Optimal Control of PDEs on networks: controllability, domain decomposition and homogenization

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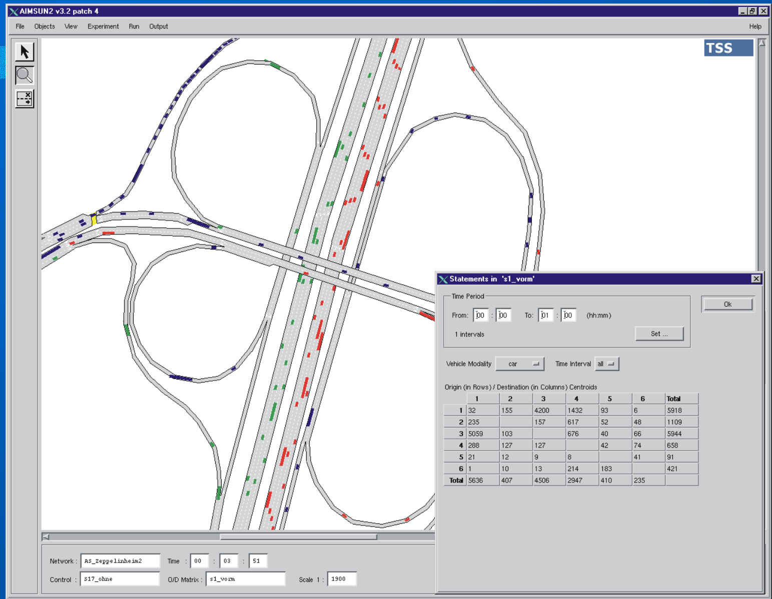
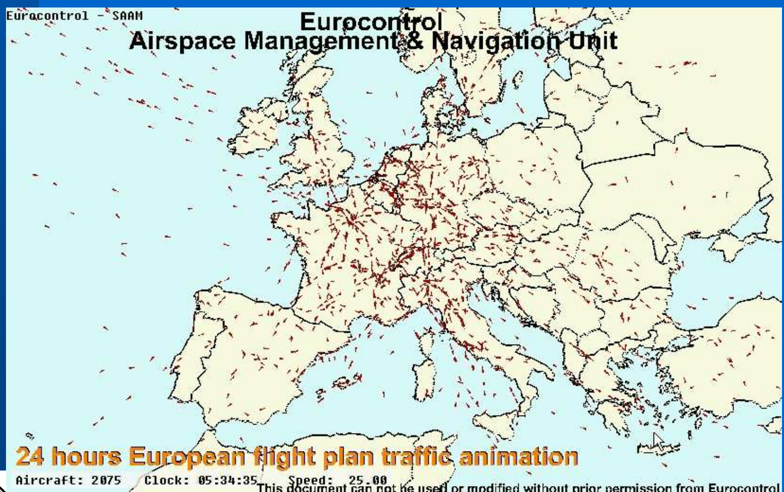
Optimization of infrastructures



