SOME OPTIMIZATION PROBLEMS IN MASS TRANSPORTATION THEORY

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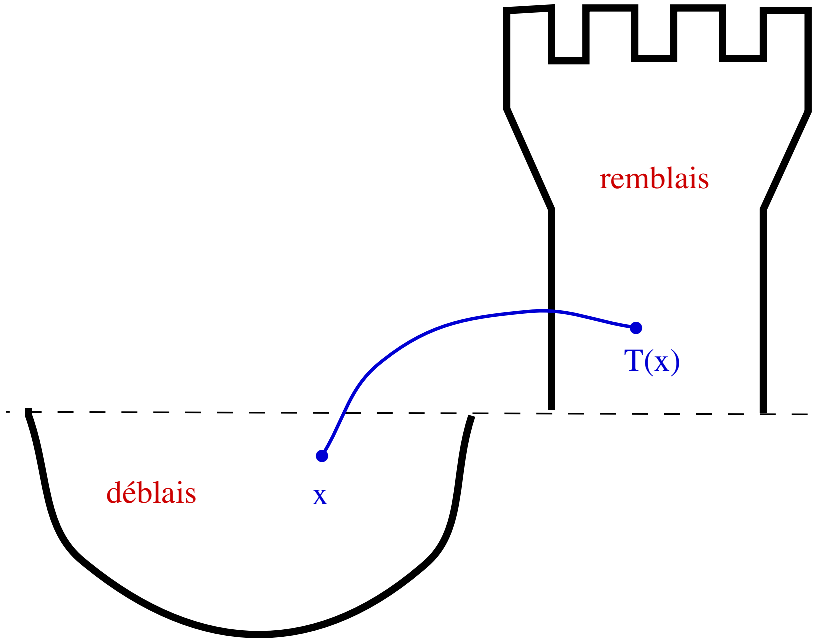
Mass transportation theory goes back to Gaspard Monge (1781) when he presented a model in a paper on *Académie de Sciences de Paris*



The elementary work to move a particle x into $T(x)$ is given by $|x - T(x)|$, so that the total work is

$$\int_{\text{déblais}} |x - T(x)|\, dx\ .$$

A map T is called admissible transport map if it maps “déblais” into “remblais”.

The Monge problem is then

$$\min\left\{\int_{\text{déblais}} |x - T(x)|\,dx \ :\ T \text{ admiss.}\right\}.$$

It is convenient to consider the Monge problem in the framework of metric spaces:

- (X, d) is a metric space;
- f^+, f^- are two probabilities on X ($f^+ =$ "déblais", $f^- =$ "remblais");
- T is an admissible transport map if $T^\# f^+ = f^-$.

The Monge problem is then

$$\min\left\{\int_X d\big(x, T(x)\big)\,dx \ :\ T\text{admiss.}\right\}.$$

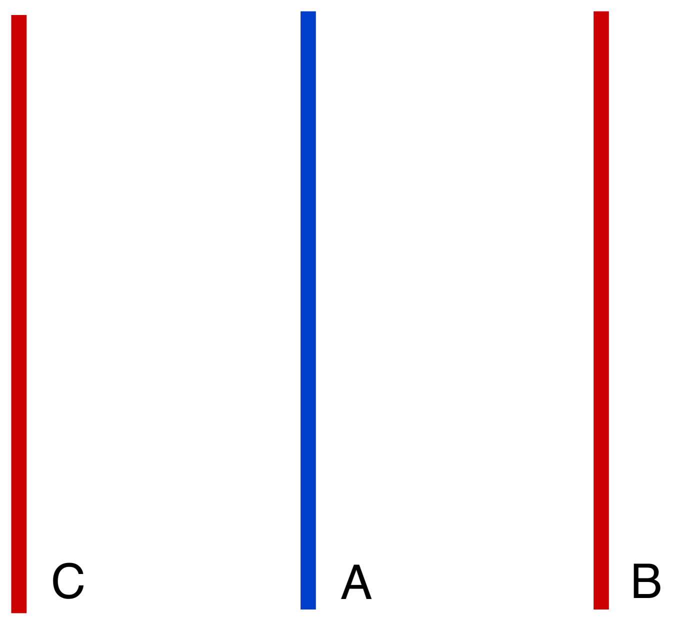
In general the problem above does not admit a solution, when the measures f^+ and f^- are singular, since the class of admissible transport maps can be empty.

Example Take $f^+ = \delta_A$ and $f^- = \frac{1}{2}\delta_B + \frac{1}{2}\delta_C$; it is clear that no map T transports f^+ into f^- so the Monge formulation above is in this case meaningless.

Example Take the measures in $\mathbf{R}^2$, still singular but nonatomic

$$f^+ = \mathcal{H}^1 \lfloor A \quad \text{and} \quad f^- = \frac{1}{2}\mathcal{H}^1 \lfloor B + \frac{1}{2}\mathcal{H}^1 \lfloor C$$

where A, B, C are the segments below.



In this case the class of admissible transport maps is nonempty but the minimum in the Monge problem is not attained.

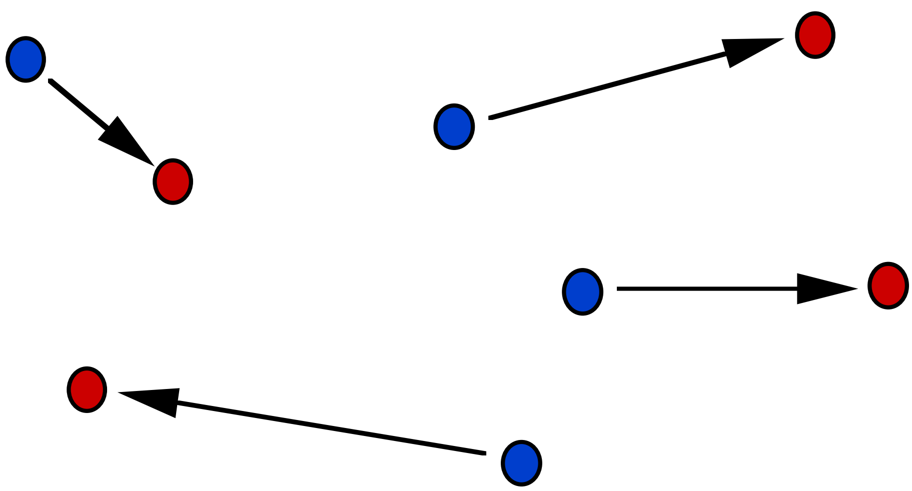
Example (book shifting) Consider in $\mathbf{R}$ the measures $f^+ = 1_{[0,a]}\mathcal{L}^1$ and $f^- = 1_{[b,a+b]}\mathcal{L}^1$. Then the two maps

$$T_1(x) = b + x \qquad \text{translation}$$

$$T_2(x) = a + b - x \qquad \text{reflection}$$

are both optimal; there are actually infinitely many optimal transport maps.

Example Take $f^+ = \sum_{i=1}^N \delta_{p_i}$ and $f^- = \sum_{i=1}^N \delta_{n_i}$. Then the optimal Monge cost is given by the minimal connection of the p_i with the n_i.



Relaxed formulation (due to Kantorovich): consider measures γ on $X \times X$

- γ is an admissible transport plan if $\pi_1^{\#}\gamma = f^+$ and $\pi_2^{\#}\gamma = f^-$.

Monge-Kantorovich problem:

$$\min\Big\{\int_{X\times X} d(x,y)\,d\gamma(x,y)\ :\ \gamma \text{ admiss.}\Big\}.$$

Wasserstein distance of exponent p: replace the cost by $\Big(\int_{X\times X} d^p(x,y)\,d\gamma(x,y)\Big)^{1/p}$.

Theorem *There exists an optimal transport plan γ_{opt}; in the Euclidean case γ_{opt} is actually a transport map T_{opt} whenever f^+ and f^- are in L^1.*

We denote by $MK(f^+, f^-, d)$ the minimum value in the Monge-Kantorovich problem. We present now some optimization problems related to mass transportation theory.

