On the calculation of $\langle VV - AA \rangle$ and other such Green's functions for all Q^2; $g-2$

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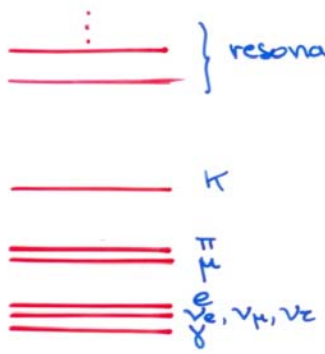
PL B424	('98)	EdR + MK
PL B443	('98)	SP + EdR + MK
PRL 83	('99)	SP + MP + EdR + MK
EJP C21	('01)	AN + MK
PRL 86	('01)	SP + BP + EdR
JHEP 0201	('02)	MG + SP + BP + EdR
JHEP 0211	('02)	SP + MP + EdR + MK
JHEP 0303	('03)	SP + MP + EdR + MK

General framework : effective theory

Concentrate on information directly relevant for low-energy physics

Two requirements

i) existence of a mass gap (clear separation of scales)



range of validity of effective theory set by mass gap

ii) masses of light degrees of freedom protected by a symmetry ('t Hooft's naturalness condition)

spin 0 → Goldstone bosons (π, K)

spin 1/2 → chiral symmetry $(\mu, e, \nu_e \cdots)$

spin 1 → gauge invariance (γ)

→ Systematic low-energy expansion

QCD Green's functions : why study them ?

- LECs in ChPT $\rightarrow$ coefficients of Taylor expansion
  L_i, C_i

- LECs in extensions of ChPT

  radiative corrections to strong interaction processes

  K_i

  radiative corrections to semileptonic decays

  X_i

  $|\Delta S| = 1$ transitions g_8, g_{27}, N_i

  $|\Delta S| = 2$ transitions B_K

  $\rightarrow$ weighted integrals of Green's functions

- Develop analytical tools that can be confronted with lattice calculations

QCD Green's functions : how to study them ?

- short distance constraints OPE
- long distance properties
- dispersion relations $\nearrow$ data

  $\searrow$ large N_c resonance approximations

A case study $\Pi_{LR}(Q^2)$ $(\langle VV-AA \rangle)$

* Simplest Green's function, with very interesting properties
* Relevant in several instances

  L_{10}

  ΔM_π^2

  Q_7 ⟷ issue of matching

  non QCD situations : S parameter in EW fits

$$\Pi_{LR}(Q^2)\,(q_\mu q_\nu - \eta_{\mu\nu} q^2) =$$

$$= 2i \int d^4x \, e^{iq\cdot x} \langle 0 | T\{ L_\mu(x) R_\nu^\dagger(0) \} | 0 \rangle$$

$$L_\mu = \bar{d}_L \gamma_\mu u_L \qquad R_\mu = \bar{d}_R \gamma_\mu u_R$$

$Q^2 = -q^2$, chiral limit

- $\Pi_{LR}(Q^2)$ → order parameter of spontaneous chiral symmetry breaking

- smooth short distance behaviour

$$\lim_{Q^2\to\infty} \Pi_{LR}(Q^2) = \frac{1}{Q^6}\left[-4\pi^2\left(\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right)\underset{\text{large } N_c}{\langle \bar{\psi}\psi \rangle^2}\right] + \mathcal{O}(1/Q^8)$$

- low energy expansion

$$\Pi_{LR}(Q^2) = -\frac{F_\pi^2}{Q^2} + 4\,\cancel{Q^2}\,L_{10} + \cdots$$

- positivity condition

$$-Q^2\,\Pi_{LR}(Q^2) \geq 0 \quad \text{for} \quad 0 \leq Q^2 \leq \infty$$

- $M^2_{\pi^\pm} - M^2_{\pi^0}$ in the chiral limit

$$\Delta M^2_\pi = -\frac{3}{4F_\pi^2} \cdot \frac{\alpha}{\pi} \int_0^\infty dQ^2\; Q^2\, \Pi_{LR}(Q^2)$$

$$0 \leq \Delta M^2_\pi < \infty$$

- unsubtracted dispersion relation

$$\Pi_{LR}(Q^2) = \int_0^\infty dt \cdot \frac{1}{t+Q^2} \cdot \frac{1}{\pi}\, \mathrm{Im}\,\Pi_{LR}(t)$$

