Carleman Estimates. Applications and open problems

Jean-Pierre Puel *

Introduction

Carleman Estimates :

- Highly technical and complicated estimates.
- Weighted Sobolev-type estimates for a solution of a PDE in terms of local information on the solution on a subdomain.
- Propagates information known on a subdomain along the gradient lines of the main weight.
- Weights are essential. Can be chosen in different ways but have to be related to the PDE under consideration.

Useful tool for :

- Unique continuation properties : If L is a PD operator, does the information

  $Lu = 0$ and $u = 0$ in an ad'hoc subdomain imply $u = 0$ everywhere?

- Backward uniqueness.

- Exact controllability (see below).

- Inverse problems.

It is essentially a linear tool !!

Here : focus on dissipative equations :

- Heat equations .

- Diffusion convection equations.

- Stokes and Navier-Stokes equations...

Exact Controllability to Trajectories

Nonlinear evolution system with a control variable v

$$\begin{cases} \frac{\partial Y}{\partial t} + LY + N(Y) = F + Bv \text{ in } (0,T), \\ Y(0) = Y_0, \end{cases} \tag{1}$$

L is for example an elliptic operator and N is a nonlinear perturbation.

Think of a nonlinear convection-diffusion equation or Navier-Stokes equations or...

On the other hand, uncontrolled trajectory of the same operator : "ideal" trajectory that we want to reach

$$\begin{cases} \frac{\partial \bar{Y}}{\partial t} + L\bar{Y} + N(\bar{Y}) = F \text{ in } (0,T), \\ \bar{Y}(0) = \bar{Y}_0, \end{cases} \tag{2}$$

Exact controllability to trajectories : can we find a control v such that

$$Y(T) = \bar{Y}(T).$$

(Linear case : null controllability : v such that $Y(T) = 0$.)

Local version : provided $(Y_0 - \bar{Y}_0)$ is "small" in a suitable norm, can we find a control v such that

$$Y(T) = \bar{Y}(T).$$

Remark : If the answer is positive and if our evolution system is well-posed, after time T we can switch off the control and the system will follow the "ideal" trajectory.

Important case :

If $\bar{Y}$ is a stationnary solution (with F independent of time t), namely

$$L\bar{Y} + N(\bar{Y}) = F. \tag{3}$$

Many important nonlinear stationnary systems of this type may have several solutions and among them unstable solutions. In this case, if $\bar{Y}$ is such an unstable solution and if the problem of exact controllability to trajectories has a positive answer, it corresponds to stabilizing (and exactly reach) an unstable solution.

Case of Navier-Stokes Equations

$(\bar{\mathbf{y}},\bar{p})$: “ideal” solution of Navier-Stokes equations (for example a stationnary solution).

$$\begin{cases} \frac{\partial \bar{\mathbf{y}}}{\partial t} - \nu \Delta \bar{\mathbf{y}} + \bar{\mathbf{y}}.\nabla \bar{\mathbf{y}} + \nabla \bar{p} = \mathbf{f} \text{ in } \Omega \times (0,T), \\ \operatorname{div} \bar{\mathbf{y}} = 0 \text{ in } \Omega \times (0,T), \\ \bar{\mathbf{y}} = 0 \text{ on } \Gamma \times (0,T) \\ \bar{\mathbf{y}}(0) = \bar{\mathbf{y}}_0 \text{ in } \Omega. \end{cases} \tag{4}$$

Consider a solution of the controlled system, starting from a different initial value

$$\begin{cases} \frac{\partial \mathbf{y}}{\partial t} - \nu \Delta \mathbf{y} + \mathbf{y}.\nabla \mathbf{y} + \nabla p = \mathbf{f} + \mathbf{v}.1\!\!\mathrm{I}_\omega \text{ in } \Omega \times (0,T), \\ \operatorname{div} \mathbf{y} = 0 \text{ in } \Omega \times (0,T), \\ \mathbf{y} = 0 \quad \text{on} \quad \Gamma \times (0,T) \\ \mathbf{y}(0) = \mathbf{y}_0 \text{ in } \Omega, \end{cases} \tag{5}$$

$1\!\!\mathrm{I}_\omega$: characteristic function of a (little) subset ω of Ω.

Exact Controllability to Trajectories :

Can we find a control $\mathbf{v}$ such that

$$\mathbf{y}(T) = \bar{\mathbf{y}}(T) \ ?$$

i.e can we reach exactly in finite time the "ideal" trajectory $\bar{\mathbf{y}}$?