Shape Optimization Problems

Given a force field f in $\mathbf{R}^n$ find the elastic body Ω whose “resistance” to f is maximal

Constraints:

- given volume, $|\Omega| = m$
- possible “design region” D given, $\Omega \subset D$
- possible support region Σ given, Dirichlet region

Optimization criterion:

- elastic compliance.

Then the shape optimization problem is

$$\min \Big\{ \mathcal{C}(\Omega) \ : \ \Omega \subset D, \ |\Omega| = m \Big\}.$$

where $\mathcal{C}(\Omega)$ denotes the compliance of the domain Ω.

More precisely, for every admissible domain Ω we consider the energy

$$\mathcal{E}(\Omega) = \inf_{u=0 \text{ on } \Sigma} \left\{ \int_\Omega j(Du)\, dx - \langle f, u \rangle \right\}$$

and the compliance, which reduces to the work of external forces

$$\mathcal{C}(\Omega) = -\mathcal{E}(\Omega) = \frac{1}{2}\langle f, u_\Omega \rangle.$$

being u_Ω the displacement of minimal energy in Ω. In linear elasticity, if $z^* = sym(z)$ and α, β are the Lamé constants,

$$j(z) = \beta |z^*|^2 + \frac{\alpha}{2} |tr z^*|^2.$$

A similar problem can be considered in the scalar case (optimal conductor), where f is a scalar function (the heat sources density) and

$$j(z) = \frac{1}{2}|z|^2.$$

The shape optimization problem above has in general no solution; in fact minimizing sequences may develop wild oscillations which give raise to limit configurations that are not in a form of a domain.

Therefore, it is convenient to consider the analogous problem where domains are replaced by densities μ of material.

Constraints:

- given mass, $\int d\mu = m$
- given design region D, i.e. $\operatorname{spt}\mu \subset D$
- Dirichlet support region Σ given.

Optimization criterium: elastic compliance

Energy $\mathcal{E}(\mu)$ defined analogously as above, and compliance $\mathcal{C}(\mu) = -\mathcal{E}(\mu)$.

There is a strong link between the mass optimization problem and the Monge-Kantorovich mass transfer problem. This is described below for simplicity in the scalar case, for a convex design region D, and for $\Sigma = \emptyset$. A general theory can be found in [BB2001]

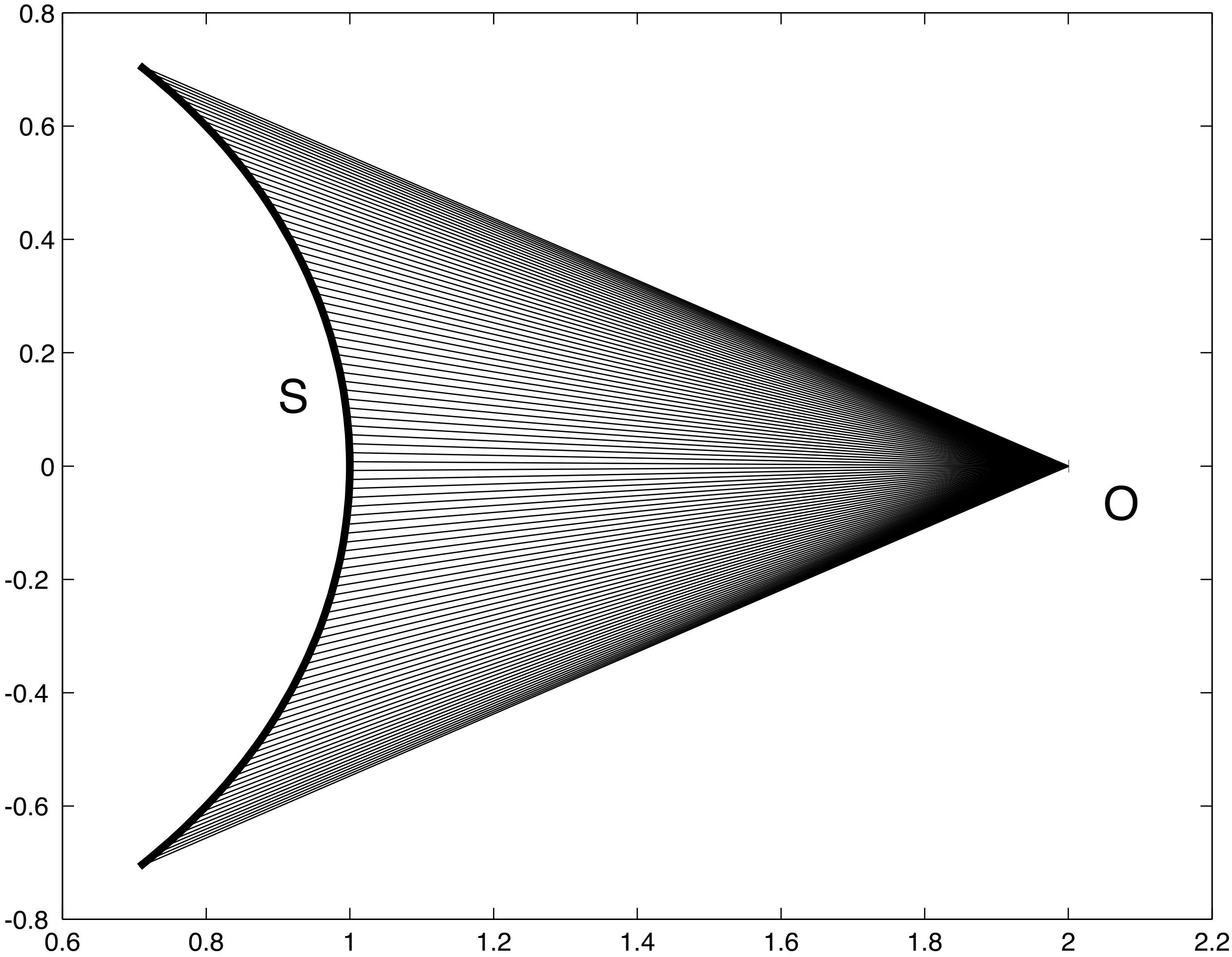
Writing $f = f^+ - f^-$ and taking the optimal transportation plan γ in the Monge-Kantorovich problem, we can obtain the optimal density μ through the formula

