Unique Continuation Principle for Systems of Parabolic Equations

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- Joint work with Luz de Teresa (Universidad Nacional Autónoma de México)

Today's talk

Introduction

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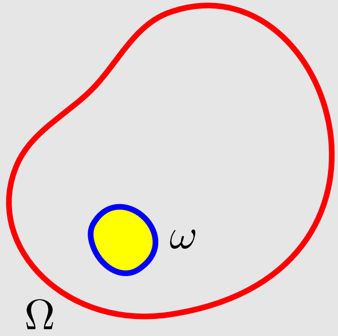
There are several types of unique continuation of interest.

- A first one is the following: let $u \in H^1(\Omega)$ satisfy

$$-\Delta u + V u = 0, \quad u \equiv 0 \text{ in } \omega \subset \Omega$$

where $V \in L^\infty(\Omega)$ and ω is an open subset of the connected open set Ω.



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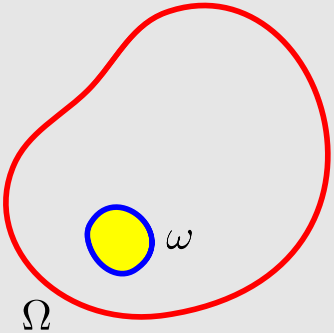

- Then one can prove that $u \equiv 0$ in Ω.

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- Now consider the Stokes equation

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- **Proof:** Observe that $\nabla p \equiv 0$ in ω , so p is constant in ω, and apply (N+1) times the result concerning the Laplacian.

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- More generally, if $a \in W^{1,\infty}(\Omega)^{N \times N}$ is a positive definite matrix and

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- This is a consequence of a convexity result: $t \mapsto \log \|u(t)\|^2$ is convex, which yields the inequality
$$(1.1) \qquad \forall\, t \in (0, T), \qquad \|u(t)\| \leq \|u_0\|^{(T-t)/T} \, \|u(T)\|^{t/T}$$

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$$c_{kj} = c_{jk} = \int_\Omega 1_{\omega_0}(x)\varphi_k(x)\varphi_j(x)dx = \int_{\omega_0} \varphi_k(x)\varphi_j(x)dx$$

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  we have that $\varphi_k\, 1_{\omega_0} = \sum_{j\geq 1} c_{kj}\varphi_j$,and

$$p(t,x)1_{\omega_0} = \sum_{k\geq 1} \beta_k(t)\varphi_k(x), \quad \text{with } \beta_k(t) = \sum_{j\geq 1} c_{kj}\alpha_j \exp(-\lambda_j t).$$

  where $p_0 = \sum_{j\geq 1} \alpha_j\varphi_j$.

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- Then one shows that u, solution to **(2.2)** is represented by the formula

$$u(t,x) = \sum_{k\geq 1}\sum_{j\neq k} c_{kj}\alpha_j \frac{\mathrm{e}^{-\lambda_k t} - \mathrm{e}^{-\lambda_j t}}{\lambda_j - \lambda_k}\varphi_k(x) + t\sum_{k\geq 1} c_{kk}\alpha_k \mathrm{e}^{-\lambda_k t}\varphi_k(x). \tag{3.4}$$

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  for all $t > 0$.

- From this one sees that if $u(t,x) = 0$ on $[0,T]\times\omega_1$, then for $x\in\omega_1$

$$-\alpha_1 c_{11}\varphi_1(x) = \sum_{k\geq 1}\sum_{j\neq k} c_{kj}\alpha_j \frac{e^{-(\lambda_k-\lambda_1)t} - e^{-(\lambda_j-\lambda_1)t}}{t(\lambda_j-\lambda_k)}\varphi_k(x)$$
$$+ \sum_{k\geq 2}\alpha_k c_{kk} e^{-(\lambda_k-\lambda_1)t}\varphi_k(x).$$

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- Letting $t \to +\infty$ one concludes that $\alpha_1 c_{11}\varphi_1 1_{\omega_1} \equiv 0$, and hence $\alpha_1 = 0$. One may repeat this argument for all $k > 1$ and conclude that $\alpha_k = 0$.

Idea of the proof

- In the same manner one shows that z, solution to **(2.3)** , is represented by (here $0 < t < T$)

$$z(t,x) = \sum_{k,j\geq 1} \mathrm{e}^{\lambda_k t} \frac{c_{kj}\alpha_j}{\lambda_j + \lambda_k} \left[\mathrm{e}^{-(\lambda_j+\lambda_k)t} - \mathrm{e}^{-(\lambda_j+\lambda_k)T} \right] \varphi_k(x). \tag{3.5}$$

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- However here it is somewhat more subtle to show

$$z \equiv 0 \text{ on } (0,T) \times \omega_1 \Longrightarrow \alpha_k = 0 \text{ for all } k \geq 1 \ldots$$

Idea of the proof

- One begins by noting that if $z \equiv 0$ on $(0,T)\times\omega_1$, the representation formula implies: for $(t,x)\in(0,T)\times\omega_1$

$$\sum_{k,j\geq 1}\frac{c_{kj}\alpha_j}{\lambda_j+\lambda_k}\mathrm{e}^{-\lambda_j t}\varphi_k(x)=\sum_{k,j\geq 1}\frac{c_{kj}\alpha_j}{\lambda_j+\lambda_k}\mathrm{e}^{-\lambda_j T}\mathrm{e}^{-\lambda_k(T-t)}\varphi_k(x). \tag{3.6}$$