Local version : same result provided $||\mathbf{y}_0 - \bar{\mathbf{y}}_0||$ is small enough.

Remark 1 *If there exists such a control* $\mathbf{v}$*, then, after time* T*, just switch off the control (*$\mathbf{v} = 0$*) and the system will stay on the "ideal" trajectory.*

Last result (Fernandez-Cara, Guerrero, Imanuvilov, Puel, Journal de Math. Pures et Appl., 2004) (dimension 3):

$$H = \{\mathbf{y} \in L^2(\Omega)^3,\ \mathrm{div}\,\mathbf{y} = 0,\ \mathbf{y}.\nu = 0 \text{ on } \Gamma\}.$$

Theorem 2 *Let us assume that*

$$\bar{\mathbf{y}}_\mathbf{0} \in H \cap L^4(\Omega)^3, \quad \bar{\mathbf{y}} \in L^\infty(\Omega \times (0,T))^3$$

and

$$\frac{\partial \bar{\mathbf{y}}}{\partial t} \in L^2(0,T;L^\sigma(\Omega))^3,\ \sigma > \frac{6}{5}$$

then there exists $\eta > 0$ *such that for every* $\mathbf{y}_0 \in H \cap L^4(\Omega)^3$ *such that* $||\mathbf{y}_0 - \bar{\mathbf{y}}_0||_{L^4(\Omega)^3} \leq \eta$, *there exists a control* $\mathbf{v} \in L^2(0,T;L^2(\omega))^3$ *and a solution* $(\mathbf{y},p)$ *of (5) such that*

$$\mathbf{y}(T) = \bar{\mathbf{y}}(T).$$

Where do Carleman Estimates come in?

Go back to general setting and linearize....

Null controllability for linearized system.

Then fixed point argument.

$$\begin{cases} \frac{\partial Y}{\partial t} + LY = Bv \text{ in } (0,T), \\ Y(0) = Y_0, \end{cases} \tag{6}$$

Look for v such that $Y(T) = 0$.

First step : $\epsilon > 0$.

Look for v such that $||Y(T)|| \leq \epsilon$.

If such a v exists there may be **many** of them. Select the one which minimizes the norm of controls :

$$\min_{v \text{ s.t. } ||Y(T)|| \leq \epsilon} \frac{1}{2} \int_0^T ||v(t)||^2 dt.$$

As $v \to Y$ is affine, it can be rewritten in following form

$$\min_v \left(\frac{1}{2}\int_0^T ||v(t)||^2 dt + \mathrm{Ind}_{B(0,\epsilon)}(Y^0(T) + \mathcal{L}(v)(T))\right)$$

where $\mathcal{L}$ is linear.

Apply Fenchel-Rockafellar duality result.

Adjoint equation :

$$\begin{cases} -\frac{\partial \Phi}{\partial t} + L\Phi = 0 \text{ in } (0, T), \\ \Phi(T) = \Phi_T, \end{cases} \tag{7}$$

Dual problem :

$$\min_{\Phi_T} J_\epsilon(\Phi_T) =: \left(\frac{1}{2}\int_0^T ||B^*\Phi(t)||^2 dt + \epsilon||\Phi_T|| + (\Phi(0), Y_0)\right).$$

It can be shown (C.Fabre,J-P.P.,E.Zuazua) that dual problem has a solution provided (in fact if and only if) unique continuation property holds

$$B^*\Phi = 0 \text{ implies } \Phi = 0 \text{ (and } \Phi_T = 0).$$

Then if the minimum is Φ_T^ϵ corresponding to a solution of (7) Φ^ϵ and if

$$v_\epsilon = B^*\Phi^\epsilon$$

then v_ϵ is solution of primal problem (minimizes the norm over admissible controls).

Second step : Passage to the limit when $\epsilon \to 0$.

Estimates on v_ϵ. We have : $\int_0^T ||v_\epsilon||^2 dt = \int_0^T ||B^*\Phi^\epsilon||^2 dt$.