$$\mu(A) = \int \mathcal{H}^1(B \cap [x, y])\, d\gamma(x, y).$$

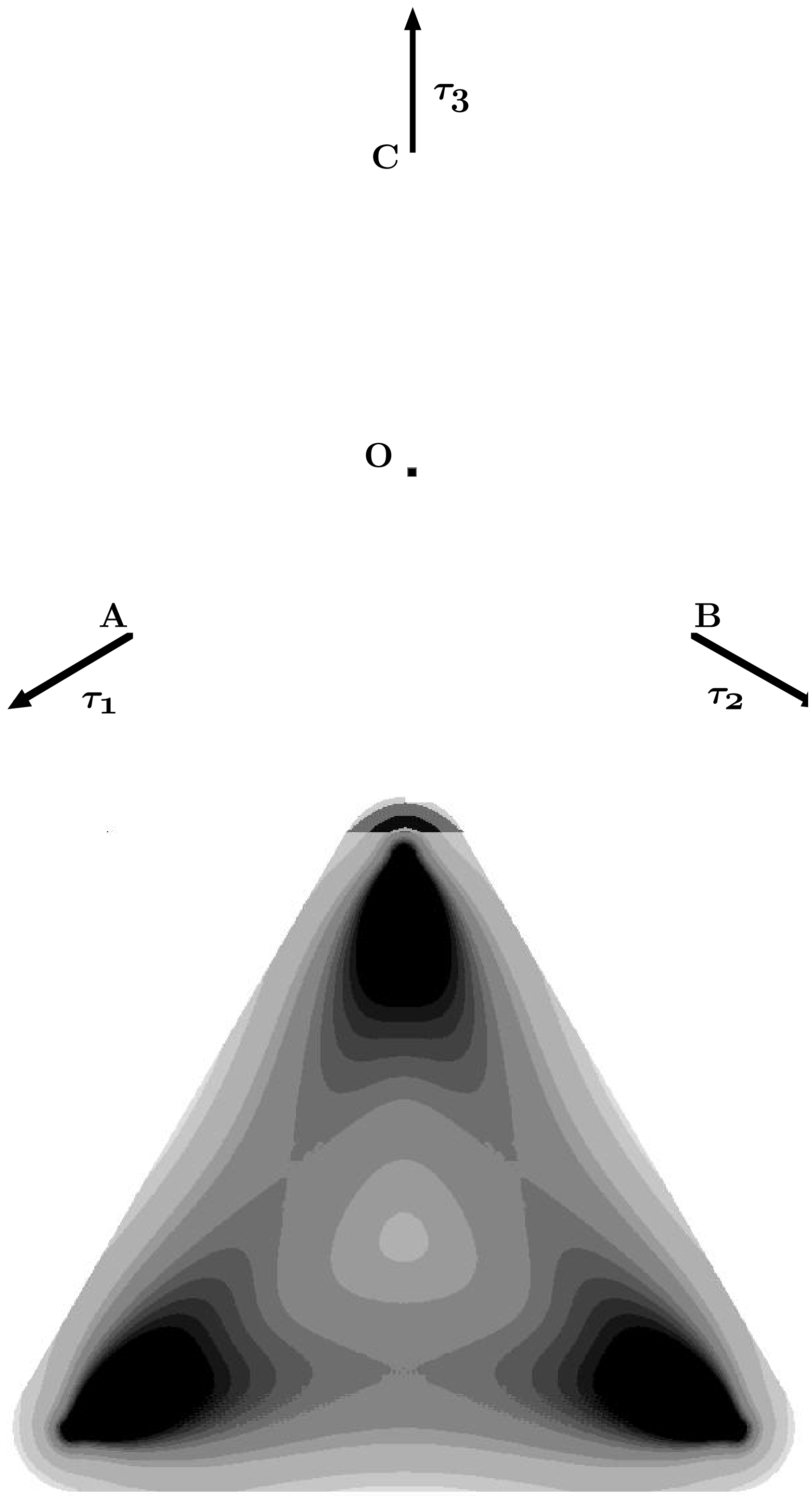
Moreover the Monge-Kantorovich PDE holds:

$$\begin{cases} -\operatorname{div}\big(\mu(x) D_\mu u\big) = f \quad \text{in } \mathbf{R}^n \setminus \Sigma \\ u \text{ is 1-Lipschitz on } D, \quad u = 0 \text{ on } \Sigma \\ |D_\mu u| = 1 \ \mu\text{-a.e. on } \mathbf{R}^n, \quad \mu(\Sigma) = 0. \end{cases}$$

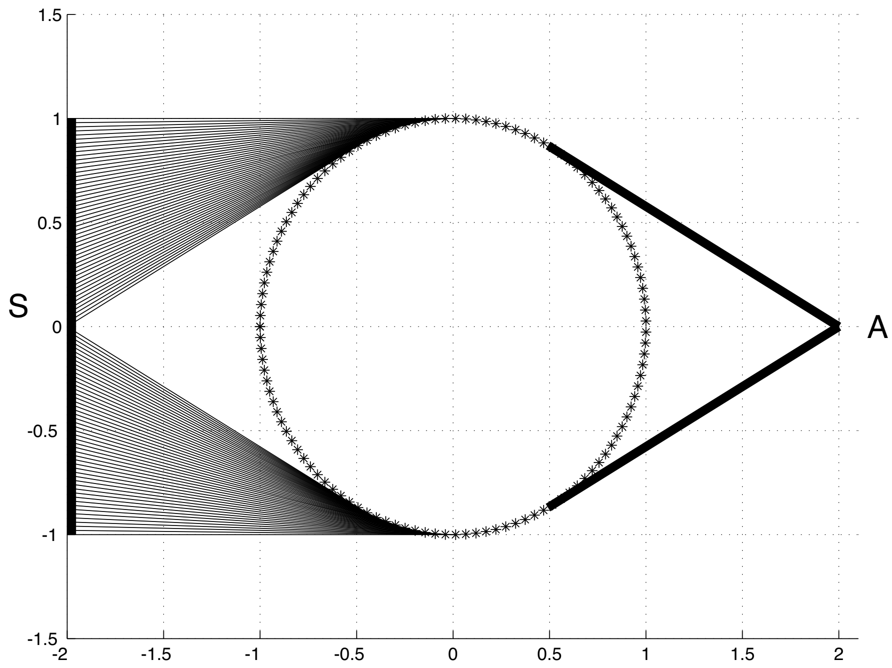
Here are some cases where the optimal mass distribution can be computed by using the Monge-Kantorovich equation (see Bouchitté-Buttazzo [JEMS '01]).



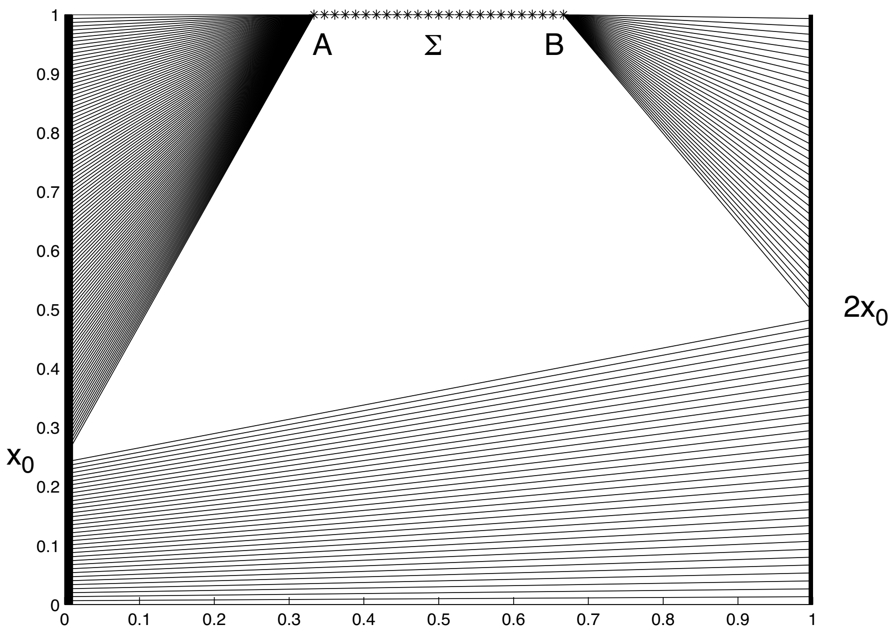
Optimal distribution of a conductor for heat sources $f = \mathcal{H}^1 \lfloor S - L\delta_O$.



Optimal distribution of an elastic material when the forces are as above.



Optimal distribution of a conductor, with an obstacle, for heat sources $f = \mathcal{H}^1 \lfloor S - 2\delta_A$.



Optimal distribution of a conductor for heat sources $f = 2\mathcal{H}^1 \lfloor S_0 - \mathcal{H}^1 \lfloor S_1$ and Dirichlet region Σ.

Problem of Optimal Networks

We consider the following model for the optimal planning of an urban transportation network (Buttazzo-Brancolini 2003).