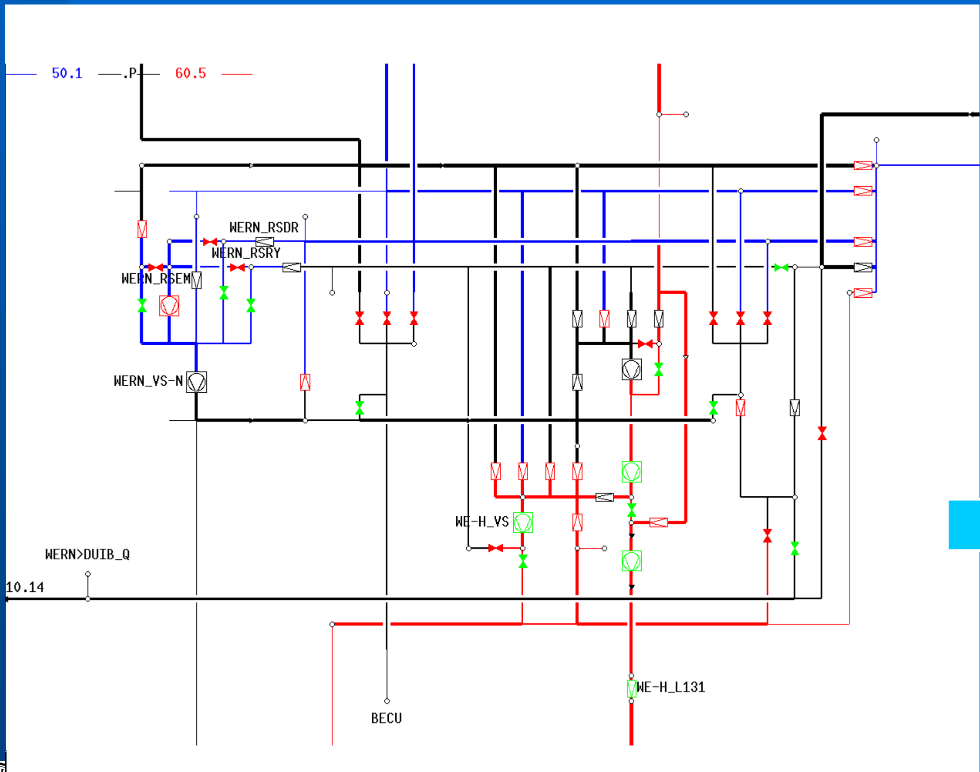


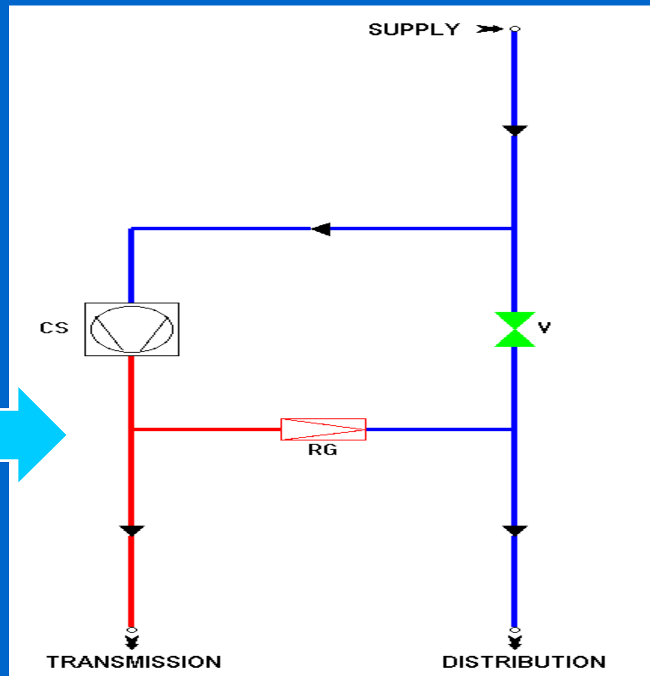
Network modelling

Gas-networks:

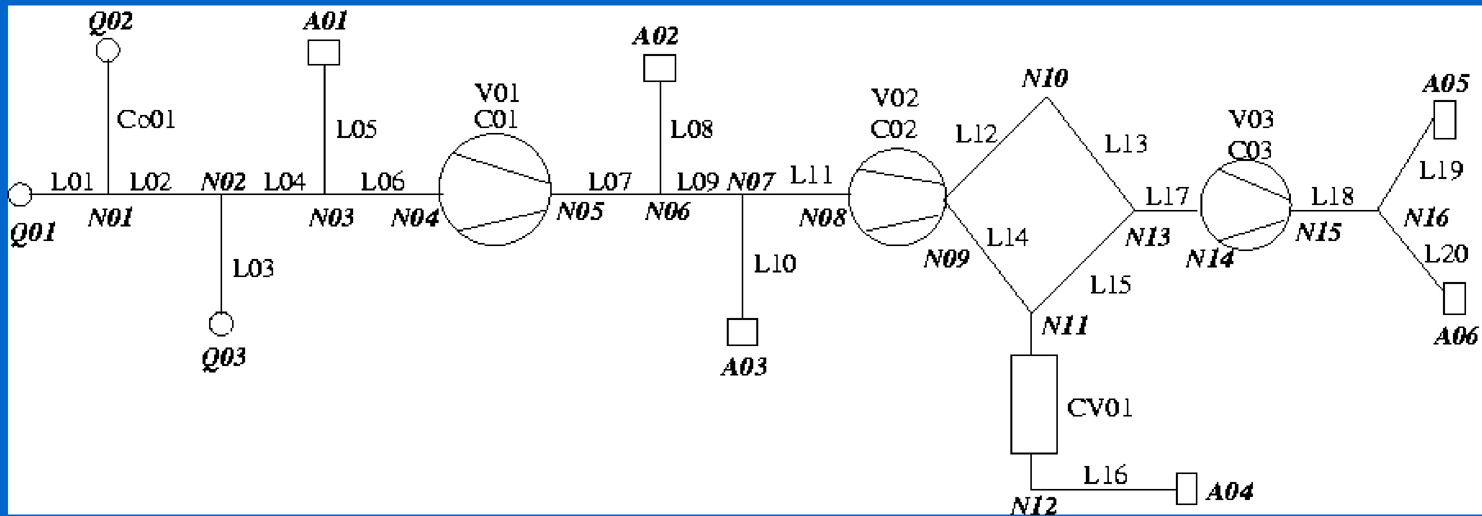
Optimal topology,

Optimal customer satisfaction





Sample data set .....

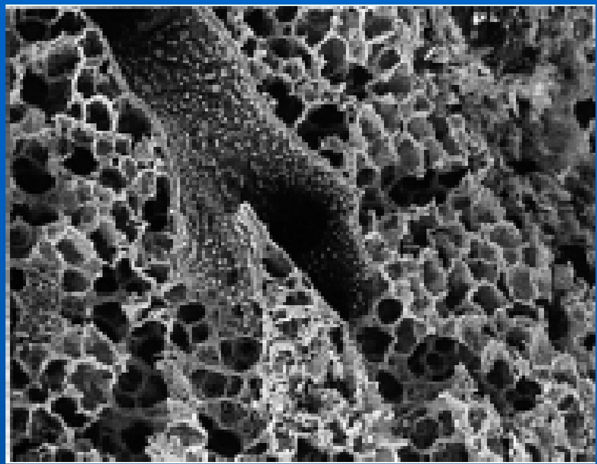


# pipes	# compressors	total length	time (ε = 0.05)	time (ε = 0.01)
11	3	920 km	1.2 Sec	2.0 Sec
20	3	1200 km	1.2 Sec	9.9 Sec
31	15	2200 km	11.5 Sec	104.4 Sec

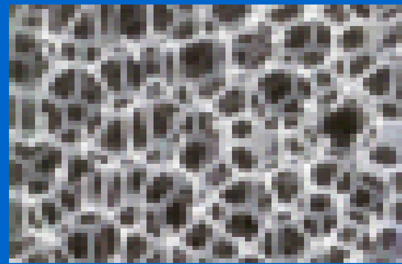
Current projects: joint work

- Traffic: A. Klar and M. Herty (Scientific Computing) Univ. Kaiserslautern
- Gas: A. Martin (discrete optimization) and J. Lang (Numerical PDE) Tech. Univ. of Darmstadt
- Water (irrigation and sewer-systems): M. Gugat Univ. of Erlangen-Nuremberg

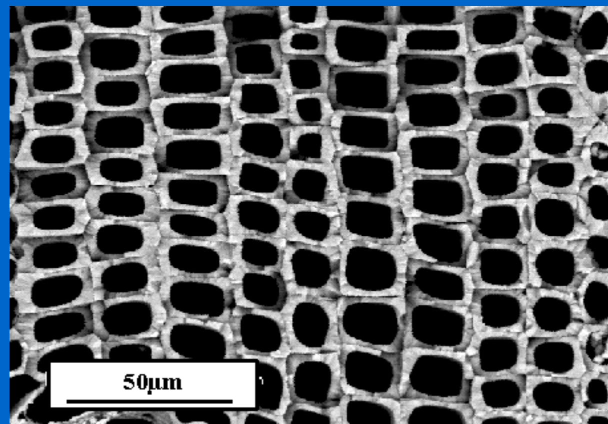
Optimization of ‚Quasi-periodic‘ material and structures