$J_\epsilon(\Phi_T^\epsilon) \leq J_\epsilon(0) = 0$. This implies

$$\frac{1}{2}\int_0^T ||B^*\Phi^\epsilon(t)||^2 dt \leq ||Y_0||.||\Phi^\epsilon(0)||.$$

If we know an observability inequality for solutions of adjoint equation like

$$||\Phi(0)||^2 \leq C\int_0^T ||B^*\Phi(t)||^2 dt,$$

we obtain an estimate on the control

$$\int_0^T ||v_\epsilon||^2 dt = \int_0^T ||B^*\Phi^\epsilon||^2 dt \leq 4C||Y_0||^2.$$

Then we can pass to the limit :

For a subsequence v_ϵ converges weakly to v,

$Y(v_\epsilon)(T) \to Y(v)(T)$ and $||Y(v_\epsilon(T)|| \leq \epsilon$ imply

$$Y(v)(T) = 0$$

and we have solved the null controllability problem.

Problem : To obtain Observability Inequality for adjoint system !!

For linearized Navier-Stokes equations around the trajectory $\bar{y}$

$$\begin{cases} \frac{\partial \mathbf{y}}{\partial t} - \nu \Delta \mathbf{y} + \nabla \cdot (\bar{\mathbf{y}} \otimes \mathbf{y} + \mathbf{y} \otimes \bar{\mathbf{y}}) + \nabla p = \mathbf{v}.1\!\!1_\omega \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega \times (0,T), \\ \operatorname{div} \mathbf{y} = 0 \quad \text{in} \quad \Omega \times (0,T), \\ \mathbf{y} = 0 \quad \text{on} \quad \Gamma \times (0,T) \\ \mathbf{y}(0) = \mathbf{y}_0 \quad \text{in} \quad \Omega, \end{cases} \tag{8}$$

(Pseudo-)adjoint system (backward equation)

$$\begin{cases} -\frac{\partial \varphi}{\partial t} - \nu \Delta \varphi - \bar{\mathbf{y}} \cdot D(\varphi) + \nabla \pi = 0 \text{ in } \Omega \times (0,T), \\ \operatorname{div} \varphi = 0 \quad \text{in} \quad \Omega \times (0,T), \\ \varphi = 0 \quad \text{on} \quad \Gamma \times (0,T) \\ \varphi(T) = \varphi_0 \quad \text{in} \quad \Omega, \end{cases} \tag{9}$$

with $D\varphi = \nabla \varphi + \nabla \varphi^T$.

We want to show the Observability Inequality

$$|\varphi(0)|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |\varphi|^2 dxdt \tag{10}$$

(no reference to the "initial" value φ_0).

How to prove observability inequality?

Consequence of a Global Carleman estimate plus standard energy estimates

Carleman estimate gives in particular

$$\int_0^T \int_\Omega \rho^2 |\varphi|^2 dxdt \leq C \int_0^T \int_\omega \overline{\rho}^\theta |\varphi|^2 dxdt$$

for some θ and with suitable weights ρ and $\bar{\rho}$ which are C^2 and > 0 on $\overline{\Omega}\times]0,T[$ and

$$\rho(t) \to 0 \text{ if } t \to T.$$

Notice that this gives the unique continuation property but gives much more !!

Estimate requires a very long multistep proof and does not deal directly with the Stokes structure...

We obtain the following precise estimate :

There exist C, $\bar{s}$, $\bar{\lambda}$ such that for every $s > \bar{s}$ and $\lambda > \bar{\lambda}$,

$$\iint_{\Omega\times(0,T)} e^{-2s\alpha}(s\lambda^2\xi|\nabla\varphi|^2 + s^3\lambda^4\xi^3|\varphi|^2)\,dx\,dt$$
$$\leq Cs^{16}\lambda^{40}\iint_{\omega\times(0,T)} e^{-8s\widehat{\alpha}+6s\alpha^*}\widehat{\xi}^{16}\,|\varphi|^2\,dx\,dt$$

where

$$\alpha(x,t) = \frac{e^{5/4\,\lambda\,m\,\|\eta^0\|_\infty} - e^{\lambda(m\,\|\eta^0\|_\infty + \eta^0(x))}}{t^4(T-t)^4}\ ,$$

$$\xi(x,t) = \frac{e^{\lambda(m\,\|\eta^0\|_\infty + \eta^0(x))}}{t^4(T-t)^4}\ ,$$

$$\widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t),$$

$$\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t),$$

$$\widehat{\xi}(t) = \max_{x\in\overline{\Omega}} \xi(x,t),$$

with $m > 4$ and $\eta^0 \in C^2(\overline{\Omega})$ such that

$$\eta^0 > 0 \text{ in } \Omega,\ \ \eta^0 = 0 \text{ on } \Gamma,\ \ |\nabla\eta^0| > 0 \text{ in } \overline{\Omega\setminus\omega'},$$

ω' being a nonempty subset of ω.

