Measuring the strong and weak low-energy constants of QCD in a small volume?

— Mikko Laine (Bielefeld) —

General ideas: L. Giusti, C. Hoelbling, M. Lüscher, H. Wittig, Comput. Phys. Commun. 153 (2003) 31 [hep-lat/0212012]

Further work: M.L., P. Hernández, L. Giusti, P. Weisz, H. Wittig, P.H. Damgaard, K. Jansen, L. Lellouch, C. Pena, J. Wennekers [hep-lat/0211020,0212014,0312012,0402002,0407007; hep-ph/0407086; ...]

The strong interaction part of the chiral Lagrangian:

$$\mathcal{L}_E = \frac{F^2}{4}\mathrm{Tr}[\partial_\mu U \partial_\mu U^\dagger] - \frac{\Sigma}{2}\mathrm{Tr}[UM + M^\dagger U^\dagger] + \ldots,$$

$U \in$ SU(3), $M =$ diag(m_u, m_d, m_s), and F, Σ are low-energy constants.

The weak interaction Hamiltonian:

$$\mathcal{H}_w^{\chi\mathsf{PT}} = 2\sqrt{2} G_F V_{ud} V_{us}^* \{g_{27}\mathcal{O}_{27} + g_8 \mathcal{O}_8 + g_8' \mathcal{O}_8'\} + \mathrm{H.c.}\ ,$$

where G_F is the Fermi constant, V_{ij} are elements of the CKM-matrix, g_{27}, g_8 and g_8' are dimensionless low-energy constants, and

$$\begin{aligned}
\mathcal{O}_{27} &\equiv \frac{1}{4}F^4\left[(\partial_\mu U U^\dagger)_{ds}(\partial_\mu U U^\dagger)_{uu} + \frac{2}{3}(\partial_\mu U U^\dagger)_{du}(\partial_\mu U U^\dagger)_{us}\right], \\
\mathcal{O}_8 &\equiv \frac{1}{4}F^4 \sum_{k=u,d,s}(\partial_\mu U U^\dagger)_{dk}(\partial_\mu U U^\dagger)_{ks}\ , \\
\mathcal{O}_8' &\equiv F^2\Sigma(UM + M^\dagger U^\dagger)_{ds}\ .
\end{aligned}$$

Reproducing $g_8/g_{27} \gg 1$ from lattice QCD is a long-standing challenge.

Bernard, Draper, Soni, Politzer, Wise 1985

What's new (i): To match for g_{27}, g_8, we can carry out simulations in a "small" volume, $2\pi/M_{\text{glueball}} \ll L \ll 2\pi/M_\pi$, with M_π physically light.

Why? χPT applies as soon as the momentum scales are below the QCD scale, e.g. $L \sim 2.0$ fm. The usual counting rules for χPT just need to be modified.

Gasser, Leutwyler 1987;
Neuberger 1988;
Hasenfratz, Leutwyler 1990;
Hansen, Leutwyler 1990, 1991

On a finite periodic lattice ($V = L^3 T,\ L_0 \equiv T,\ L_i \equiv L$),

$$p_\mu = \frac{2\pi}{L_\mu} n_\mu, \quad n_\mu \in \mathbb{Z} \ .$$

Writing $U = \exp(2i\xi/F)$, the propagator is

$$\langle\, \xi_p\, \xi_{-p}\,\rangle \sim \frac{1}{p^2 + M_\pi^2} \ .$$

For $L^2 \ll (2\pi/M_\pi)^2$, the zero-modes $p = 0$ become dominant and have to be summed to all orders. This is the so-called ϵ-regime of χPT.

In the ϵ-regime, one can write

$$U = \exp\left(i\frac{2\xi}{F}\right)\, U_0, \quad \int_x \xi(x) = 0\ .$$

The non-zero momentum modes are treated perturbatively as in usual χPT.

Left over are non-perturbative zero-mode integrals. Going from the θ-vacuum to a fixed topology ν, they are of the type

$$\begin{aligned} \frac{1}{2}\left\langle \mathrm{Tr}\left[U_0 + U_0^\dagger\right]\right\rangle_{U_0\in U(N_\mathrm{f})} &= \frac{\mathrm{d}}{\mathrm{d}\mu}\ln\det[I_{\nu+j-i}(\mu)]|_{i,j=1,\ldots,N_\mathrm{f}}\ , \\ \mu &\equiv m\Sigma V \sim 1\ . \end{aligned}$$

The great strength of the ϵ-regime is that NLO corrections can be computed without introducing any new low-energy constants, unlike in the usual "p-regime" where $L \gtrsim 2\pi/M_\pi$!

What's new (ii): Start with $m_c = m_u = m_d = m_s$, so that the theory has an exact $\mathrm{SU}(4)_\mathrm{L} \times \mathrm{SU}(4)_\mathrm{R}$ symmetry in the chiral limit.

Why? To disentangle the role of the charm quark, i.e., tell apart effects due to the mass scale m_c ($\sim$ 1 GeV) from soft gluon exchange ($\sim$ 250 MeV).

