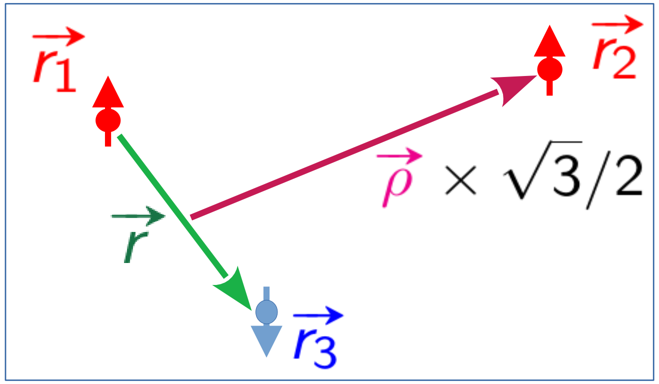

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$(r, \rho) \longrightarrow (R, \alpha)$:

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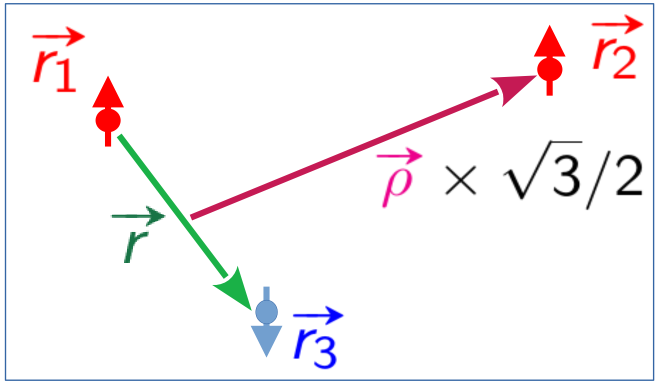

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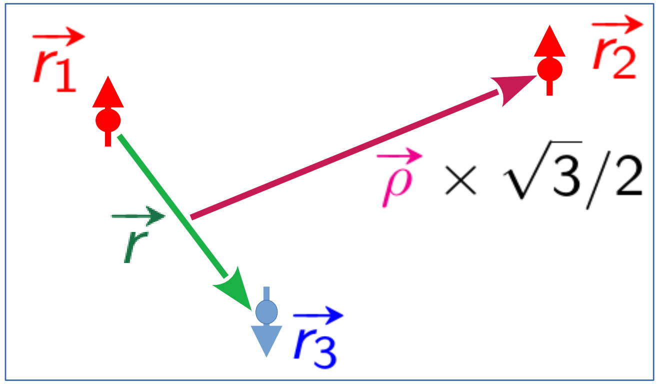

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$$\chi_0(r,\rho) = F(R)\,\varphi(\alpha)$$

$$\Rightarrow \quad \psi(\vec{r},\vec{\rho}) = \frac{F(R)}{R^2}\,(1-\hat{P}_{12})\,\frac{\varphi(\alpha)}{\sin\alpha\ \cos\alpha}\, Y_\ell^m(\hat{\rho})$$

$$\psi(\vec{r},\vec{\rho}) \underset{r\to 0}{=} \frac{1}{r} A(\vec{\rho}) + O(r)$$

$$-\frac{\hbar^2}{m}\left(\Delta_{\vec{r}} + \Delta_{\vec{\rho}}\right)\psi + \frac{m\,\omega^2}{4}(r^2+\rho^2)\,\psi = E\,\psi$$

Exactly solvable!

$$\psi(\vec{r},\vec{\rho}) = (1-\hat{P}_{12})\,\chi(r,\vec{\rho})$$

$$\chi(r,\vec{\rho}) = \frac{\chi_0(r,\rho)}{r\,\rho}\,Y_\ell^{\mathrm{m}}(\hat{\rho})$$

$(r,\rho) \longrightarrow (R,\alpha)$:

$$r = R\,\sin\alpha$$
$$\rho = R\,\cos\alpha$$

$$\chi_0(r,\rho) = F(R)\,\varphi(\alpha)$$

$$\Rightarrow \quad \psi(\vec{r},\vec{\rho}) = \frac{F(R)}{R^2}\,(1-\hat{P}_{12})\,\frac{\varphi(\alpha)}{\sin\alpha\,\cos\alpha}\,Y_\ell^{\mathrm{m}}(\hat{\rho})$$

$$\psi(\vec{r}, \vec{\rho}) \underset{r \to 0}{=} \frac{1}{r} A(\vec{\rho}) + O(r) \qquad \text{(CC)}$$

$$-\frac{\hbar^2}{m} (\Delta_{\vec{r}} + \Delta_{\vec{\rho}}) \psi + \frac{m \omega^2}{4} (r^2 + \rho^2) \psi = E \psi \qquad \text{(Schrö)}$$

Exactly solvable!

$$\psi(\vec{r}, \vec{\rho}) = (1 - \hat{P}_{12}) \chi(r, \vec{\rho})$$

$$\chi(r, \vec{\rho}) = \frac{\chi_0(r, \rho)}{r \rho} Y_\ell^m(\hat{\rho})$$

$(r, \rho) \longrightarrow (R, \alpha)$:

$$r = R \sin \alpha$$
$$\rho = R \cos \alpha$$

$$\chi_0(r, \rho) = F(R) \varphi(\alpha)$$

$$\Rightarrow \psi(\vec{r}, \vec{\rho}) = \frac{F(R)}{R^2} (1 - \hat{P}_{12}) \frac{\varphi(\alpha)}{\sin \alpha \cos \alpha} Y_\ell^m(\hat{\rho})$$

$$\psi(\vec{r},\vec{\rho}) \underset{r\to 0}{=} \frac{1}{r} A(\vec{\rho}) + O(r) \qquad \text{(CC)}$$

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$$\Longrightarrow \quad \psi(\vec{r},\vec{\rho}) = \frac{F(R)}{R^2}\,(1-\hat{P}_{12})\,\frac{\varphi(\alpha)}{\sin\alpha\,\cos\alpha}\,Y_\ell^m(\hat{\rho})$$

$$\text{(CC)} \qquad \varphi'(0) - \frac{4}{\sqrt{3}}\,(-1)^\ell\,\varphi\left(\frac{\pi}{3}\right) = 0$$

$$\psi(\vec{r},\vec{\rho}) \underset{r\to 0}{=} \frac{1}{r}\,A(\vec{\rho}) + O(r) \qquad \text{(CC)}$$

$$-\frac{\hbar^2}{m}\,(\Delta_{\vec{r}} + \Delta_{\vec{\rho}})\,\psi + \frac{m\,\omega^2}{4}(r^2+\rho^2)\,\psi = E\,\psi \qquad \text{(Schrö)}$$

Exactly solvable!

$$\psi(\vec{r},\vec{\rho}) = (1-\hat{P}_{12})\,\chi(r,\vec{\rho})$$

$$\chi(r,\vec{\rho}) = \frac{\chi_0(r,\rho)}{r\,\rho}\,Y_\ell^m(\hat{\rho})$$

$(r,\rho) \longrightarrow (R,\alpha)$:

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(CC) $\quad \varphi'(0) - \frac{4}{\sqrt{3}}\,(-1)^\ell\,\varphi\left(\frac{\pi}{3}\right) = 0 \qquad \varphi\left(\frac{\pi}{2}\right) = 0$

$$\psi(\vec{r},\vec{\rho}) \underset{r\to 0}{=} \frac{1}{r} A(\vec{\rho}) + O(r) \qquad \text{(CC)}$$

$$-\frac{\hbar^2}{m}\left(\Delta_{\vec{r}} + \Delta_{\vec{\rho}}\right)\psi + \frac{m\,\omega^2}{4}(r^2+\rho^2)\,\psi = E\,\psi \qquad \text{(Schrö)}$$

Exactly solvable!