Some references for the nonlinear convection diffusion equations.

Linear case : Lebeau-Robbiano, Fursikov-Imanuvilov

Nonlinear case, local results : Fursikov-Imanuvilov

Nonlinear case, global results : Fernandez Cara-Zuazua (nonlinearity slightly superlinear), Anita-Tataru (nice and surprising result for more superlinear nonlinearities), Doubova-Osses-P. (transmission problem), Fernandez Cara-Gonzales Burgos-Guerrero-P. (nonlinear Fourier boundary conditions),....

Extensions :

Boussinesq system (Navier-Stokes coupled with an energy equation) (Guerrero)

Alternative approach with a second control in the divergence term in a first step (Gonzales Burgos, Guerrero, P.)

Reducing the number of controls for Navier-Stokes or Boussinesq (Fernandez Cara, Guerrero, Imanuvilov, P.)

Controllability (to zero) for a fluid structure interacting system (M.Boulakia, A.Osses preprint)

Among open problems :

Can the result be global ? Can we use a more “nonlinear” method ?

Can we get rid of the L^∞ condition on $\bar{y}$?

What is the situation for compressible viscous fluids ? Completely open question !!

$$\left\{\begin{array}{l} \frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho.\mathbf{y}) = 0, \\ \frac{\partial \mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + \mathbf{y}.\nabla\mathbf{y} + \nabla p = \mathbf{f} + \mathbf{v}.1\!\!1_\omega, \\ \mathbf{y} = 0 \quad \text{on} \quad \Gamma\times(0,T) \\ \mathbf{y}(0) = \mathbf{y}_0, \\ p = C\rho^\gamma \end{array}\right. \tag{11}$$

with γ as close as possible to 1.4 !!

For heat-type equations, how robust are these estimates (and their consequences)? What happens when we have oscillating coefficients ? What happens for discretized problems in connexion with numerical approximations ? (some work by E. Zuazua and his group)

In convection diffusion equations, what happens when the diffusion parameter tends to zero ? (some interesting results by Coron-Guerrero and Lebeau-Guerrero).

Backward uniqueness

Classical results by J-L.Lions-B.Malgrange. Also by A.Friedman

Very nice recent result by Escauriaza-Seregin-Sverak (revisited by S.Ervedoza). Used for proving regularity and uniqueness results for Navier-Stokes equations in exterior domains in the class $L^\infty(0,T;L^3(\Omega))$

Ω is an exterior domain. If u grows at most exponentially in space and satisfies the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f(u, \nabla u) \text{ in } \Omega \times (0,T)$$

with $|f(s,p)| \leq C(1 + |s| + |p|)$ (NO BOUNDARY CONDITION)

and

$$u(T) = 0$$

then u is identically zero !!

Use of Carleman estimates with particular weights (not the same as for controllability....)

The result is NOT TRUE in bounded domains.... Open question : Is this result also true for Stokes system ? for Navier-Stokes ?

Data assimilation problem

(Non standard approach)

We consider a (linearized) Navier-Stokes system on a time interval $(-T_0, 0)$ (from yesterday to to-day)

$$\begin{cases} \frac{\partial \mathbf{y}}{\partial t} - \nu \Delta \mathbf{y} + \nabla \cdot (\bar{\mathbf{y}} \otimes \mathbf{y} + \mathbf{y} \otimes \bar{\mathbf{y}}) + \nabla p = \mathbf{f} \\ \qquad\qquad\qquad\qquad \text{in } \Omega \times (-T_0, 0), \\ \operatorname{div} \mathbf{y} = 0 \quad \text{in} \quad \Omega \times (-T_0, 0), \\ \mathbf{y} = 0 \quad \text{on} \quad \Gamma \times (-T_0, 0) \end{cases} \tag{12}$$

$$\mathbf{y}(-T_0) = \mathbf{y}_0 \quad \text{in} \quad \Omega, \tag{13}$$

where the initial data $\mathbf{y}_0$ is unknown. On the other hand we know some measurments of the solution on a subdomain ω during the time period $(-T_0, 0)$

$$\mathbf{y} = \mathbf{h} \text{ in } \omega \times (-T_0, 0).$$

Goal : to be able to predict $\mathbf{y}$ on $(0, T)$ (from to-day to to-morrow).

Usual method (variational data assimilation):

Try to recover $\mathbf{y}_0$ and then solve (simulate) system (12) on $(-T_0, T)$ (from yesterday to to-morrow).