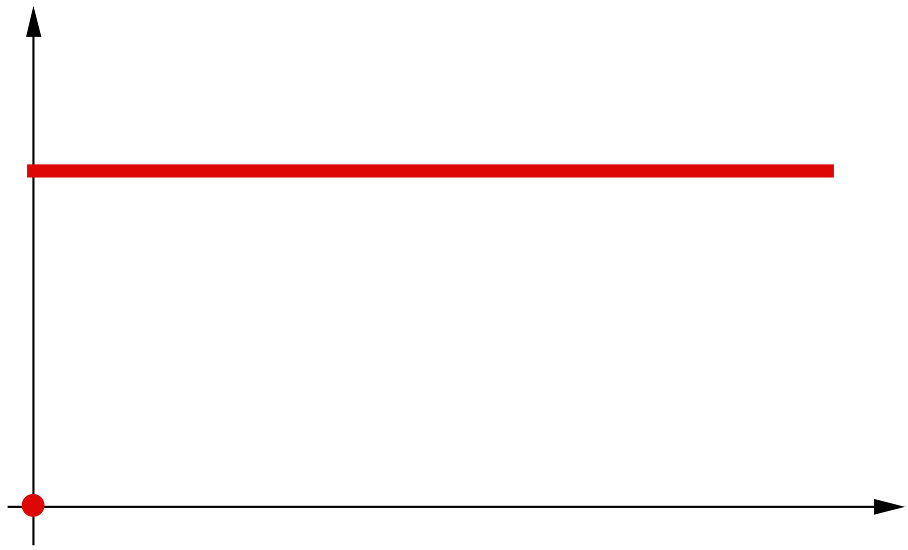
- Ω *the geographical region or urban area*
  a compact regular domain of $\mathbf{R}^N$
- f^+ *the density of residents*
  a probability measure on Ω
- f^- *the density of working places*
  a probability measure on Ω
- Σ *the transportation network*
  a closed connected 1-dimensional subset of Ω, the unknown.

The goal is to introduce a cost functional $F(\Sigma)$ and to minimize it on a class of admissible choices.

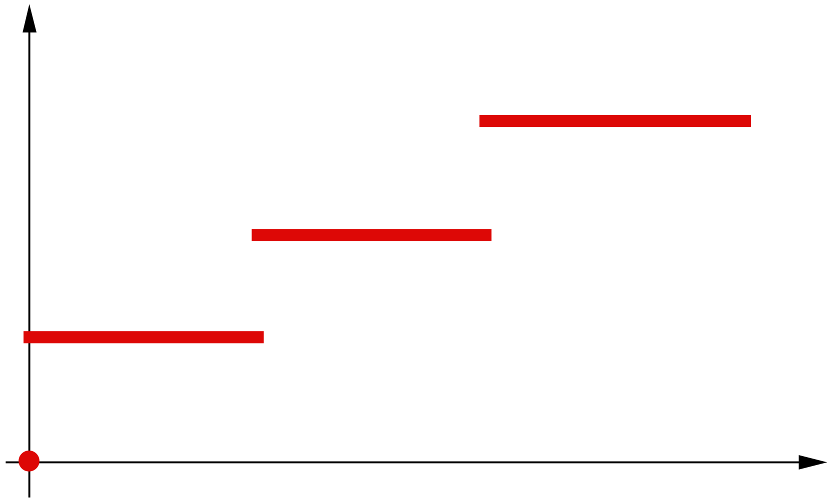
Consider two functions:

$A : \mathbf{R}^+ \to \mathbf{R}^+$ continuous and increasing; $A(t)$ represents the cost to cover a length t by one's own means (walking, time consumption, car fuel, ...);

$B : \mathbf{R}^+ \to \mathbf{R}^+$ l.s.c. and increasing; $B(t)$ represents the cost to cover a length t by using the transportation network (ticket, time consumption, ...).



Small town policy: only one ticket price



Large town policy: several ticket prices

We define

$$d_\Sigma(x,y) = \inf\Big\{A\big(\mathcal{H}^1(\Gamma\setminus\Sigma)\big) + B\big(\mathcal{H}^1(\Gamma\cap\Sigma)\big)\ :\ \Gamma \text{ connects } x \text{ to } y\Big\}.$$

The cost of the network Σ is defined via the Monge-Kantorovich functional:

$$F(\Sigma) = MK(f^+, f^-, d_\Sigma)$$

and the admissible Σ are simply the closed connected sets with $\mathcal{H}^1(\Sigma) \leq L$.

There is a strong link between the convergences of distances and of the associated Hausdorff measures (Buttazzo-Schweizer 2005).

Therefore the optimization problem is

$$\min\Big\{F(\Sigma)\ :\ \Sigma \text{ cl. conn.}, \mathcal{H}^1(\Sigma)\le L\Big\}.$$

Theorem *There exists an optimal network Σ_{opt} for the optimization problem above.*

In the special case $A(t) = t$ and $B \equiv 0$ (communist model) some necessary conditions of optimality on Σ_{opt} have been derived (Buttazzo-Stepanov 2003). For instance:

- no closed loops;
- at most triple point junctions;
- $120°$ at triple junctions;
- no triple junctions for small L;
- asymptotic behavior of Σ_{opt} as $L \to +\infty$ (Mosconi-Tilli 2003);
- regularity of Σ_{opt} is an open problem.

Problem of Optimal Pricing Policies

With the notation above, we consider the measures f^+, f^- fixed, as well as the transportation network Σ. The unknown is the pricing policy the manager of the network has to choose through the l.s.c. monotone increasing function B. The goal is to maximize the total income, a functional $F(B)$, which can be suitably defined (Buttazzo-Pratelli-Stepanov, in preparation) by means of the Monge-Kantorovich transport plans.

Of course, a too low ticket price policy will not be optimal, but also a too high ticket price policy will push customers to use their own transportation means, decreasing the total income of the company.

The function B can be seen as a control variable and the corresponding transport plan as a state variable, so that the optimization problem we consider:

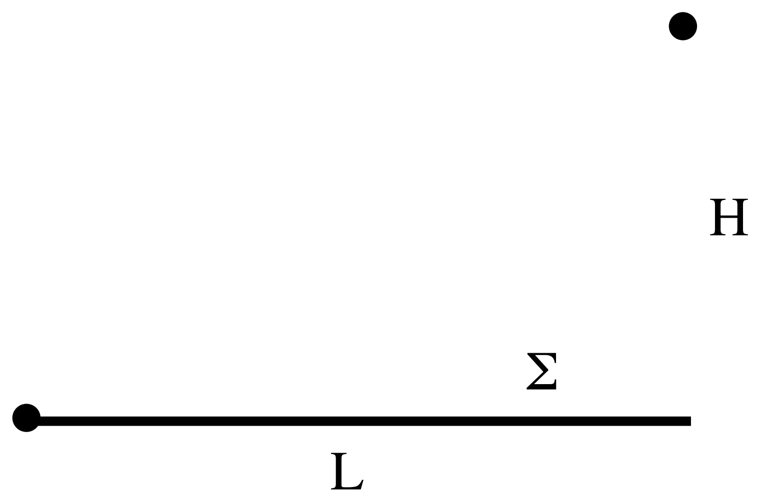
$$\min\left\{F(B)\ :\ B \text{ l.s.c. increasing, } B(0)=0\right\}$$

can be seen as an optimal control problem.

Theorem *There exists an optimal pricing policy B_{opt} solving the maximal income problem above.*

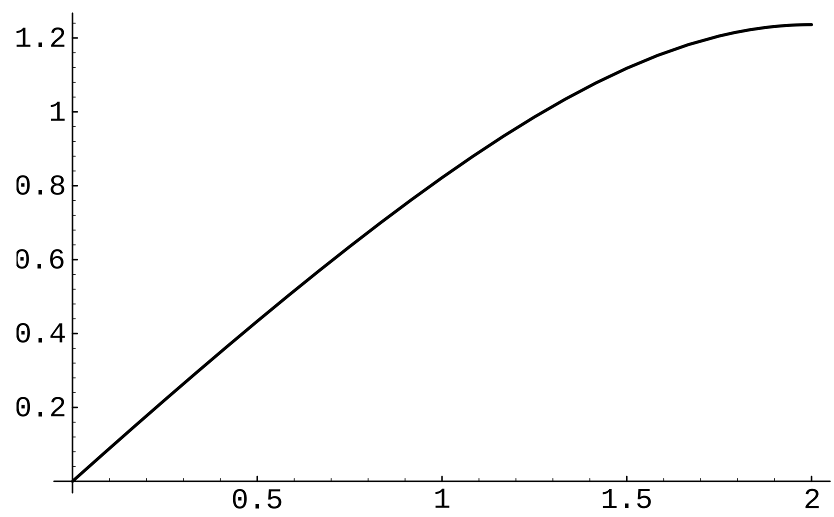
Also in this case some necessary conditions of optimality can be obtained. In particular, the function B_{opt} turns out to be continuous, and its Lipschitz constant can be bounded by the one of A (the function measuring the own means cost).