$$\psi(\vec{r},\vec{\rho}) = (1-\hat{P}_{12})\,\chi(r,\vec{\rho})$$

$$\chi(r,\vec{\rho}) = \frac{\chi_0(r,\rho)}{r\,\rho}\,Y_\ell^m(\hat{\rho})$$

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$$\text{(CC)} \quad \varphi'(0) - \frac{4}{\sqrt{3}}\,(-1)^\ell\,\varphi\left(\frac{\pi}{3}\right) = 0$$

$$\varphi\left(\frac{\pi}{2}\right) = 0 \;\Leftarrow\; \left(\begin{array}{c}\psi \text{ finite} \\ \text{for } \rho\to 0\end{array}\right)$$

$$\psi(\vec{r},\vec{\rho}) \underset{r\to 0}{=} \frac{1}{r} A(\vec{\rho}) + O(r) \qquad \text{(CC)}$$

$$-\frac{\hbar^2}{m}(\Delta_{\vec{r}} + \Delta_{\vec{\rho}})\psi + \frac{m\omega^2}{4}(r^2+\rho^2)\psi = E\,\psi \qquad \text{(Schrö)}$$

Exactly solvable!

$$\psi(\vec{r},\vec{\rho}) = (1-\hat{P}_{12})\,\chi(r,\vec{\rho})$$

$$\chi(r,\vec{\rho}) = \frac{\chi_0(r,\rho)}{r\,\rho}\, Y_\ell^m(\hat{\rho})$$

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$$\varphi'(0) - \frac{4}{\sqrt{3}}(-1)^\ell\,\varphi\left(\frac{\pi}{3}\right) = 0$$

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$$\psi(\vec{r},\vec{\rho}) \underset{r\to 0}{=} \frac{1}{r}\, A(\vec{\rho}) + O(r) \qquad \text{(CC)}$$

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$$\varphi'(0) - \frac{4}{\sqrt{3}}\,(-1)^\ell\,\varphi\left(\frac{\pi}{3}\right) = 0 \qquad \varphi\left(\frac{\pi}{2}\right) = 0$$

(Schrö)

$$-\varphi''(\alpha) + \frac{\ell(\ell+1)}{\cos^2\alpha}\,\varphi(\alpha) = s^2\,\varphi(\alpha)$$

$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R}\,F'(R)\right] + \left(\frac{\hbar^2\,s^2}{m\,R^2} + \frac{m\,\omega^2}{4}\,R^2\right)F(R) = E\,F(R)$$

$$\varphi'(0) - \frac{4}{\sqrt{3}} (-1)^{\ell} \varphi\left(\frac{\pi}{3}\right) = 0$$

$$\varphi\left(\frac{\pi}{2}\right) = 0$$

$$-\varphi''(\alpha) + \frac{\ell(\ell+1)}{\cos^2 \alpha} \varphi(\alpha) = s^2 \varphi(\alpha)$$

$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R} F'(R)\right] + \left(\frac{\hbar^2 s^2}{m R^2} + \frac{m\omega^2}{4} R^2\right) F(R) = E F(R)$$

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$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R} F'(R)\right] + \left(\frac{\hbar^2 s^2}{m R^2} + \frac{m \omega^2}{4} R^2\right) F(R) = E F(R)$$

$$\boxed{s} \longleftarrow \begin{cases} \varphi'(0) - \dfrac{4}{\sqrt{3}} (-1)^{\ell} \varphi\left(\dfrac{\pi}{3}\right) = 0 \qquad \varphi\left(\dfrac{\pi}{2}\right) = 0 \\ -\varphi''(\alpha) + \dfrac{\ell(\ell+1)}{\cos^2 \alpha} \varphi(\alpha) = s^2 \varphi(\alpha) \end{cases}$$

$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R} F'(R)\right] + \left(\frac{\hbar^2 s^2}{m R^2} + \frac{m \omega^2}{4} R^2\right) F(R) = E F(R)$$

$$\ell = 0 \qquad \varphi(\alpha) = \sin\left[s\left(\frac{\pi}{2} - \alpha\right)\right]$$

$$\Rightarrow \quad s\,\cos\left(\frac{s\,\pi}{2}\right) + \frac{4}{\sqrt{3}}\,\sin\left(\frac{s\,\pi}{6}\right) = 0$$

smallest solution: $s_{\ell=0} = 2.166221977\ldots$

$$s \longleftarrow \begin{cases} \varphi'(0) - \dfrac{4}{\sqrt{3}}\,(-1)^{\ell}\,\varphi\left(\dfrac{\pi}{3}\right) = 0 \qquad \varphi\left(\dfrac{\pi}{2}\right) = 0 \\ -\varphi''(\alpha) + \dfrac{\ell(\ell+1)}{\cos^2\alpha}\,\varphi(\alpha) = s^2\,\varphi(\alpha) \end{cases}$$

$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R}\,F'(R)\right] + \left(\frac{\hbar^2\,s^2}{m\,R^2} + \frac{m\,\omega^2}{4}\,R^2\right)F(R) = E\,F(R)$$

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$$\ell = 1 \qquad s_{\ell=1} = 1.772724267\ldots$$

$$s \longleftarrow \begin{cases} \varphi'(0) - \dfrac{4}{\sqrt{3}}\,(-1)^{\ell}\,\varphi\left(\dfrac{\pi}{3}\right) = 0 \qquad \varphi\left(\dfrac{\pi}{2}\right) = 0 \\ -\varphi''(\alpha) + \dfrac{\ell(\ell+1)}{\cos^2\alpha}\,\varphi(\alpha) = s^2\,\varphi(\alpha) \end{cases}$$

$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R}\,F'(R)\right] + \left(\frac{\hbar^2\,s^2}{m\,R^2} + \frac{m\,\omega^2}{4}\,R^2\right)\,F(R) = E\,F(R)$$

$\ell = 0$

$$\varphi(\alpha) = \sin\left[s\left(\frac{\pi}{2} - \alpha\right)\right]$$

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$$s \longleftarrow \begin{cases} \varphi'(0) - \dfrac{4}{\sqrt{3}}(-1)^{\ell}\,\varphi\left(\dfrac{\pi}{3}\right) = 0 \qquad \varphi\left(\dfrac{\pi}{2}\right) = 0 \\ -\varphi''(\alpha) + \dfrac{\ell(\ell+1)}{\cos^2\alpha}\,\varphi(\alpha) = s^2\,\varphi(\alpha) \end{cases}$$

1-body Schrö. eq.
fictitious particle
in 2D

$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R}F'(R)\right] + \left(\frac{\hbar^2\,s^2}{m\,R^2} + \frac{m\,\omega^2}{4}R^2\right)F(R) = E\,F(R)$$

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1-body Schrö. eq.
fictitious particle
in 2D

$$-\frac{\hbar^2}{m}\underbrace{\left[F''(R) + \frac{1}{R}\, F'(R)\right]}_{\text{2D Laplacian}} + \left(\frac{\hbar^2\, s^2}{m\, R^2} + \frac{m\,\omega^2}{4} R^2\right) F(R) = E\, F(R)$$

$\ell = 0$

$$\varphi(\alpha) = \sin\left[s\left(\frac{\pi}{2} - \alpha\right)\right]$$

$$\Rightarrow \quad s\,\cos\left(\frac{s\,\pi}{2}\right) + \frac{4}{\sqrt{3}}\,\sin\left(\frac{s\,\pi}{6}\right) = 0$$