- Weinberg sum rules

$$\int_0^\infty dt\; \mathrm{Im}\,\Pi_{LR}(t) = 0$$

$$\int_0^\infty dt\; t\, \mathrm{Im}\,\Pi_{LR}(t) = 0$$

Large Nc representation

$$\frac{1}{\pi}\, \mathcal{I}m\, \Pi_{LR}(t) = \sum_V f_V^2 M_V^2\, \delta(t - M_V^2) - \sum_A f_A^2 M_A^2\, \delta(t - M_A^2) - F_\pi^2\, \delta(t)$$

$$-Q^2\, \Pi_{LR}(Q^2) = F_\pi^2 + \sum_A f_A^2 M_A^2 \frac{Q^2}{M_A^2 + Q^2} - \sum_V f_V^2 M_V^2 \frac{Q^2}{M_V^2 + Q^2}$$

$$\sum f_V^2 M_V^2 - \sum f_A^2 M_A^2 = F_\pi^2$$

$$\sum f_V^2 M_V^4 - \sum f_A^2 M_A^4 = 0$$

$$\sum f_V^2 M_V^6 - \sum f_A^2 M_A^6 = -4\pi^2 \left(\frac{\alpha_s}{\pi} + \cdots\right) \langle \bar\psi \psi \rangle^2$$

Inverse moment dispersion relations

$$-4 L_{10} = \int_0^\infty dt\, \frac{1}{t} \left[\frac{1}{\pi}\, \mathcal{I}m\, \Pi_V(t) - \frac{1}{\pi}\, \mathcal{I}m\, \Pi_A(t) \right]$$

$$= \sum f_V^2 - \sum f_A^2$$

Minimal hadronic approximation (MHA) to large N_c

→ keep only one resonance per channel

$$f_V^2 = \frac{F_\pi^2}{M_A^2 - M_V^2} \cdot \frac{M_A^2}{M_V^2} \, , \quad f_A^2 = \frac{F_\pi^2}{M_A^2 - M_V^2} \cdot \frac{M_V^2}{M_A^2}$$

$$L_{10} = -\frac{F_\pi^2}{4}\left(\frac{1}{M_V^2} + \frac{1}{M_A^2}\right) \sim -5 \times 10^{-3}$$

$$-F_\pi^2 M_V^2 M_A^2 = -4\pi^2\left(\frac{\alpha_s}{\pi} + O(\alpha_s^2)\right) \langle \bar\psi \psi \rangle^2$$

$$\longrightarrow \quad \langle \bar\psi\psi \rangle\Big|_{1\,\mathrm{GeV}} \sim -(300\,\mathrm{MeV})^3$$

How good is the MHA?

→ compare to data

→ study MHA in a toy model

Comparison with data

Finite energy moments

$$\mathcal{M}_n = \int_0^{s_0} dt\, t^n \frac{1}{\pi} \operatorname{Im} \Pi_{LR}(t)$$

$$n = -2, -1, 0, 1, 2, 3$$

s_0 fixed by the requirement that $\langle VV \rangle$ and $\langle AA \rangle$ have no $1/Q^2$ term in the OPE

$$\rightarrow \frac{N_c}{16\pi^2} \times \frac{2}{3} \times s_0 \times \left[1 + \mathcal{O}(\alpha_s)\right] = \frac{F_0^2 M_A^2}{M_A^2 - M_V^2}$$

$$\mathcal{M}_n^{exp} \simeq \mathcal{M}_n^{MHA}$$

Toy model for $\langle VV \rangle$

$$\frac{1}{\pi} \operatorname{Im} \Pi_V(t) = \sigma^2 \sum_{n=0}^{\infty} \delta(t - M_0^2 - n\sigma^2)$$

$$\mathcal{A}(Q^2) \equiv -Q^2 \frac{d\Pi_V}{dQ^2}(Q^2) = \int_0^{\infty} dt \frac{Q^2}{(t+Q^2)^2} \frac{1}{\pi} \operatorname{Im} \Pi_V(t)$$

$$M_0^2 = \sigma^2/2 \quad (\text{no } 1/Q^2 \text{ term in OPE})$$

successive approximations

$$\frac{1}{\pi} \operatorname{Im} \Pi_V(t) \Big|_{MHA}^{(m+1)} = \sigma^2 \sum_{n=0}^{m} \delta(t - \tilde{M}_n^2) + \Theta(t - s_0)$$

$$s_0 = (m+1)\sigma^2 \quad (\text{no } 1/Q^2 \text{ in OPE})$$

ex: $m=0$ $\qquad \tilde{M}_0 = \frac{\sigma^2}{2}\left(1 + \frac{1}{12}\right)$

More complicated objects : 3 pt functions

$\langle VVS \rangle$, $\langle VAP \rangle$, $\langle VVP \rangle$, $\langle AAP \rangle$, $\langle VVA$

$$(\Pi_{VVP})^{abc}_{\mu\nu}(p,q) = \int d^4x \int d^4y \, e^{ip\cdot x} e^{iq\cdot y}$$

$$\langle 0 | T\{ V^a_\mu(x) V^b_\nu(y) P^c(0) \} | 0 \rangle$$

$$= \epsilon_{\mu\nu\alpha\beta} \, p^\alpha q^\beta \, d^{abc} \, \mathcal{H}_V(p^2, q^2, (p+q)^2)$$