Here is the case of a service pole at the origin, with a residence pole at (L, H), with a network Σ. We take $A(t) = t$.



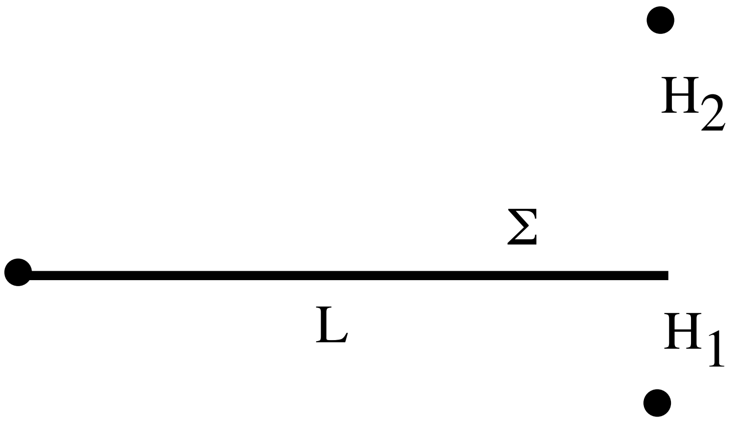
The optimal pricing policy $B(t)$ is given by

$$B(t) = (H^2 + L^2)^{1/2} - (H^2 + (L - t)^2)^{1/2}.$$



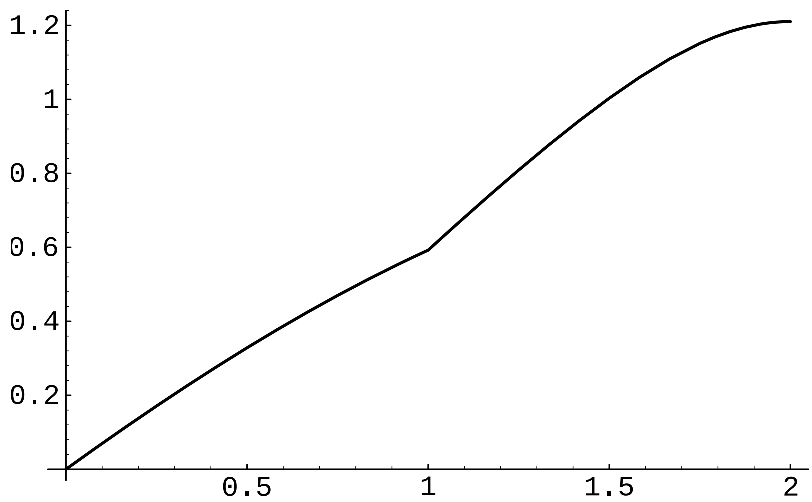
The case $L = 2$ and $H = 1$.

Here is another case, with a single service pole at the origin, with two residence poles at (L, H_1) and (L, H_2), with a network Σ.



The optimal pricing policy $B(t)$ is then

$$B(t) = \begin{cases} B_2(t) & \text{in } [0, T] \\ B_2(T) - B_1(T) + B_1(t) & \text{in } [T, L] \end{cases}.$$



The case $L = 2$, $H_1 = 0.5$, $H_2 = 2$.

Problem of Optimal City Structures

We consider the following model for the optimal planning of an urban area (Buttazzo-Santambrogio 2003).

- Ω *the geographical region or urban area*
  a compact regular domain of $\mathbf{R}^N$
- f^+ *the density of residents*
  a probability measure on Ω
- f^- *the density of services*
  a probability measure on Ω.

Here the distance d in Ω is fixed (for simplicity the Euclidean one) while the unknowns are f^+ and f^- that have to be determined in an optimal way taking into account the following facts:

- there is a transportation cost for moving from the residential areas to the services poles;
- people desire not to live in areas where the density of population is too high;
- services need to be concentrated as much as possible, in order to increase efficiency and decrease management costs.

The transportation cost will be described through a Monge-Kantorovich mass transportation model; it is indeed given by a p-Wasserstein distance ($p \geq 1$) $W_p(f^+, f^-)$, being $p = 1$ the classical Monge case.

The total unhappiness of citizens due to high density of population will be described by a penalization functional, of the form

$$H(f^+) = \begin{cases} \int_\Omega h(u)\,dx & \text{if } f^+ = u\,dx \\ +\infty & \text{otherwise,} \end{cases}$$

where h is assumed convex and superlinear (i.e. $h(t)/t \to +\infty$ as $t \to +\infty$). The increasing and diverging function $h(t)/t$ then represents the unhappiness to live in an area with population density t.

Finally, there is a third term $G(f^-)$ which penalizes sparse services. We force f^- to be a sum of Dirac masses and we consider $G(f^-)$ a functional defined on measures, of the form studied by Bouchitté-Buttazzo in 1990:

$$G(f^-) = \begin{cases} \sum_n g(a_n) & \text{if } f^- = \sum_n a_n \delta_{x_n} \\ +\infty & \text{otherwise,} \end{cases}$$

where g is concave and with infinite slope at the origin. Every single term $g(a_n)$ in the sum represents here the cost for building and managing a service pole of dimension a_n, located at the point $x_n \in \Omega$.

We have then the optimization problem

$$\min\Big\{W_p(f^+,f^-)+H(f^+)+G(f^-)\ :\ f^+,f^-\text{ probabilities on }\Omega\Big\}.$$

Theorem *There exists an optimal pair* (f^+,f^-) *solving the problem above.*

Also in this case we obtain some necessary conditions of optimality. In particular, if Ω is sufficiently large, the optimal structure of the city consists of a finite number of disjoint subcities: circular residential areas with a service pole at the center.

Problem of Optimal Riemannian Metrics

Here the domain Ω and the probabilities f^+ and f^- are given, whereas the distance d is supposed to be conformally flat, that is generated by a coefficient $a(x)$ through the formula

$$d_a(x,y) = \inf\Big\{ \int_0^1 a\big(\gamma(t)\big)|\gamma'(t)|\,dt \ : \\ \gamma \in \mathrm{Lip}(]0,1[;\Omega),\ \gamma(0)=x,\ \gamma(1)=y \Big\}.$$

We can then consider the cost functional

$$F(a) = MK(f^+, f^-, d_a).$$

The goal is to prevent as much as possible the transportation of f^+ onto f^- by maximizing the cost $F(a)$ among the admissible coefficients $a(x)$. Of course, increasing $a(x)$ would increase the values of the distance d_a

and so the value of the cost $F(a)$. The fact is that the class of admissible controls is taken as

$$\mathcal{A} = \Big\{ a(x) \text{ Borel measurable } : \\ \alpha \le a(x) \le \beta,\ \int_\Omega a(x)\,dx \le m \Big\}.$$

In the case when $f^+ = \delta_x$ and $f^- = \delta_y$ are Dirac masses concentrated on two fixed points $x, y \in \Omega$, the problem of maximizing $F(a)$ is nothing else than that of proving the existence of a conformally flat Euclidean metric whose geodesics joining x and y are as long as possible.

This problem has several natural motivations; indeed, in many concrete examples, one can be interested in making as difficult as possible the communication between some masses f^+ and f^-. For instance, it is easy to imagine that this situation may arise in economics, or in medicine, or simply in traffic planning, each time the connection between two *"enemies"* is undesired. Of course, the problem is made non trivial by the integral constraint $\int_\Omega a(x)\,dx \le m$, which has a physical meaning: it prescribes the quantity of material at one's disposal to solve the problem.