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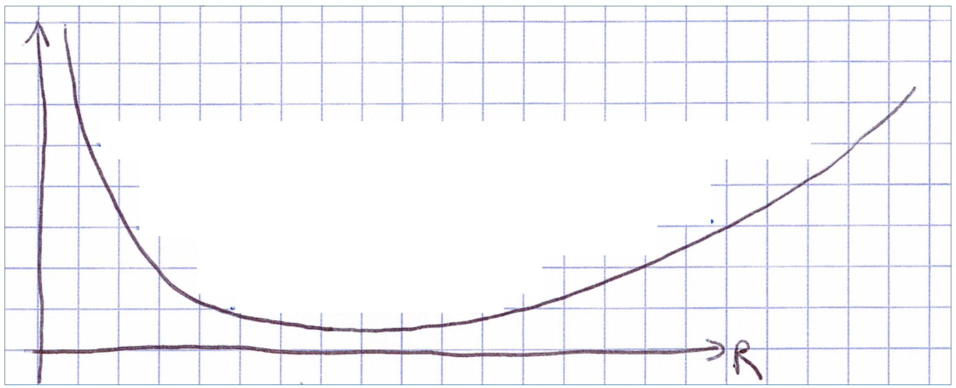
1-body Schrö. eq.
fictitious particle
in 2D

$$-\frac{\hbar^2}{m}\underbrace{\left[F''(R) + \frac{1}{R}\,F'(R)\right]}_{\text{2D Laplacian}} + \underbrace{\left(\frac{\hbar^2\,s^2}{m\,R^2} + \frac{m\,\omega^2}{4}\,R^2\right)}_{U_{\text{eff}}(R)}\,F(R) = E\,F(R)$$



$\ell = 0$

$$\varphi(\alpha) = \sin\left[s\left(\frac{\pi}{2} - \alpha\right)\right]$$

$$\Rightarrow \quad s\cos\left(\frac{s\,\pi}{2}\right) + \frac{4}{\sqrt{3}}\sin\left(\frac{s\,\pi}{6}\right) = 0$$

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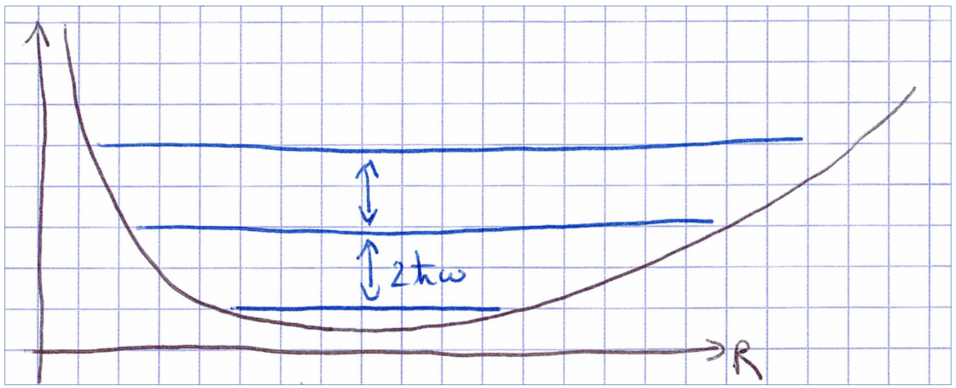
1-body Schrö. eq.
fictitious particle
in 2D

$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R}F'(R)\right] + \left(\frac{\hbar^2 s^2}{m\,R^2} + \frac{m\,\omega^2}{4}R^2\right)F(R) = E\,F(R)$$

2D Laplacian $\qquad U_{\text{eff}}(R)$

$$E = (s + 1 + 2q)\,\hbar\omega \qquad q = 0, 1, 2\ldots$$



$\ell = 0$ $\varphi(\alpha) = \sin\left[s\left(\frac{\pi}{2} - \alpha\right)\right]$

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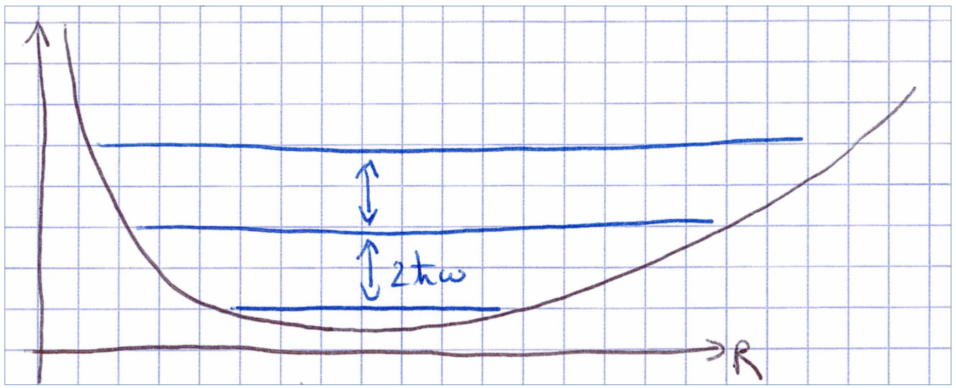
1-body Schrö. eq.
fictitious particle
in 2D

$$\underbrace{-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R}\,F'(R)\right]}_{\text{2D Laplacian}} + \underbrace{\left(\frac{\hbar^2\,s^2}{m\,R^2} + \frac{m\,\omega^2}{4}\,R^2\right)}_{U_{\text{eff}}(R)}\,F(R) = E\,F(R)$$

$E = (s + 1 + 2q)\,\hbar\omega \qquad q = 0, 1, 2\ldots$

ground state: $\ell = 1, \quad E = 2.7727\ \hbar\omega$



1-body Schrö. eq.
fictitious particle
in 2D

$$-\frac{\hbar^2}{m}\underbrace{\left[F''(R) + \frac{1}{R}F'(R)\right]}_{\text{2D Laplacian}} + \underbrace{\left(\frac{\hbar^2 s^2}{m R^2} + \frac{m\omega^2}{4}R^2\right)}_{U_{\text{eff}}(R)} F(R) = E\,F(R)$$

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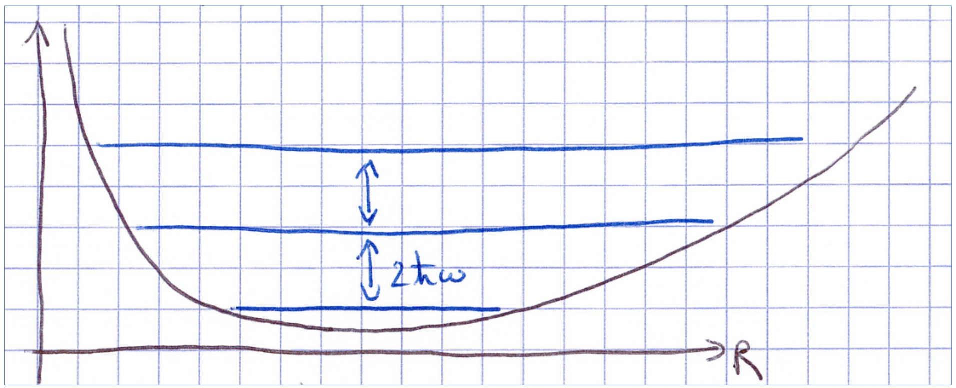
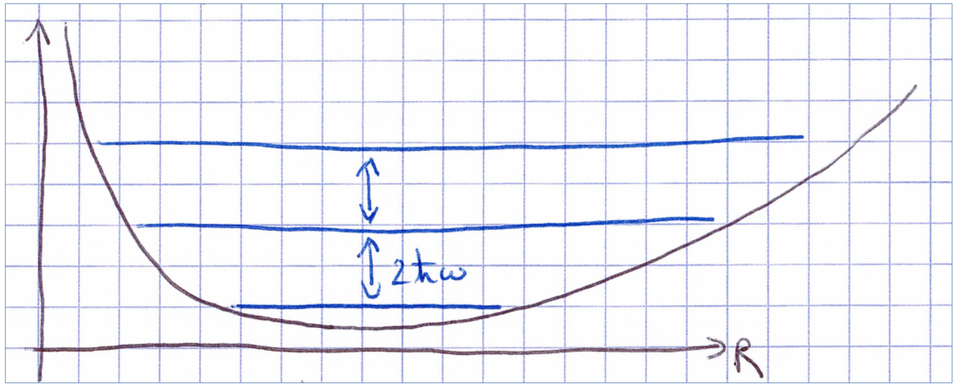