Define

$$J(\mathbf{y}_0) = \frac{1}{2}\int_0^T \int_\omega |\mathbf{y} - \mathbf{h}|^2 dx dt + \alpha ||\mathbf{y}_0||^2$$

where $\alpha > 0$ is a Tychonov regularization parameter.

To find $\bar{\mathbf{y}}_\mathbf{0}$ such that

$$J(\bar{\mathbf{y}}_\mathbf{0}) = \min_{\mathbf{y}_0} J(\mathbf{y}_0)$$

This problem is known to be unstable when $\alpha \to 0$ (ill-posed when $\alpha = 0$).

Non standard approach : Try to recover $\mathbf{y}(0)$ (value to-day) from measurements between $-T_0$ and 0 (without knowing anything on $\mathbf{y}(-T_0)$ of course).

Claim : Well-posed problem.

Reason : Global Carleman inequality and observability inequality (valid if $\bar{\mathbf{y}} \in W^{1,\infty}$) .

$$|\mathbf{y}(0)|^2_{L^2(\Omega)} \leq C \int_{-T_0}^{0} \int_\Omega |f|^2 dxdt + C \int_{-T_0}^{0} \int_\omega |\mathbf{h}|^2 dxdt \tag{14}$$

if

$$\mathbf{h} = \mathbf{y} \text{ on } \omega \times (-T_0, 0).$$

This implies stability.

Consider a controllability problem for the adjoint system : if $\varphi_0 \in H = \{z \in L^2(\Omega)^3,\ \operatorname{div} z = 0,\ z.\nu = 0 \text{ on } \Gamma\}$, find $\mathbf{v} = \mathbf{v}(\varphi_0)$ such that the solution of

$$\begin{cases} -\frac{\partial \varphi}{\partial t} - \nu \Delta \varphi + \bar{\mathbf{y}} \cdot D(\varphi) + \nabla \pi = \mathbf{v}.1\!\!1_\omega \text{ in } \Omega \times (-T_0, 0), \\ \operatorname{div} \varphi = 0 \quad \text{in} \quad \Omega \times (-T_0, 0), \\ \varphi = 0 \quad \text{on} \quad \Gamma \times (-T_0, 0) \\ \varphi(0) = \varphi_0 \quad \text{in} \quad \Omega, \end{cases} \tag{15}$$

satisfies

$$\varphi(-T_0) = 0.$$

This is possible because of (14). Now multiplying (12) by φ we obtain

$$\forall \varphi_0 \in H,\ \int_\Omega \mathbf{y}(0).\varphi_0 dx = \int_{-T_0}^{0}\int_\Omega \mathbf{f}.\varphi dx dt + \int_{-T_0}^{0}\int_\omega \mathbf{h}.\mathbf{v}(\varphi_0) dx dt.$$

Therefore we can recover for example all the coefficients of $\mathbf{y}(0)$ on a Hilbert basis, at the price of solving a controllability problem for each element of the basis.

Of course we can obtain an approximation of $\mathbf{y}(0)$ by considering a finite number of elements φ_0.

Importance of considering reduced basis.

Notice that we can estimate the sensitivity of this procedure to errors in the measures $\mathbf{h}$.

We can also use an "optimal control" approximation.

Consider, for φ_0 fixed, the functional

$$J_\alpha(\mathbf{v}) = \frac{1}{\alpha}\int_\Omega |\varphi(-T_0)|^2 dx + \frac{1}{2}\int_{-T_0}^{0}\int_\omega |\mathbf{v}|^2 dx dt$$

where φ is solution to (15), and the optimal control problem :

$$\min_{\mathbf{v}\in L^2(-T_0,0;L^2(\omega))^3} J_\alpha(\mathbf{v})$$

which gives a solution $\mathbf{v}_\alpha(\varphi_0)$ and a corresponding solution φ_α to (15) for which we can define

$$l_\alpha(\varphi_0) = \int_{-T_0}^{0}\int_\Omega \mathbf{f}.\varphi_\alpha dx dt + \int_{-T_0}^{0}\int_\omega \mathbf{h}.\mathbf{v}_\alpha(\varphi_0) dx dt.$$

It can be shown that when $\alpha \to 0$,

$$l_\alpha \to \int_\Omega \mathbf{y}(0).\varphi_0 dx.$$

Example of application with numerical experiments to large-scale ocean circulation model by G. Garcia, A. Osses, J.-P. Puel with very promising numerical results. To be continued ....