The analogous problem of minimizing the cost functional $F(a)$ over the class $\mathcal{A}$, which corresponds to favor the transportation of f^+ into f^-, is trivial, since

$$\inf\Big\{F(a)\ :\ a\in\mathcal{A}\Big\} = F(\alpha).$$

The existence of a solution for the maximization problem

$$\max\Big\{F(a)\ :\ a\in\mathcal{A}\Big\}$$

is a delicate matter. Indeed, maximizing sequences $\{a_n\} \subset \mathcal{A}$ could develop an oscillatory behavior producing only a relaxed solution. This phenomenon is well known; basically what happens is that the class $\mathcal{A}$ is not closed with respect to the natural convergence

$$a_n \to a \quad \Longleftrightarrow \quad d_{a_n} \to d_a \text{ uniformly}$$

and actually it can be proved that $\mathcal{A}$ is dense in the class of all geodesic distances (in particular, in all the Riemannian ones).

Nevertheless, we were able to prove the following existence result.

Theorem *The maximization problem above admits a solution in $a_{opt} \in \mathcal{A}$.*

Several questions remain open:

- Under which conditions is the optimal solution unique?

- Is the optimal solution of bang-bang type? In other words do we have $a_{opt} \in \{\alpha, \beta\}$ or intermediate values (homogenization) are more performant?

- Can we characterize explicitely the optimal coefficient a_{opt} in the case $f^+ = \delta_x$ and $f^- = \delta_y$?

Some Numerical Computations

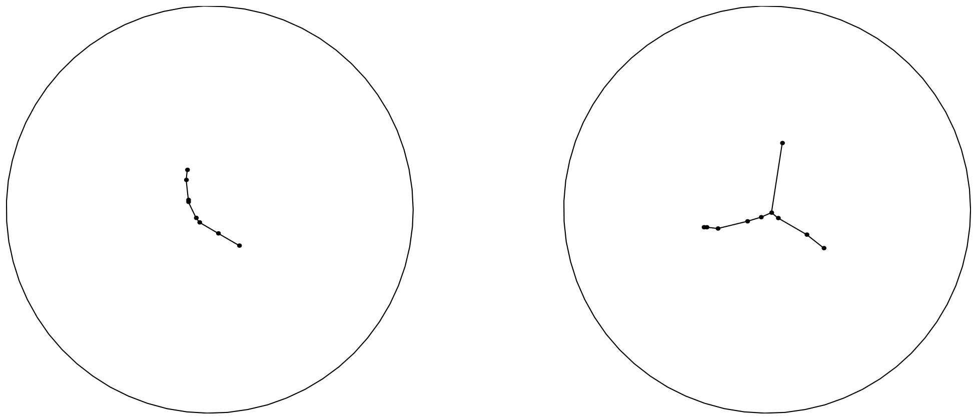
Here are some numerical computations performed (Buttazzo-Oudet-Stepanov 2002) in the simpler case of the so-called problem of optimal irrigation.

This is the optimal network Problem 1 in the case $f^- \equiv 0$, where customers only want to minimize the averaged distance from the network.

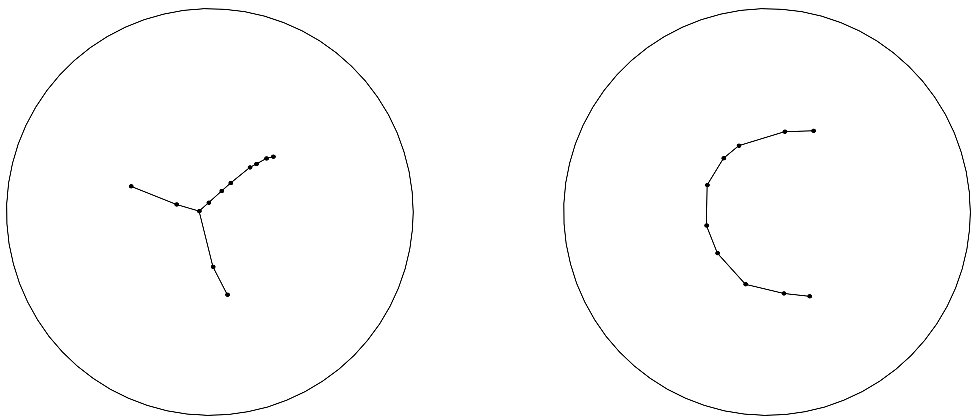
In other words, the optimization criterion becomes simply

$$F(\Sigma) = \int_\Omega \mathrm{dist}(x, \Sigma)\, df^+(x) \ .$$

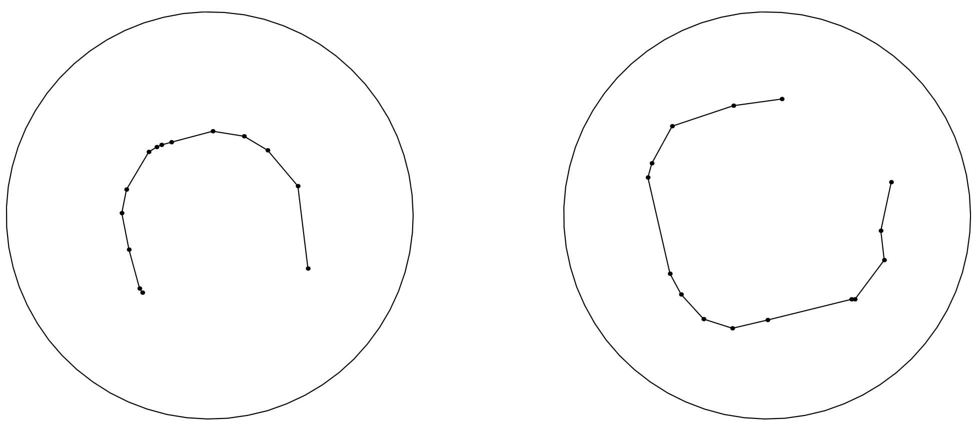
Due to the presence of many local minima the method is based on a genetic algorithm.



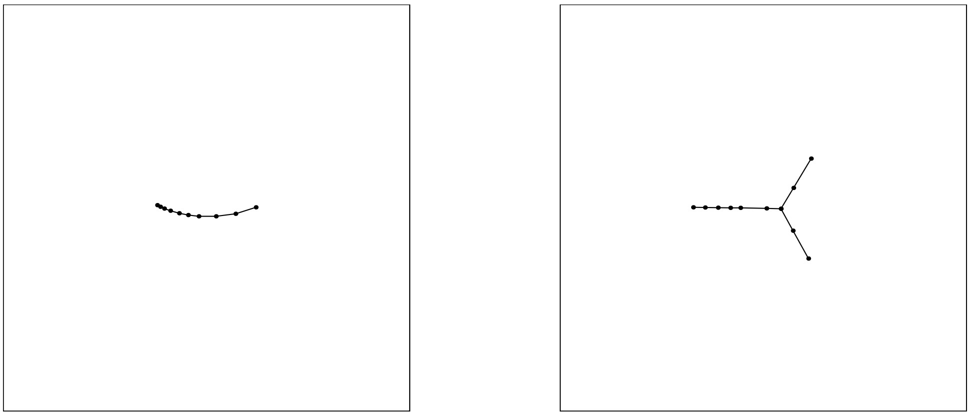
Optimal sets of length 0.5 and 1 in a unit disk



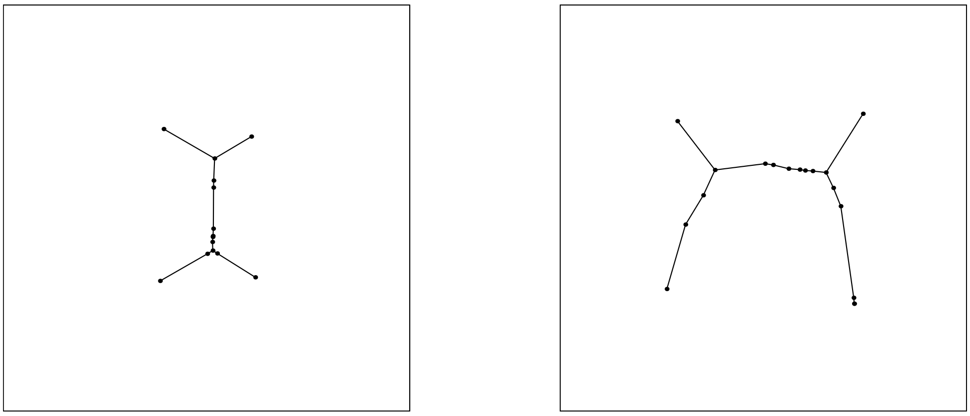
Optimal sets of length 1.25 and 1.5 in a unit disk