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  Upon choosing $t := \frac{T}{2} + \mathrm{i}s$ one gets for all $s \in \mathbb{R}$

$$\sum_{n\geq 1} b_{1n}(x) \mathrm{e}^{-\mathrm{i}\lambda_n s} = \sum_{n\geq 1} b_{2n}(x) \mathrm{e}^{\mathrm{i}\lambda_n s}$$

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where we have set

$$\begin{aligned} b_{1j}(x) &:= \sum_{k\geq 1}\frac{c_{kj}\alpha_j}{\lambda_j+\lambda_k}\mathrm{e}^{-\lambda_j T/2}\varphi_k(x)1_{\omega_1}(x)\\ b_{2k}(x) &:= \sum_{j\geq 1}\frac{c_{kj}\alpha_j}{\lambda_j+\lambda_k}\mathrm{e}^{-(\lambda_k+2\lambda_j)T/2}\varphi_k(x)1_{\omega_1}(x). \end{aligned} \tag{3.7}$$

Idea of proof

Lemma. Let $(b_n)_{n\geq 1}$ be complex numbers such that $\sum_{n\geq 1} |b_n| < \infty$, and let $(\lambda_n)_{n\geq 1}$ be distinct real numbers. If for all $s \in \mathbb{R}$

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- **Proof.** If k is the least integer $n \geq 1$ such that $b_n \neq 0$, multiply by $\mathrm{e}^{-\mathrm{i}\lambda_k s}$ and integrate over $[-L, L]$ to get

$$0 = b_k + \sum_{n\geq k+1} b_n \frac{1}{2L} \int_{-L}^{+L} \mathrm{e}^{\mathrm{i}(\lambda_n - \lambda_k)s} ds = b_k + \sum_{n\geq k+1} b_n \frac{\sin((\lambda_n - \lambda_k)L)}{(\lambda_n - \lambda_k)L}.$$

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- **Proof.** If k is the least integer $n \geq 1$ such that $b_n \neq 0$, multiply by $\mathrm{e}^{-\mathrm{i}\lambda_k s}$ and integrate over $[-L, L]$ to get

$$0 = b_k + \sum_{n\geq k+1} b_n \frac{1}{2L}\int_{-L}^{+L} \mathrm{e}^{\mathrm{i}(\lambda_n - \lambda_k)s} ds = b_k + \sum_{n\geq k+1} b_n \frac{\sin((\lambda_n - \lambda_k)L)}{(\lambda_n - \lambda_k)L}.$$

Letting $L \to +\infty$ it follows that $b_k = 0$.

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Then b_{1n}, b_{2n} defined by **(3.7)** can be written as

$$\begin{aligned} b_{1n}(x) &= \alpha_n \psi_n(x) 1_{\omega_1}(x) \mathrm{e}^{-\lambda_n T/2} \\ b_{2n}(x) &= (\psi_n | p(T)) \varphi_n(x) 1_{\omega_1}(x) \mathrm{e}^{-\lambda_n T/2}. \end{aligned} \tag{3.8}$$

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- From **(3.8)** one concludes that for all $n \geq 1$: $\alpha_n \psi_n(x) \equiv 0$ in ω_1 and

$$(\psi_n | p(T)) = \sum_{j \geq 1} \frac{c_{nj}}{\lambda_j + \lambda_n} \, \alpha_j \, \mathrm{e}^{-\lambda_j T} = 0. \tag{3.9}$$

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- Then L is a bounded self-adjoint operator and **(3.9)** means $(\varphi_n | Lp(T)) = 0$ for all $n \geq 1$.
- We have thus $Lp(T) = 0$. And next we show that this implies that $p(T) = 0$.

Idea of proof

- One shows a representation formula for L: for all $f \in L^2(\Omega)$ one has

$$(Lf|f) = \int_0^\infty \|1_{\omega_0} S(t) f\|_2^2 dt \tag{3.10}$$

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- Thus **(3.9)** implies that $S(t)p(T) \equiv 0$ on $(0,\infty) \times \omega_0$, and the unique continuation principle for the heat equation implies $p(T) \equiv 0$ in Ω.

Proof of (3.10)

- For $t > 0$ define

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- Since for $g \in L^2(\Omega)$

$$\int_{\omega_0} g(x)^2 dx = \sum_{n,k\geq 1} (g|\varphi_n)(g|\varphi_k) \int_{\omega_0} \varphi_n(x)\varphi_k(x) dx = \sum_{n,k\geq 1} c_{nk}(g|\varphi_n)(g|\varphi_k)$$

Proof of (3.10)

- So one sees that

$$\int_0^\infty \|1_{\omega_0} S(t) f\|_2^2 dt = \int_0^\infty \left[\sum_{n,k\geq 1} c_{nk} \mathrm{e}^{-\lambda_n t} (f|\varphi_n) \mathrm{e}^{-\lambda_k t} (f|\varphi_k) \right] dt$$

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- So one sees that

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$$= \sum_{n,k\geq 1} c_{nk} (f|\varphi_n)(f|\varphi_k) \int_0^\infty \mathrm{e}^{-(\lambda_n+\lambda_k)t} dt$$
$$= \sum_{n,k\geq 1} \frac{c_{nk}}{\lambda_n+\lambda_k} (f|\varphi_n)(f|\varphi_k) = (Lf|f).$$

- This recalls the known exercises about Hilbert matrices: prove that the m-dimensional Hilbert matrix

$$\left(\frac{1}{i+j-1} \right)_{1\leq i,j\leq m}$$

is a self-adjoint positive definite matrix...