Furthermore, group theory becomes simpler: "the GIM cancellation takes place", i.e. no penguin contractions are needed, and there are only two operators rather than three $(27 \leftrightarrow 80 \equiv +; 8 \leftrightarrow 20 \equiv -)$.

$$
\begin{aligned}
H_\mathrm{w}^\mathsf{QCD} &= \sqrt{2} G_\mathrm{F} V_{ud} V_{us}^* \sum_{\sigma=\pm} k_1^\sigma O_1^\sigma \ , \\
O_1^\pm &= \{(\bar{s}\gamma_\mu P_\mathrm{L} u)(\bar{u}\gamma_\mu P_\mathrm{L} d) \pm (\bar{s}\gamma_\mu P_\mathrm{L} d)(\bar{u}\gamma_\mu P_\mathrm{L} u)\} - (u \to c) \ ,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_\mathrm{w}^{\chi\mathsf{PT}} &= \sqrt{2} G_\mathrm{F} V_{ud} V_{us}^* \sum_{\sigma=\pm} g_1^\sigma \mathcal{O}_1^\sigma \ , \\
\mathcal{O}_1^\pm &= \frac{F^4}{4} \{(U\partial_\mu U^\dagger)_{us}(U\partial_\mu U^\dagger)_{du} \pm (U\partial_\mu U^\dagger)_{ds}(U\partial_\mu U^\dagger)_{uu}\} \\
&\qquad - (u \to c) \ .
\end{aligned}
$$

To match for $g_1^{\pm}$, define the correlators

$$[\mathcal{C}_1^{\pm}(x_0,y_0)]^{ab} \quad = \quad \int \mathrm{d}^3x \int \mathrm{d}^3y \left\langle \mathcal{J}_0^a(x)[\mathcal{O}_1^{\pm}(0)]\mathcal{J}_0^b(y) \right\rangle \ ,$$

where $\mathcal{J}_\mu^a$ is the left-handed current, $(\mathcal{J}_\mu^a)^{\mathsf{QCD}} = \bar{\psi}\gamma_\mu P_{\mathrm{L}} T^a \psi$.

On the chiral theory side, we obtain in the ϵ-regime (for $x_0, y_0 \neq 0$)

$$H(x_0,y_0) \equiv \frac{\mathcal{C}_1^-(x_0,y_0)}{\mathcal{C}_1^+(x_0,y_0)} = 1 - \frac{4}{F^2T^2}\rho^3 \left\{ \beta_1(\rho)\rho^{-3/2} - k_{00}(\rho) \right\} ,$$

where $\rho = T/L$, and β_1, k_{00} are certain known shape coefficients.

$$\Longrightarrow \qquad \frac{g_1^-}{g_1^+} = \frac{k_1^-}{k_1^+} \frac{(C_1^-)^{\mathsf{QCD}}(x_0,y_0)}{(C_1^+)^{\mathsf{QCD}}(x_0,y_0)} \frac{1}{H(x_0,y_0)} \ .$$

What's new (iii): Use Ginsparg-Wilson fermions (the Neuberger Dirac operator).

The renormalisation and mixings are like in the continuum: no power-divergent subtractions, and measurements can be easily carried out at a fixed topology.

What's new (iv): "Low-mode averaging".

Low eigenvalues ($|\lambda_1| \sim (\Sigma V)^{-1}$) of the massless Dirac operator tend to make the signal noisy (or "spikey"), if $m \lesssim (\Sigma V)^{-1}$:

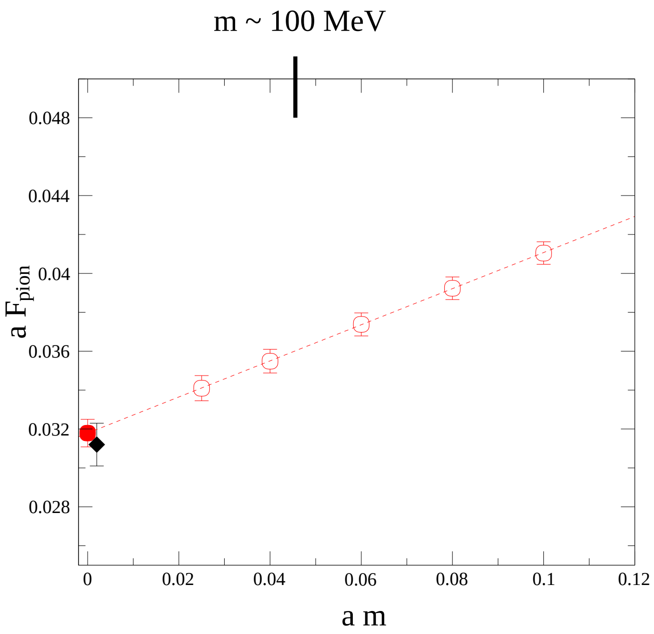
$$\langle \psi(x) \bar{\psi}(y) \rangle = \sum_n \frac{v_n(x) v_n^\dagger(y)}{\lambda_n + m} \ .$$

To avoid these fluctuations, a certain number of low modes, n_{low}, are treated separately: we take the volume average of their contributions to the correlators.