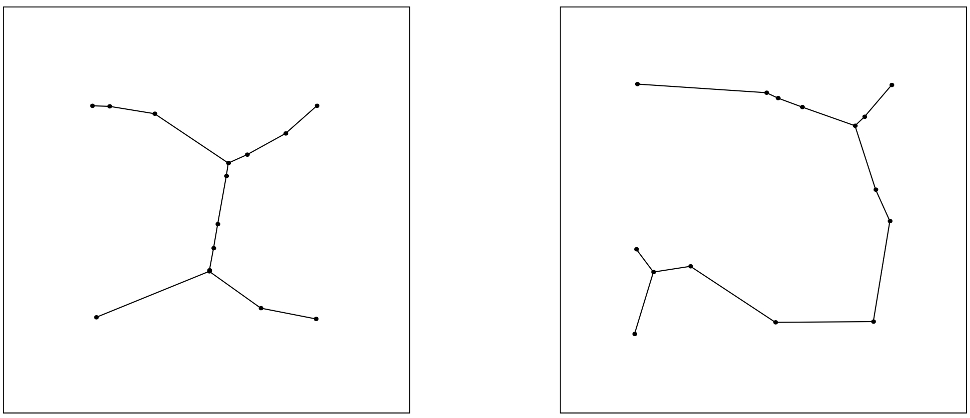
Optimal sets of length 2 and 3 in a unit disk



Optimal sets of length 0.5 and 1 in a unit square

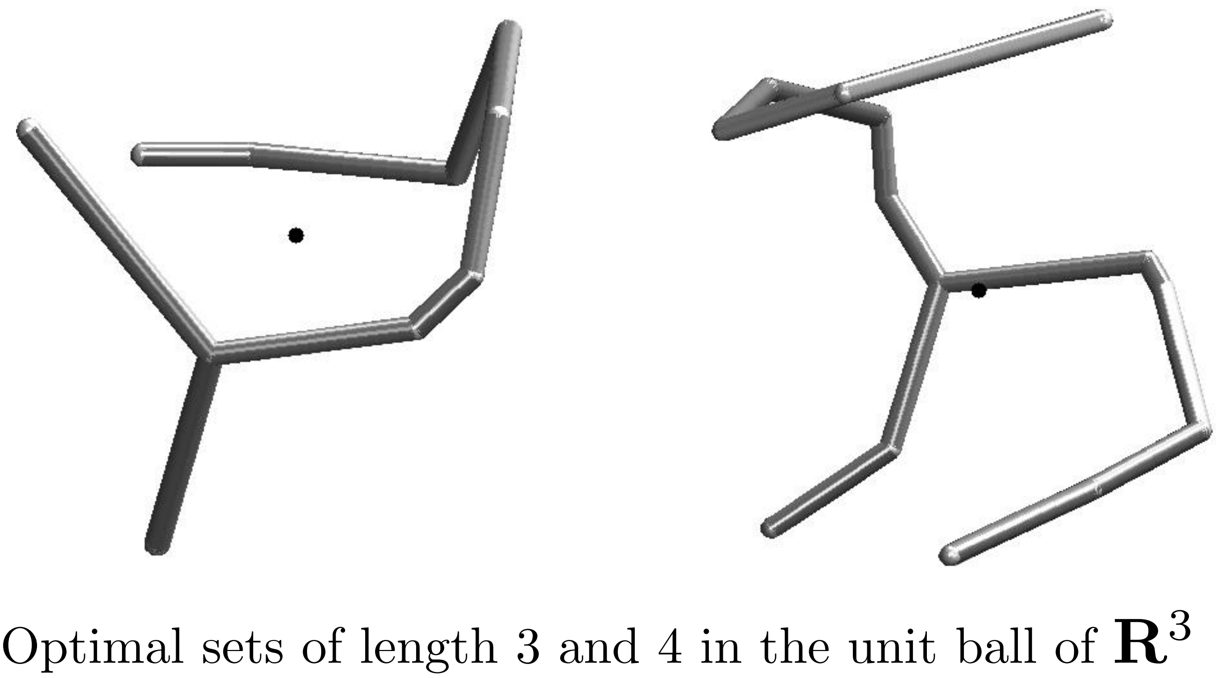


Optimal sets of length 1.5 and 2.5 in a unit square



Optimal sets of length 3 and 4 in a unit square

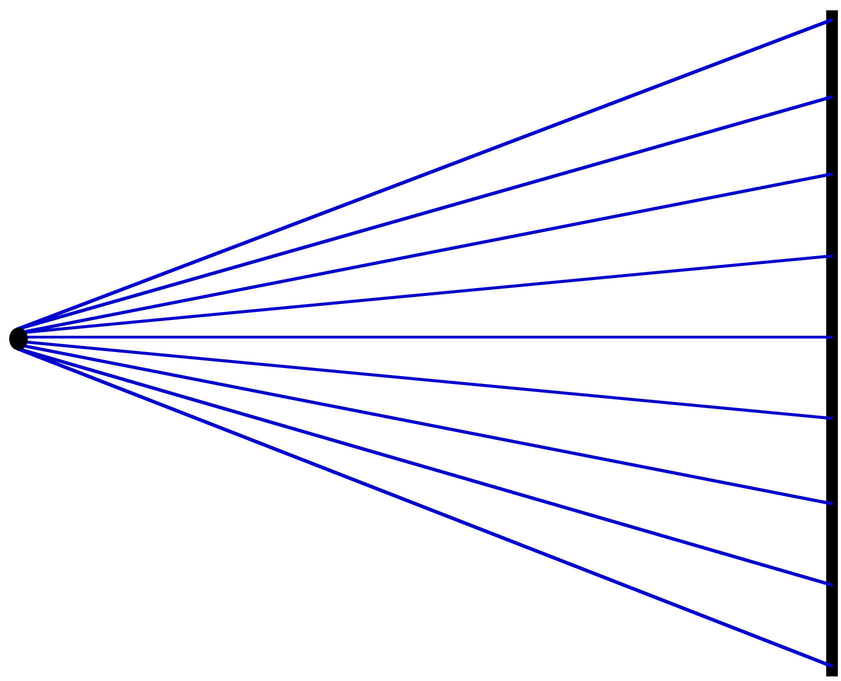
Optimal sets of length 1 and 2 in the unit ball of $\mathbf{R}^3$



Optimal sets of length 3 and 4 in the unit ball of $\mathbf{R}^3$

The usual Monge-Kantorovich theory does not however provide an explaination to several natural structures (see figures) presenting some interesting features that should be interpreted in terms of mass transportation.

For instance, if the source is a Dirac mass and the target is a segment, as in figure below, the Monge-Kantorovich theory provides a behaviour quite different from what expected.



The Monge transport rays.

Various approaches have been proposed to give more appropriated models:

- Q. Xia (Comm. Cont. Math. 2003) by the minimization of a suitable functional defined on currents;
- V. Caselles, J. M. Morel, S. Solimini, … (Preprint 2003 `http://www.cmla.ens-cachan.fr/Cmla/`, Interfaces and Free Boundaries 2003, PNLDE **51** 2002) by a kind of analogy of fluid flow in thin tubes.
- A. Brancolini, G. Buttazzo, E. Oudet, E. Stepanov, (see `http://cvgmt.sns.it`) by a variational model for irrigation trees.

Here we propose a different approach based on a definition of path length in a Wasserstein space (A. Brancolini, G. Buttazzo, F. Santambrogio 2004).