1-body Schrö. eq.
fictitious particle
in 2D

$$-\frac{\hbar^2}{m}\underbrace{\left[F''(R) + \frac{1}{R}F'(R)\right]}_{\text{2D Laplacian}} + \underbrace{\left(\frac{\hbar^2 s^2}{m R^2} + \frac{m\omega^2}{4}R^2\right)}_{U_{\text{eff}}(R)} F(R) = E\,F(R)$$



$$\boxed{E = (s + 1 + 2q)\,\hbar\omega} \qquad q = 0, 1, 2\ldots$$

ground state: $\ell = 1, \quad E = 2.7727\ \hbar\omega$

1-body Schrö. eq.
fictitious particle
in 2D

$$-\frac{\hbar^2}{m}\underbrace{\left[F''(R) + \frac{1}{R}\,F'(R)\right]}_{\text{2D Laplacian}} + \underbrace{\left(\frac{\hbar^2\,s^2}{m\,R^2} + \frac{m\,\omega^2}{4}\,R^2\right)}_{U_{\text{eff}}(R)} F(R) = E\,F(R)$$

$$\boxed{E = (s + 1 + 2q)\,\hbar\omega} \qquad q = 0, 1, 2\ldots$$

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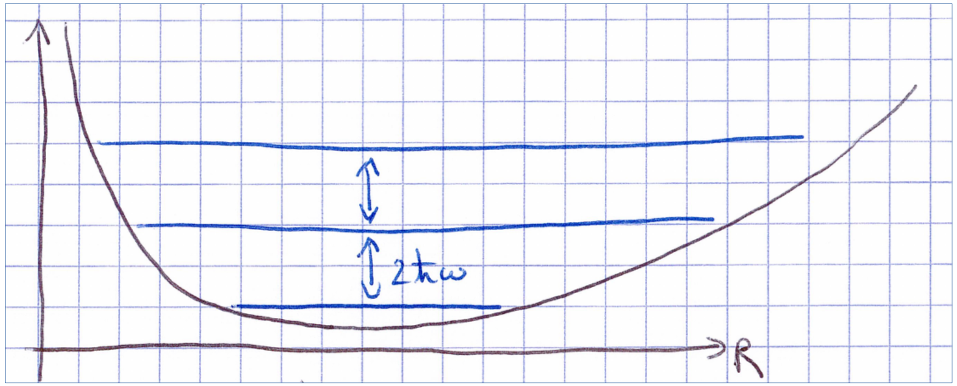
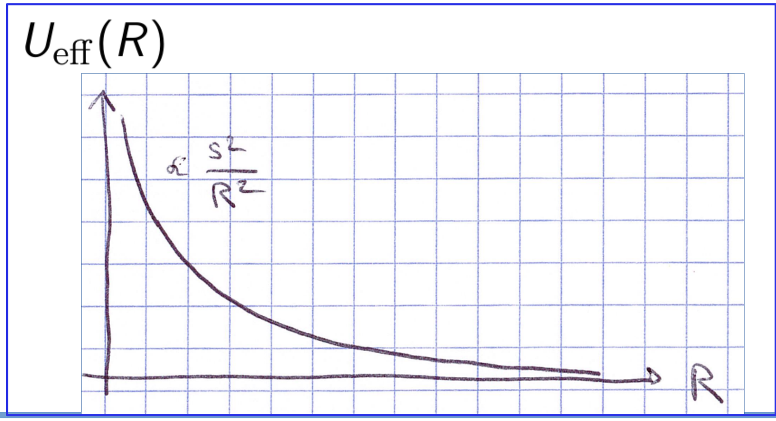


Free space: $\omega = 0$



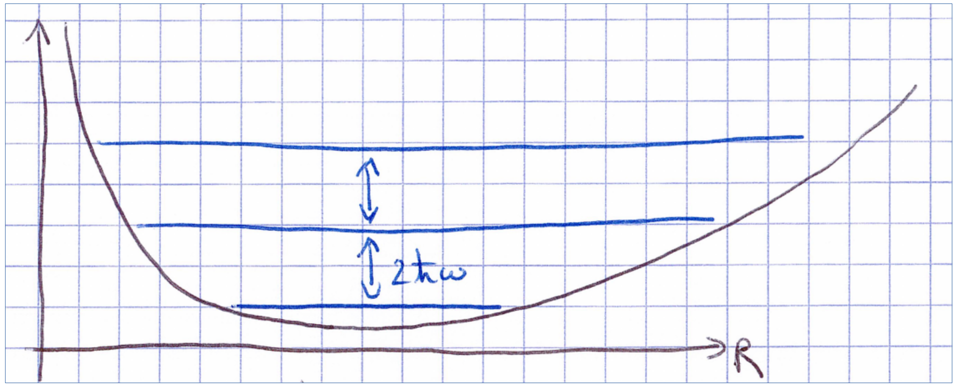
$$E = 0:$$
$$F(R) = R^s$$

1-body Schrö. eq.
fictitious particle
in 2D

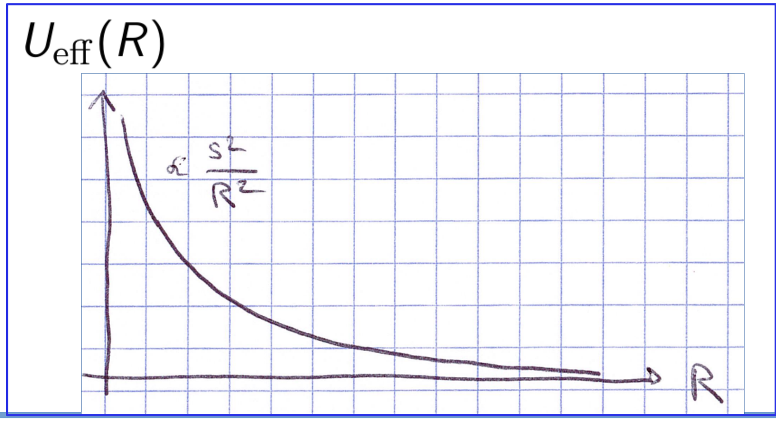
$$-\frac{\hbar^2}{m}\underbrace{\left[F''(R) + \frac{1}{R}\,F'(R)\right]}_{\text{2D Laplacian}} + \underbrace{\left(\frac{\hbar^2\, s^2}{m\,R^2} + \frac{m\,\omega^2}{4}\,R^2\right)}_{U_{\rm eff}(R)} F(R) = E\,F(R)$$

$$\boxed{E = (s + 1 + 2q)\,\hbar\omega} \qquad q = 0, 1, 2\ldots$$

ground state: $\ell = 1, \quad E = 2.7727\ \hbar\omega$



Free space: $\omega = 0$



$E = 0:$
$F(R) = R^s$

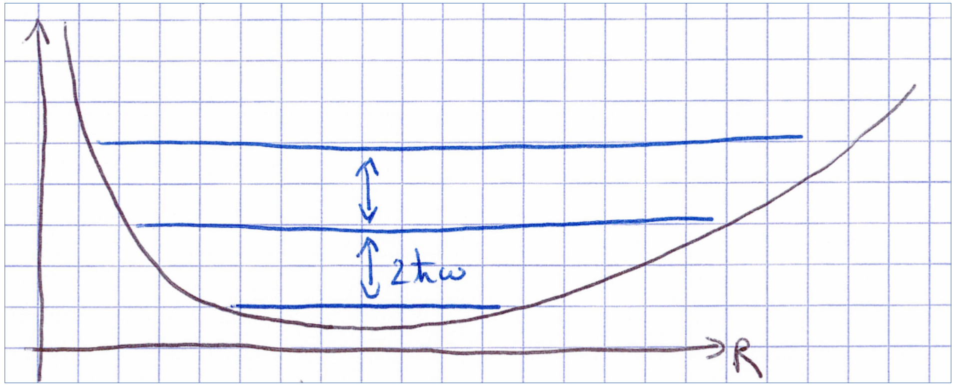
All this remains true for arbitrary N
Only values of s are different