See also: Edwards 2002;
DeGrand, Schaefer 2003

Result for F from $\int \mathrm{d}^3 x \langle \mathcal{J}_0^a(x) \mathcal{J}_0^b(0) \rangle$

	β	T/L	L[fm]	am	configs.
ϵ-regime	6.0	16/16	1.49	0.005...0.010	203
p-regime	6.0	24/16	1.49	0.025...0.100	113

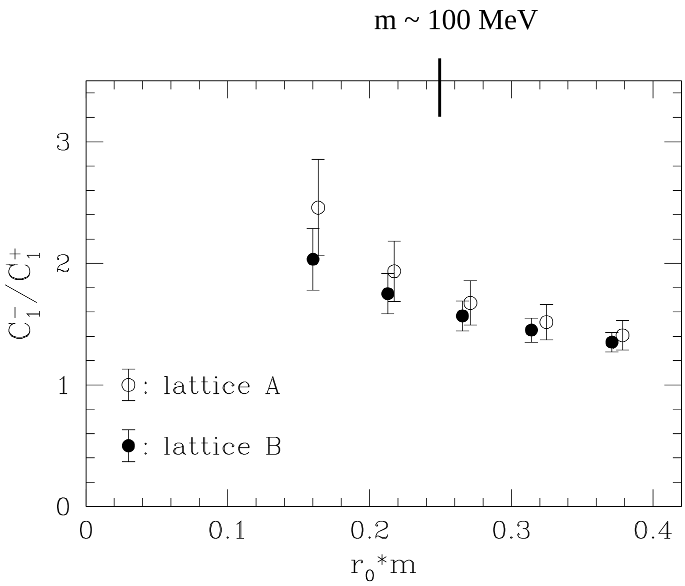


$\Rightarrow$ Numerical signal is good, and the determinations in the p-regime (large m followed by chiral extrapolation) and ϵ-regime (directly at small m) agree.

In physical units, at this V and a, the quenched $F \sim 103(4)$ MeV.

Result for g_1^-/g_1^+?

	β	T/L	L[fm]	am	configs.
A	6.0	40/12	1.12	0.030...0.070	751
B	5.8485	30/12	1.49	0.040...0.092	638

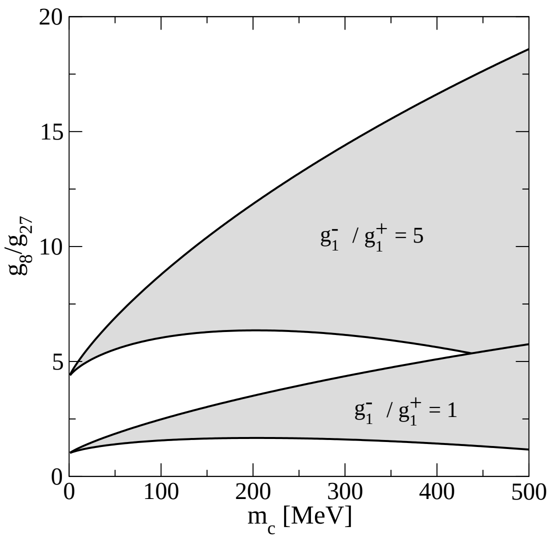


Simulating directly in the ϵ-regime here requires a higher n_{low} in the low-mode averaging procedure $\Rightarrow$ in progress. On the other hand, for a fixed volume, small m, and NLO in ChPT, the ratio could also be fit to a Taylor series in m!

$\Rightarrow$ One gets an enhancement, but this is at least partly cancelled by the Goldstone-mode factor $H(x_0, y_0)$ (≈ 2.3 for lattice B). Further systematics needed to see whether there is an effect in the SU(4) limit already.

How does the system behave for $m_c > m_u = m_d = m_s$? In ChPT:

$$\frac{g_8}{g_{27}} = \frac{1}{6}\left\{\left[1 + \frac{15 m_c \Sigma}{32\pi^2 F^4}\ln\frac{\Lambda_1}{m_c}\right] + \frac{g_-}{g_+}\left[5 + \frac{15 m_c \Sigma}{32\pi^2 F^4}\ln\frac{\Lambda_2}{m_c}\right]\right\}.$$



Here the higher order LECs enter, and there is no firm prediction from χPT. Lattice needed: domain wall fermion measurements from RBC collaboration suggest (?) m_c has little effect, so maybe SU(4) is all we need!

Conclusions

Philosophy: in order to understand from which scale the enhancement comes from, let us try to factorise the problem into parts and inspect one physics scale at a time, with controlled systematic errors, rather than everything at once.

Conceptual points: (i) Usually $V \to \infty$, $m \to 0$, here $m \to 0, V \to \infty$.
(ii) Start with the SU(4) degenerate limit.

Technical points: (iii) Use Ginsparg-Wilson fermions.
(iv) Implement low-mode averaging.

Initial tests suggest that a numerical signal can be obtained this way.

Challenges: (i) Three-point functions at smaller m.
(ii) $L \gtrsim 2.0$ fm for ChPT convergence in the ϵ-regime.
(iii) Check the effect of $m_c > m_u, m_d, m_s$.
(iv) Unquenching...