We also give a model where the opposite feature occurs: instead of favouring the concentration of transport rays, the variational functional gives a lower cost to diffused measures. We do not know of natural phenomena where this diffusive behaviour occurs.

The framework is a metric space (X, d); we assume that closed bounded subsets of X are compact. Consider the path functional

$$\mathcal{J}(\gamma) = \int_0^1 J(\gamma(t))|\gamma'|(t)\, dt$$

for all Lipschitz curves $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

- $J : X \to [0, +\infty]$ is a given mapping;
- $|\gamma'|(t)$ is the metric derivative of γ at the point t, i.e.

$$|\gamma'|(t) = \limsup_{s \to t} \frac{d\big(\gamma(s), \gamma(t)\big)}{|s - t|} .$$

Theorem *Assume:*

- J *is lower semicontinuous in* X*;*
- $J \geq c$ *with* $c > 0$*, or more generally* $\int_0^{+\infty} \left(\inf_{B(r)} J\right) dr = +\infty$.

Then, for every $x_0, x_1 \in X$ *there exists an optimal path for the problem*

$$\min\left\{\mathcal{J}(\gamma) \;:\; \gamma(0) = x_0,\; \gamma(1) = x_1\right\}$$

provided there exists a curve γ_0*, connecting* x_0 *to* x_1*, such that* $\mathcal{J}(\gamma_0) < +\infty$.

The application of the theorem above consists in taking as X a Wasserstein space $\mathcal{W}_p(\Omega)$ ($p \geq 1$), being Ω a compact subset of $\mathbf{R}^N$.

When Ω is compact, $\mathcal{W}_p(\Omega)$ is the space of all Borel probability measures on Ω, and the p-Wasserstein distance is equivalent to the weak* convergence.

The behaviour of geodesics curves for the length functional $\mathcal{J}$ will be determined by the choice of the "coefficient" functional J.

We take for J a l.s.c. functional on the space of measures, of the kind considered by Bouchitté and Buttazzo (Nonlinear Anal. 1990, Ann. IHP 1992, Ann. IHP 1993):

$$J(\mu) = \int_\Omega f(\mu^a)dx + \int_\Omega f^\infty(\mu^c) + \int_\Omega g(\mu(x))d\#.$$

- $\mu = \mu^a \cdot dx + \mu^c + \mu^\#$ is the decomposition of μ into absolutely continuous, Cantor, and atomic parts;
- $f : \mathbf{R} \to [0, +\infty]$ is convex, l.s.c., proper;
- f^∞ is the recession function of f;
- $g : \mathbf{R} \to [0, +\infty]$ is l.s.c. and subadditive, with $g(0) = 0$;
- $\#$ is the counting measure;
- f and g verify the compatibility condition

$$\lim_{t\to+\infty} \frac{f(ts)}{t} = \lim_{t\to 0^+} \frac{g(ts)}{t} .$$

The functional J is l.s.c. for the weak* convergence of measures and, if $f(s) > 0$ for $s > 0$ and $g(1) > 0$, the assumptions of the abstract framework are fulfilled. In order to obtain an optimal path for $\mathcal{J}$ between two fixed probabilities μ_0 and μ_1, it remains to ensure that $\mathcal{J}$ is not identically $+\infty$.

We now study two prototypical cases.

Concentration $f \equiv +\infty$, $g(z) = |z|^r$ with $r \in]0, 1[$. We have then

$$J(\mu) = \int_\Omega |\mu(x)|^r \, d\# \qquad \mu \text{ atomic}$$

Diffusion $f(z) = |z|^q$ with $q > 1$, $g \equiv +\infty$. We have then

$$J(\mu) = \int_\Omega |u(x)|^q \, dx \qquad \mu = u \cdot dx, \; u \in L^q.$$

Concentration case. The following facts in the concentration case hold:

- If μ_0 and μ_1 are convex combinations (also countable) of Dirac masses, then they can be connected by a path $\gamma(t)$ of finite minimal cost $\mathcal{J}$.

- If $r > 1 - 1/N$ then every pair of probabilities μ_0 and μ_1 can be connected by a path $\gamma(t)$ of finite minimal cost $\mathcal{J}$.

- The bound above is sharp. Indeed, if $r \leq 1 - 1/N$ there are measures that cannot be connected by a finite cost path (for instance a Dirac mass and the Lebesgue measure).

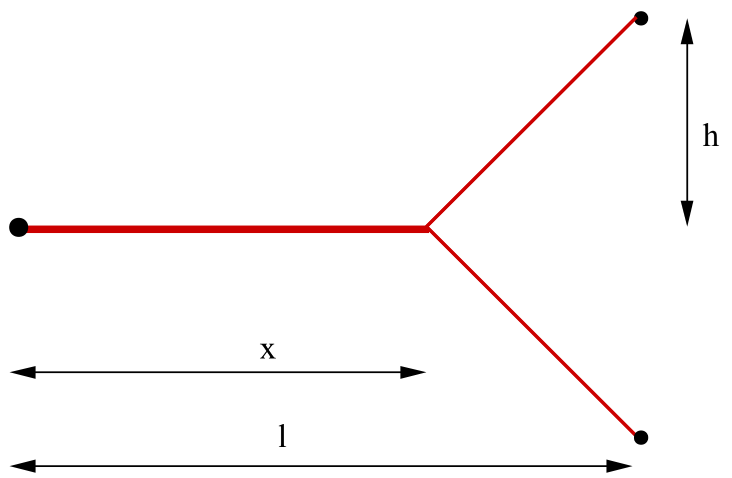
Example. (Y-shape versus V-shape). We want to connect (concentration case $r < 1$ fixed) a Dirac mass to two Dirac masses (of weight $1/2$ each) as in figure below, l and h are fixed. The value of the functional $\mathcal{J}$ is given by

$$\mathcal{J}(\gamma) = x + 2^{1-r}\sqrt{(l-x)^2 + h^2}.$$

Then the minimum is achieved for

$$x = l - \frac{h}{\sqrt{4^{1-r} - 1}}.$$

When $r = 1/2$ we have a Y-shape if $l > h$ and a V-shape if $l \leq h$.



A Y-shaped path for $r = 1/2$.

DIFFUSION CASE. The following facts in the diffusion case hold:

- If μ_0 and μ_1 are in $L^q(\Omega)$, then they can be connected by a path $\gamma(t)$ of finite minimal cost $\mathcal{J}$. The proof uses the displacement convexity (McCann 1997) which, for a functional F and every μ_0, μ_1, is the convexity of the map $t \mapsto F(T(t))$, being $T(t) = [(1-t)\mathrm{Id} + tT]^{\#}\mu_0$ and T an optimal transportation between μ_0 and μ_1.
- If $q < 1 + 1/N$ then every pair of probabilities μ_0 and μ_1 can be connected by a path $\gamma(t)$ of finite minimal cost $\mathcal{J}$.
- The bound above is sharp. Indeed, if $q \geq 1 + 1/N$ there are measures that cannot be connected by a finite cost path (for instance a Dirac mass and the Lebesgue measure).

A more refined approach in necessary when Ω is unbounded; indeed, in this case several assumptions fail:

- the Wasserstein spaces $\mathcal{W}_p(\Omega)$ are not even locally compact;
- $\mathcal{W}_p(\Omega)$ do not contain all the probabilities on Ω but only those with finite momentum of order p

$$\int_\Omega |x|^p \, d\mu < +\infty \, ;$$

- the Wasserstein convergence does not coincide with the weak* convergence.

However, some of the previous results can be generalized to the unbounded setting, even if the analysis in not so complete as in the case Ω compact.

Open Problems

- linking two L^q measures in the diffusion-unbounded case;
- concentration case in unbounded setting;
- Ω unbounded but not necessarily the whole space;
- working with the space $\mathcal{W}_\infty(\Omega)$;
- comparing this model to the ones by Xia and by Morel, Solimini, ...;
- numerical computations;
- evolution models?

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