1-body Schrö. eq.
fictitious particle
in 2D

$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R}F'(R)\right] + \left(\frac{\hbar^2 s^2}{m R^2} + \frac{m\omega^2}{4}R^2\right)F(R) = E\,F(R)$$

2D Laplacian

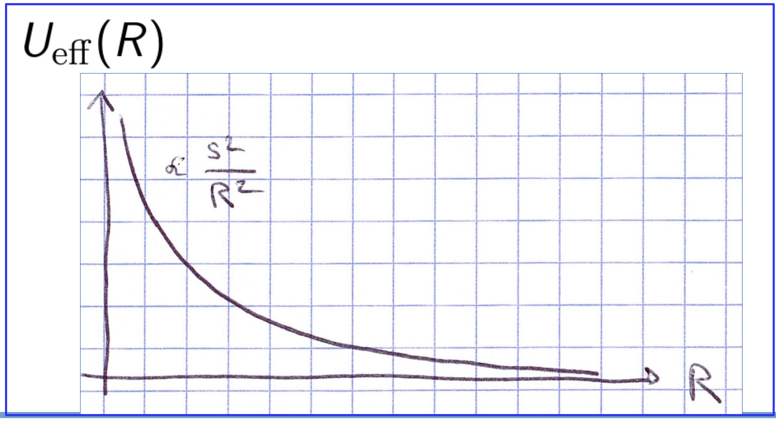
$U_{\rm eff}(R)$



$$E = (s + 1 + 2q)\,\hbar\omega \qquad q = 0, 1, 2\ldots$$

~~ground state: $\ell = 1$, $E = 2.7727\,\hbar\omega$~~

Free space: $\omega = 0$



$E = 0$:
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All this remains true for arbitrary N
Only values of s are different

Chapter 3

Symmetry properties of the unitary N-body problem

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- Y. Castin, Comptes Rendus Physique **5**, 407 (2004)
- S. Tan, arXiv:cond-mat/0412764
- FW & Y. Castin, PRA **74**, 053604 (2006)

Chapter 3

Symmetry properties of the unitary N-body problem

ZRM , $a = \infty$

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$\Psi(\vec{r_1}, \ldots, \vec{r_N})$

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ZRM , $a = \infty$

$$\Psi(\vec{r_1}, \ldots, \vec{r_N})$$

$$\begin{cases} -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_{\vec{r_i}} \Psi + \sum_{i=1}^{N} U(r_i)\, \Psi = \mathcal{E}\, \Psi \\ \end{cases}$$

Chapter 3

Symmetry properties of the unitary N-body problem

ZRM , $a = \infty$ $\Psi(\vec{r_1}, \ldots, \vec{r_N})$

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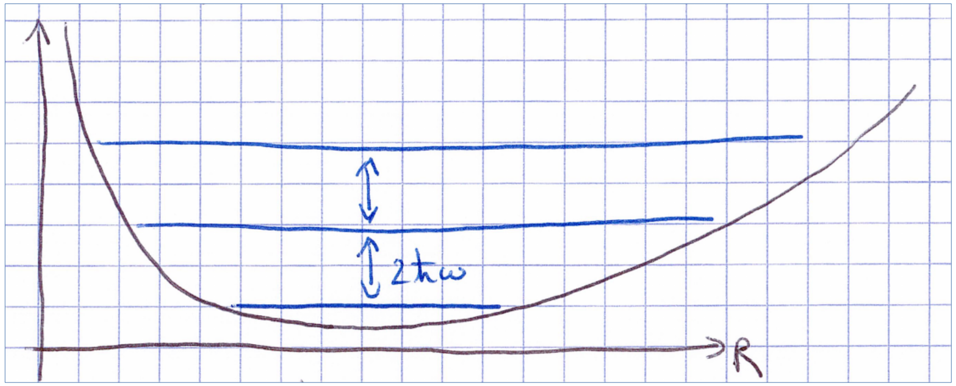

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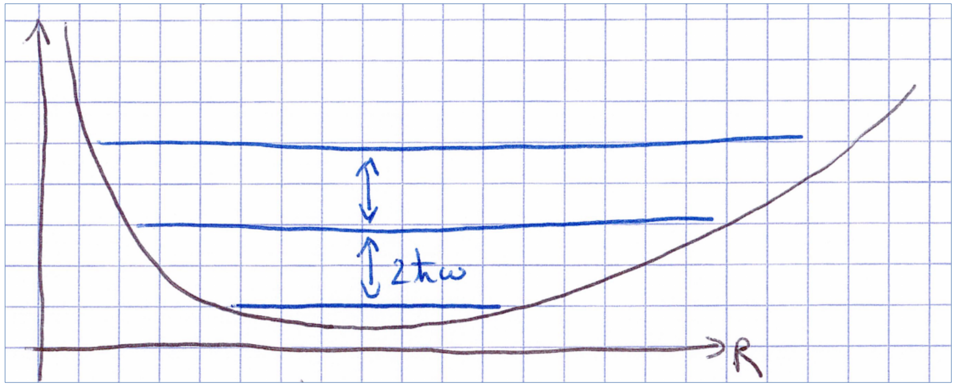
[SO(2,1) dynamical symmetry]



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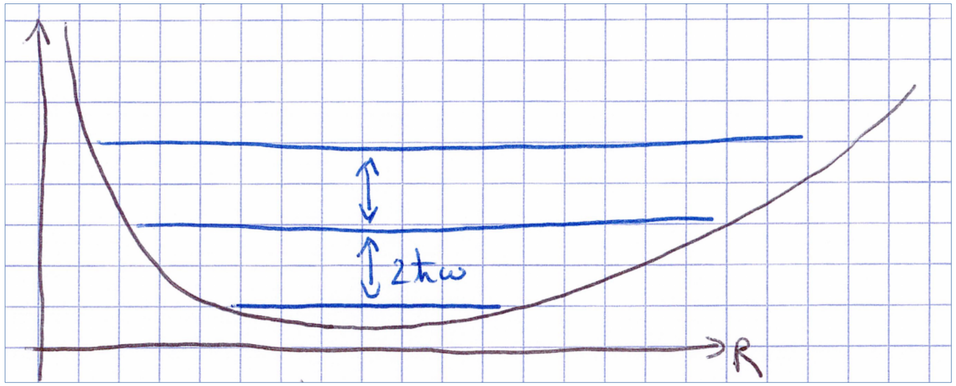

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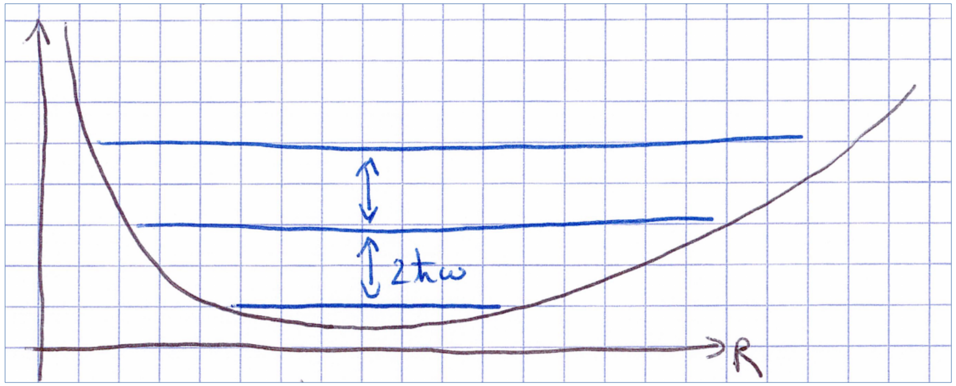