Several short distance limits

$x \sim y \sim 0$

$$\lim_{\lambda\to\infty} \mathcal{H}_V((\lambda p)^2, (\lambda q)^2, (\lambda p + \lambda q)^2) =$$

$$= -\frac{1}{2\lambda^4} \langle \bar\psi\psi \rangle_0 \frac{p^2+q^2+(p+q)^2}{p^2 q^2 (p+q)^2} + O\left(\frac{1}{\lambda^6}\right)$$

$x \sim y \not\sim 0$

$$\lim_{\lambda\to\infty} \mathcal{H}_V((\lambda p)^2, (q-\lambda p)^2, q^2) =$$

$$= -\frac{1}{\lambda^2} \cdot \frac{1}{p^2} \Pi_{AP}(q^2) + O\left(\frac{1}{\lambda^3}\right)$$

$x \sim 0 \not\sim y$

$$\lim_{\lambda\to\infty} \mathcal{H}_V((\lambda p)^2, q^2, (q+\lambda p)^2) =$$

$$= +\frac{1}{\lambda^2} \cdot \frac{1}{p^2} \Pi_{VT}(q^2) + O\left(\frac{1}{\lambda^3}\right)$$

- Involves 2 pt functions

$\Pi_{AP} \leftrightarrow \langle AP \rangle$

$\Pi_{VT} \leftrightarrow \langle VT \rangle$

- long distance

$$\mathcal{H}_V(p^2, q^2, (p+q)^2) = \frac{N_c}{8\pi^2} \cdot \frac{\langle \bar\psi\psi \rangle_0}{F_0^2} \cdot \frac{1}{(p+q)^2} + \cdots$$

d

$$\int d^4x\, e^{ip\cdot x} \langle 0|T\{A^a_\mu(x) P^b(0)\}|0\rangle$$
$$= p_\mu \delta^{ab} \Pi_{AP}(p^2)$$
$$\Pi_{AP}(p^2) = \frac{\langle\bar\psi\psi\rangle_0}{p^2}$$

$$\int d^4x\, e^{ip\cdot x} \langle 0|T\{V^a_\mu(x)(\bar\psi\sigma_{\rho\tau}\frac{\lambda^b}{2}\psi)(0)\}|0\rangle$$
$$= (p_\rho \eta_{\mu\tau} - p_\tau \eta_{\mu\rho})\delta^{ab}\Pi_{VT}(p^2)$$
$$\lim_{\lambda\to\infty} \Pi_{VT}((\lambda p)^2) = -\frac{1}{\lambda^2}\frac{\langle\bar\psi\psi\rangle_0}{p^2} + O\left(\frac{1}{\lambda^4}\right)$$

MHA

$$\mathcal{H}_V(p^2, q^2, (p+q)^2) =$$
$$= -\frac{1}{2}\langle\bar\psi\psi\rangle_0 \frac{p^2+q^2+(p+q)^2 - c_V}{(p^2-M_V^2)(q^2-M_V^2)(p+q)^2}$$
$$c_V = \frac{N_c}{4\pi^2}\cdot\frac{M_V^4}{F_0^2}$$
$$\Pi_{VT}(p^2) = -\langle\bar\psi\psi\rangle_0 \frac{1}{p^2-M_V^2}$$

→ solves all the short distance constraints considered so far

→ possible to incorporate more resonances
  → need more terms in the OPE

Applications

- Resonance estimates of LECs

$\langle VVP \rangle$

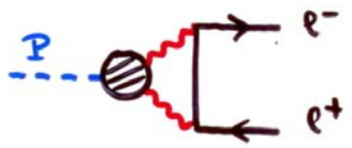
$\langle AAP \rangle$

$\langle VAP \rangle \rightarrow C_{78}\,,\ C_{82}\,,\ C_{87}\,,\ C_{88}\,,\ C_{89}\,,\ C_{90}$