$$s(N_\uparrow, N_\downarrow) := \text{lowest } s \qquad s(2,1) = 1.7727$$

Excursion: s for $N \to \infty$

$$s\left(\frac{N}{2}, \frac{N}{2}\right) \underset{N\to\infty}{\sim} \quad ?$$

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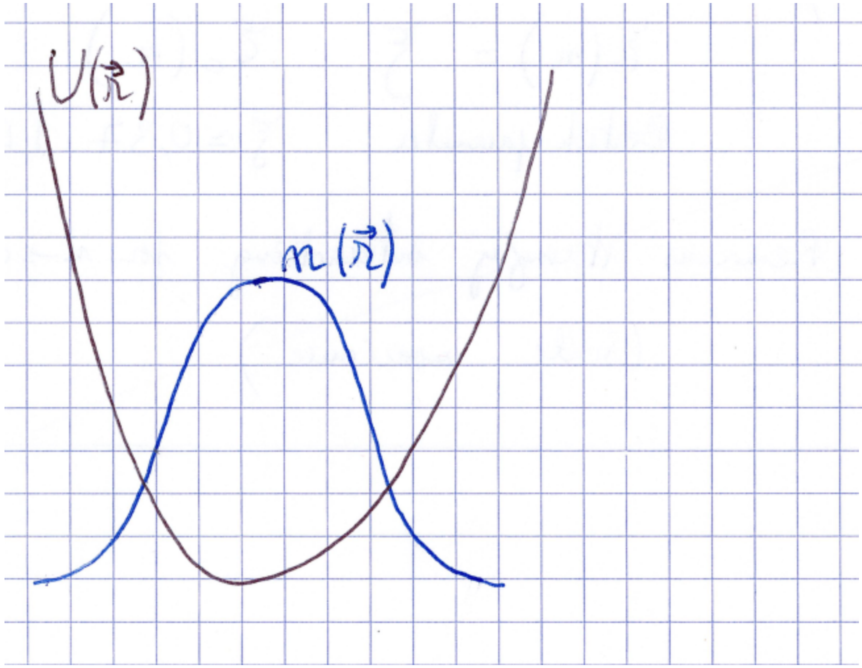

Local Density Approximation (= hydrostatics)

Locally : $\simeq$ homogeneous gas,

$$\mu_{local}(\vec{r}) = \mu_0 - U(\vec{r})$$

Exact for $N \to \infty$.

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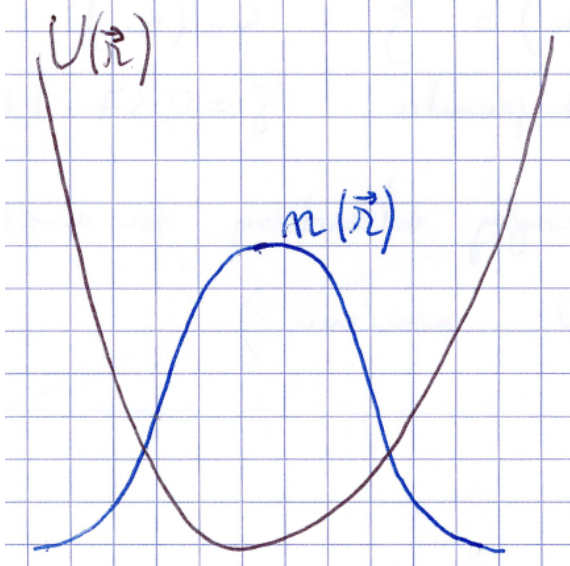

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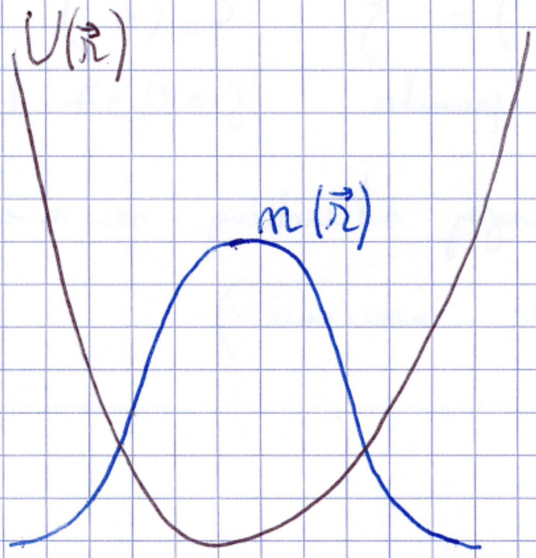

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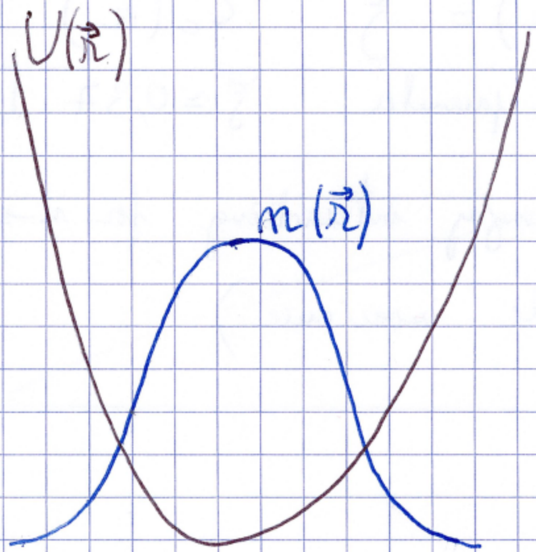

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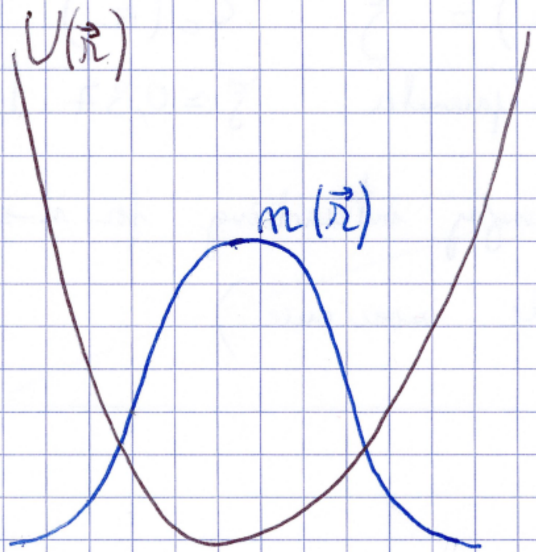

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NB: Free space, $E = 0$: $\psi(\vec{r_1}, \ldots, \vec{r_N}) = R^{\,s - \frac{3N-5}{2}}\, \phi(\vec{\Omega})$

Part 2: Short-distance scaling law

Arbitrary a and $U(\vec{r})$.

$\Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ eigenstate of ZRM.

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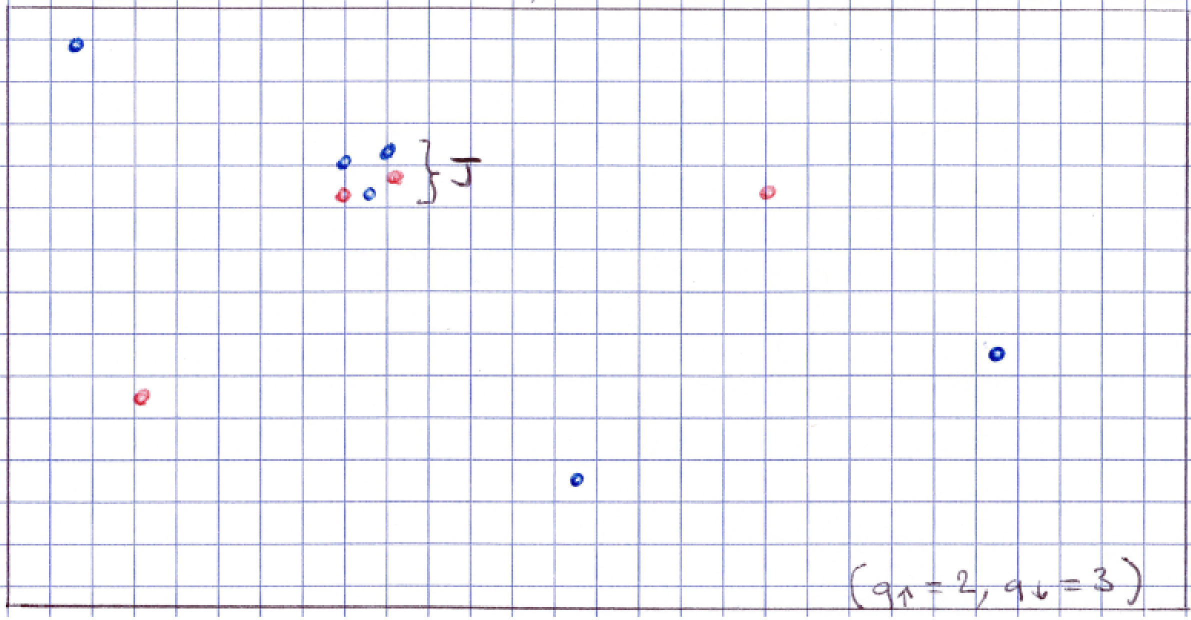

$J \subset \{1, 2, \dots, N\}$

with $\begin{cases} q_\uparrow & \text{spin-}\uparrow \\ q_\downarrow & \text{spin-}\downarrow \end{cases}$

$q = q_\uparrow + q_\downarrow \geq 3$

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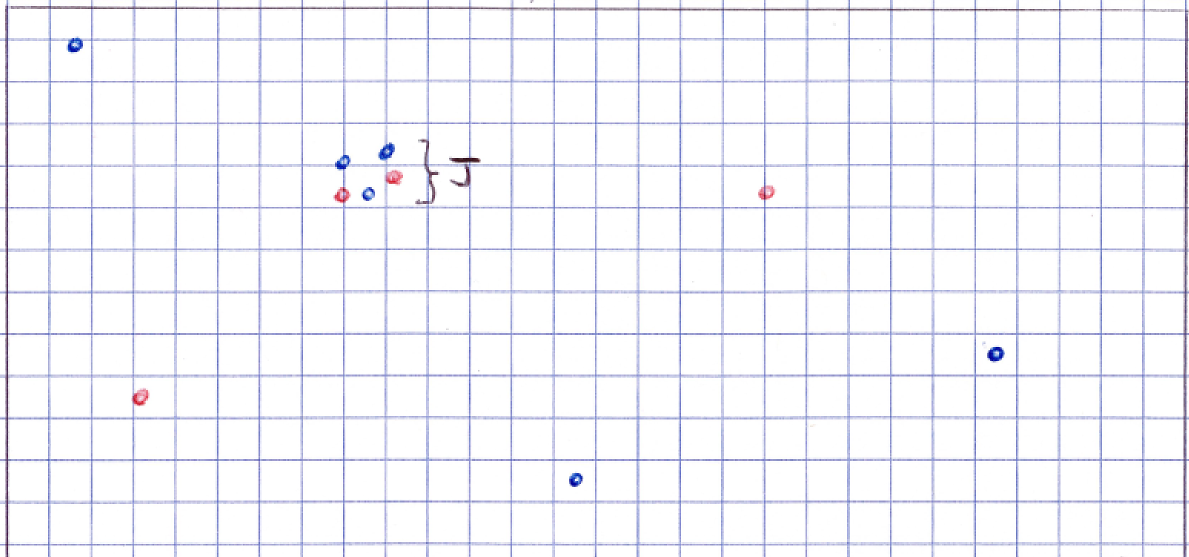

$(q_\uparrow = 2, q_\downarrow = 3)$

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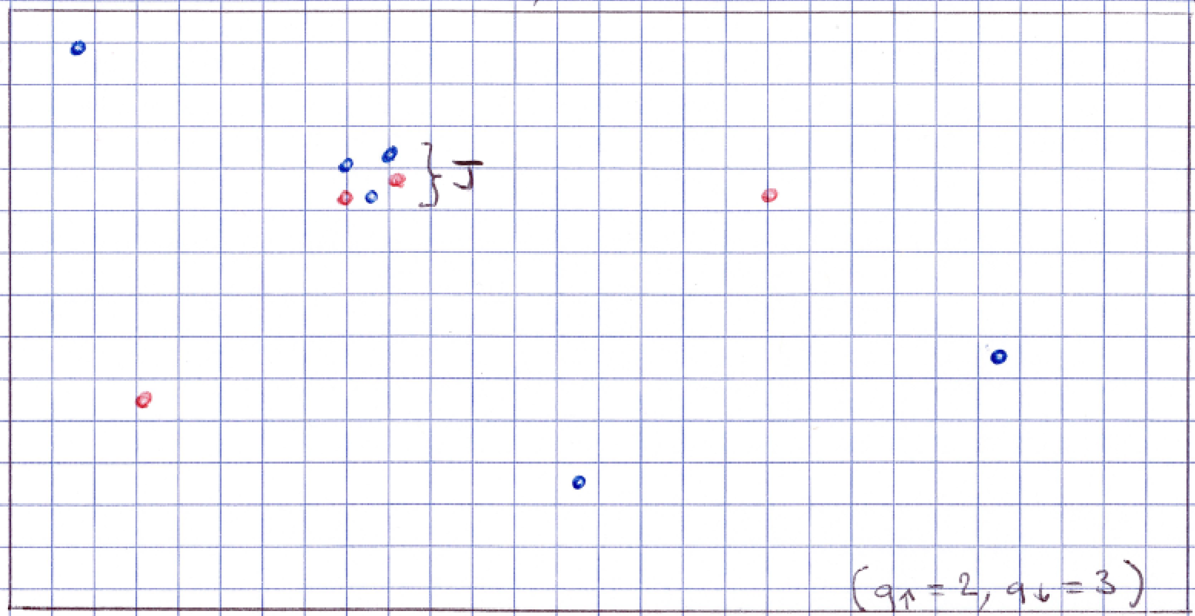

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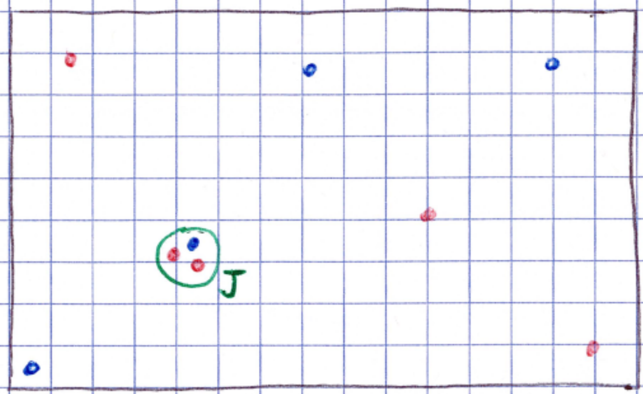

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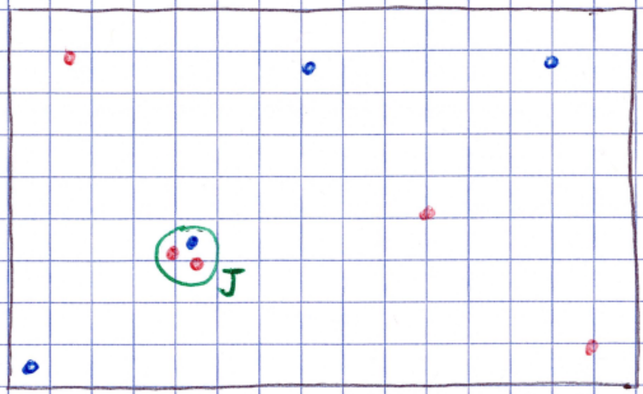