- $P \to \ell^+\ell^-$ decay modes

$$\frac{Br\,(P\to\ell^+\ell^-)}{Br\,(P\to\gamma\gamma)} = 2\left(\frac{\alpha\, m_\ell}{\pi\, M_P}\right)^2 \beta_\ell\,(M_P^2)\,|\mathcal{A}\,(M_P^2)|^2$$

$$\mathcal{A}(s) = \chi_P(\mu) + \frac{N_c}{3}\left[-\frac{5}{2} + \frac{3}{2}\ln\frac{m_\ell^2}{\mu^2} + C(s)\right]$$



$$\langle \ell^-(p')\,|\,P^3(0)\,|\,\ell^-(p)\rangle = -\frac{ie^4}{32\pi^4}\times\frac{\mathcal{A}(t)}{t}\times\frac{m_\ell\,\langle\bar\psi\psi\rangle_0}{F_0^{\,2}} \times \bar{u}(p')\,\gamma_5\, u(p)$$

$$= ie^4\int\frac{d^4q}{(2\pi)^4}\,\bar{u}(p')\,\gamma^\mu\,[\,\not p' - \not q + m_\ell\,]\,\gamma^\nu\, u(p) \times \frac{2}{3}\,\epsilon_{\mu\nu\alpha\beta}\, q^\alpha\,(p'-p)^\beta\, \mathcal{H}_V\,(q^2, (p'-p-q)^2, t)$$

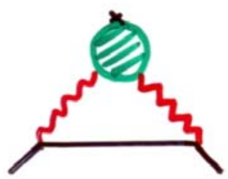
$$\chi\,(\mu = M_V)\Big|_{MHA} = 2.2 \pm 0.9$$

Muon g-2

Hadronic Green's functions appear in several places

$\langle VV \rangle$

$\langle VVVV \rangle$



$\langle VVA \rangle$

$$a_\mu^{hvp;1} = 2\pi^2 \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx \, \frac{(1-x)(2-x)}{x} \, \mathcal{A}\left(\frac{x^2}{1-x} m_\mu^2\right)$$

$$\mathcal{A}(Q^2) = -Q^2 \frac{d\Pi_V}{dQ^2}(Q^2)$$

$$= \int_0^\infty dt \, \frac{Q^2}{(t+Q^2)^2} \, \frac{1}{\pi} \mathrm{Im}\,\Pi_V(t)$$

$$\frac{1}{\pi} \mathrm{Im}\,\Pi_V(t) = \frac{2}{3} f_V^2 M_V^2 \, \delta(t - M_V^2)$$
$$+ \frac{2}{3} \frac{N_c}{12\pi^2} \{1 + O(\alpha_s)\} \, \Theta(t - s_0)$$

$$2 f_V^2 M_V^2 = \frac{N_c}{12\pi^2} s_0 \left(1 + \frac{3}{8} \frac{\alpha_s(s_0)}{\pi} + \cdots\right)$$

$$\rightarrow a_\mu^{hvp;1}\Big|_{MHA} = (570 \pm 170) \times 10^{-10}$$

Determinations from e^+e^- data

$$a_\mu^{hvp;1}(e^+e^-) = 683.8\,(7.5) \times 10^{-10}$$

1% level !

Two loop EW corrections

$$\frac{1}{2} d^{abc} W_{\mu\nu\rho}(q_1, q_2) =$$

$$= i \int d^4x_1 \int d^4x_2 \, e^{i(q_1 \cdot x_1 + q_2 \cdot x_2)}$$

$$\langle 0 | T\{ V^a_\mu(x_1) V^b_\nu(x_2) A^c_\rho(0)\} | 0 \rangle$$

$$= \frac{1}{8\pi^2} w_L(q_1^2, q_2^2, (q_1+q_2)^2)(q_1+q_2)_\rho \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta$$

+ transverse part

$$w_L(q_1^2, q_2^2, (q_1+q_2)^2) = -\frac{2N_c}{(q_1+q_2)^2}$$

$$W_{\mu\nu\rho}(k \pm q, k) = \frac{1}{8\pi^2} \Big\{ w_L(Q^2) q_\rho \epsilon_{\mu\nu\alpha\beta} q^\alpha k^\beta$$

$$+ w_T(Q^2) [q^2 \epsilon_{\mu\nu\rho\alpha} k^\alpha - q_\mu \epsilon_{\nu\rho\alpha\beta} q^\alpha k^\beta - q_\rho \epsilon_{\mu\nu\alpha\beta} q^\alpha k^\beta]$$

$$+ \mathcal{O}(k^2) \Big\}$$

$$w_L(Q^2)\Big|_{pQCD} = 2 w_T(Q^2)\Big|_{pQCD} = \frac{1}{Q^2}$$

proof : $\langle LVR \rangle$ is an order parameter