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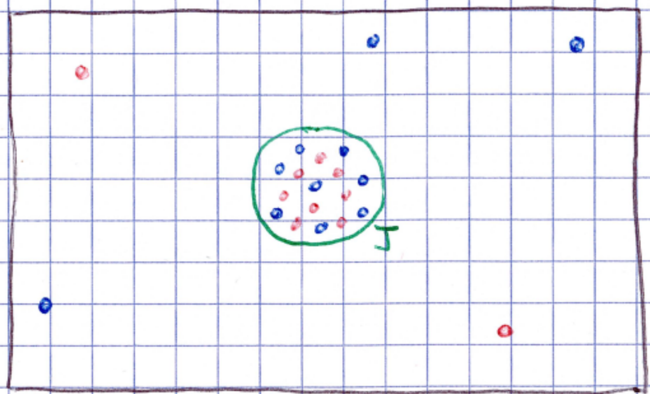
$(q_\uparrow, q_\downarrow) = \left(\frac{q}{2}, \frac{q}{2}\right)$

$q \gg 1$



$$\Psi(\vec{r}_1 \ldots \vec{r}_N) \underset{R_J \to 0}{\propto} R_J^{\,s\left(\frac{q}{2}, \frac{q}{2}\right) - \frac{3q-5}{2}}$$

$$s\left(\frac{q}{2}, \frac{q}{2}\right) \simeq \sqrt{\xi} \left(\frac{3q}{4}\right)^{\frac{4}{3}}$$

Part 3 : The Castin mode

Back to $\boxed{a = \infty}$. Isotropic harmonic trap, $\omega(t)$.

$\vec{X} := (\vec{r}_1, \ldots, \vec{r}_N)$.

ZRM, t-dependent : $\Psi(\vec{X}, t)$

$$\begin{cases} \bullet \; i\hbar \frac{\partial}{\partial t} \Psi(\vec{X}, t) = -\frac{\hbar^2}{2m} \Delta_{\vec{X}} \Psi(\vec{X}, t) + \frac{1}{2} m\, \omega(t)^2 X^2 \times \Psi(\vec{X}, t) \\ \bullet \; \Psi(\vec{X}, t) \text{ satisfies (CC) for } a = \infty \end{cases}$$

$$\hookrightarrow \Psi(\vec{X}, t) \underset{r \to 0}{=} \frac{1}{r} A(\vec{c}; \vec{r}_2, \vec{r}_4, \ldots \vec{r}_N; t) + O(r)$$

Consider : $\omega(t) = \begin{cases} \omega_0 & , \; t < 0 \\ \text{arbitrary} & , \; t > 0. \end{cases}$

Scaling solution [Y. Castin, 2004]

if $\Psi(\vec{X}, t=0)$ stationary state of ZRM (ω_0)

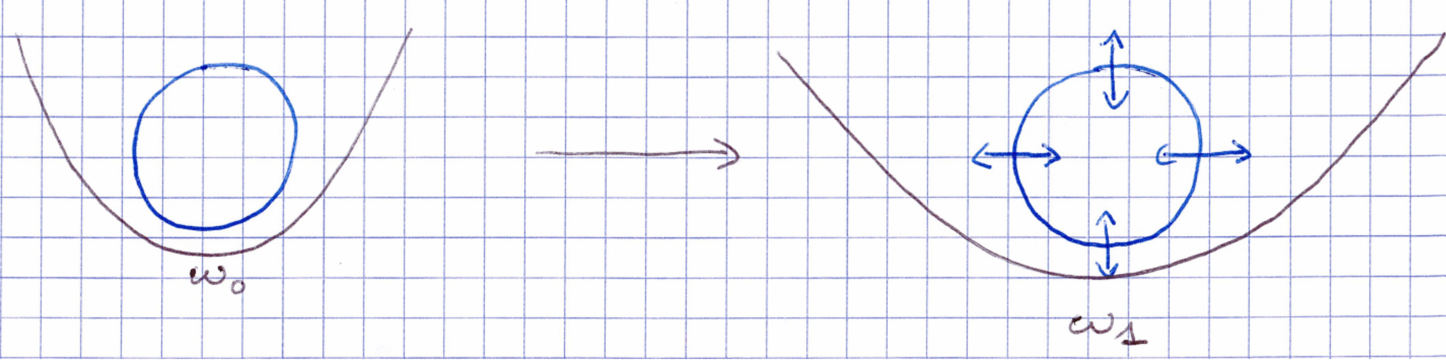
then : $$\boxed{\Psi(\vec{X}, t) = \Psi\left(\frac{\vec{X}}{\lambda(t)}, 0\right) e^{i \frac{\dot{\lambda}(t)}{\lambda(t)} X^2 \frac{m}{2\hbar}} \frac{e^{i\theta(t)}}{\lambda(t)^{3N/2}}}$$

with $\lambda(t)$ s.t. $$\ddot{\lambda} = -\frac{d}{d\lambda}\left[\frac{\omega_0^2}{2\lambda^2} + \frac{\omega(t)^2}{2}\lambda^2\right] \quad (*)$$

$$[\lambda(0) = 1 \;, \; \dot{\lambda}(0) = 0]$$

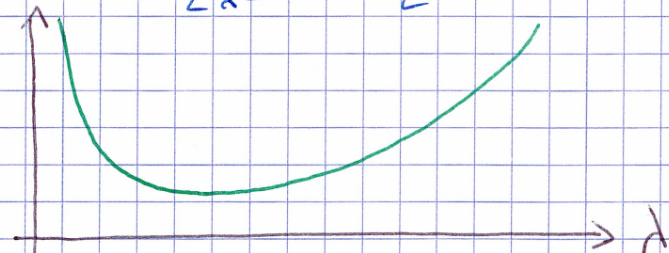
Proof : $\begin{cases} \bullet \text{ (CC) : satisfied by } \Psi(\vec{X}, 0) \Rightarrow \text{also by } \Psi(\frac{\vec{X}}{\lambda}, 0) \\ \bullet \text{ Schrödinger eq : explicit computation (same than non-interacting case)} \end{cases}$

Breathing mode : $\omega(t) = \omega_1$, $t > 0$.



$\lambda(t)$ periodic, frequency $2\omega_1$

indeed: $(*) \Leftrightarrow$ a classical particle in $V_{eff}(\lambda) = \frac{\omega_0^2}{2\lambda^2} + \frac{\omega_1^2}{2}\lambda^2$



$\frac{\omega_0^2}{\lambda^2} \Leftrightarrow$ centrifugal potential

$\leadsto$ just harmonic oscillator (ω_1)

$$n(\vec{r}, t) = n\left(\frac{\vec{r}}{\lambda(t)}, 0\right) \times \frac{1}{\lambda(t)^3}$$

$\Rightarrow$ Breathing motion, frequency $= 2\omega_1$, undamped!

Experimental observation: Wuhan, Kaijun Jiang et al.
PRL 132, 243403 (2024)
PRA 111, 053317 (2025)

frequency: $\omega_B \simeq 2.010 \times \omega_1$

damping: $\frac{\Gamma_B}{\omega_B} = 2 \times 10^{-3}$, mainly due to residual anisotropy + technical noise, also damping CM mode [Min + Peng, PRA 2026]

Consequence: in hydrodynamics, 2 viscosities:
- shear (no effect here)
- bulk ζ $(T > T_c)$

entropy: $\dot{S} = \int d^3r \, \frac{\zeta}{T} \| \vec{\nabla} \cdot \vec{v} \|^2 \geq 0$

Undamped breathing: $S(t)$ periodic

$\Rightarrow S(t) =$ constant $\Rightarrow$ $\boxed{\zeta = 0}$

[D.T. Son, PRL **98**, 020604 (2007)]