Lung tissue



Metal foam

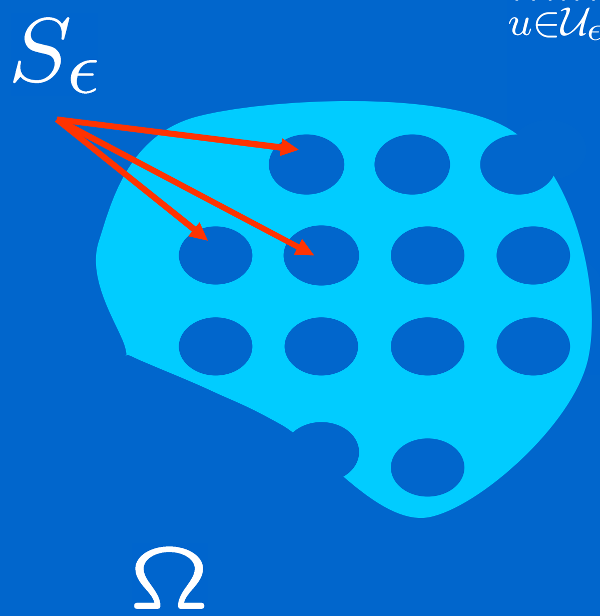


wood

Flexible structres

Mathematical modelling

Perforated domain paradigm or fattened graphs



$$\min_{u\in\mathcal{U}_\epsilon} J_\epsilon(u,y) := \int_{\Omega\setminus S_\epsilon} (y-z_d)^2 dx + \frac{1}{2}\|u\|^2_{\mathcal{U}_\epsilon} \quad (J_\epsilon)$$

subject to (E_ϵ) :

$$-\operatorname{div} A^\epsilon \nabla y = f_\epsilon + u$$

$$y \in H^1_0(\Omega), \quad u \in \mathcal{U}_\epsilon$$

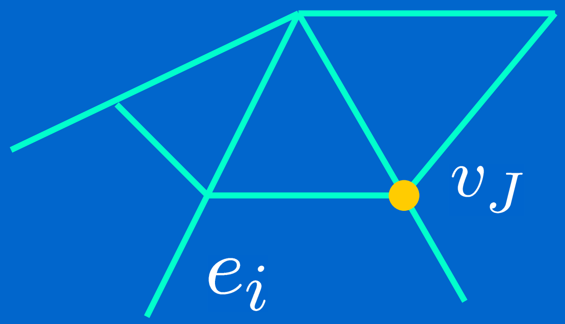
$$y = \Psi_\epsilon \quad \text{q.e. on } S_\epsilon$$

2-D or 3-D models, reticulated structures
Cioranescu, Saint-Jean Paulin 1999….
P.Kogut and G.L. 2005

Mathematical modelling

Graph modelling

$$\mathcal{G} = (V, E)$$



$$\min \frac{1}{2} \sum_{i=1}^{n} \int_0^{\ell_i} |y_i - y_i^d|^2 dx + \frac{\alpha}{2} \sum_{J\mathcal{J}} |u(v_J)|^2$$

subject to

$$\begin{aligned}
-y_i'' + \sigma y_i &= f_i & &\in (0,\ell_i) \times [0,T] \\
y_i'(v_N) &= u_i & &\forall i \in \mathcal{I}_N,\ t \in (0,T) \\
y_i(v_J) &= y_j(v_J) & &i,j \in \mathcal{I}_J,\ t \in (0,T) \\
\sum_{i \in \mathcal{I}_J} d_{iJ} y_i'(v_J) &= 0 & &J \in \mathcal{J}_M,\ t \in (o,T)
\end{aligned}$$

E.J.P.G. Schmidt, J.E. Lagnese and G.L. 1994

Shallow water equations

Equations of motion (de St. Venant 1871)

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ V \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} AV \\ \frac{1}{2}V^2 + gh(x,A) + gY_b(x) \end{pmatrix} = \begin{pmatrix} 0 \\ gI_f \end{pmatrix}$$

Notation: $U = \begin{pmatrix} A \\ V \end{pmatrix}$, $F(U) = \begin{pmatrix} AV \\ \frac{1}{2}V^2 + gh(x,A) + gY_b(x) \end{pmatrix}$,

$$\boxed{\partial_t U + \partial_x F(U) = S(U)}$$

Modelhierarchy

Diffusion-Advection equation:

$$\frac{1}{g}(\partial V_t + V\partial_x V) \longrightarrow 0, \qquad K \text{ 'conveyance'}$$

$$\partial_x h = I_S - I_R = -\partial_x Y_b - \frac{Q|Q|}{K^2}, \qquad K(h) = \frac{1}{n}\cdot\left(\frac{bh}{b+2h}\right)^{\frac{2}{3}}$$

$$b\partial_t h + \left(\frac{d}{dh}K\sqrt{I_S - \partial_x h}\right)\partial_x h = \left(\frac{1}{2}\frac{K}{\sqrt{I_S-\partial_x h}}\right)\partial_{xx} h$$

Modelhierarchy

Kinematic wave equation:

neglect $\partial_x h$ against $\partial_x Y_b$ (bed-slope)

$Q = \frac{b}{n} R(H)^{\frac{2}{3}} \sqrt{I_S} h$ (uniform flow)

$$b\partial_t h + \Big(\frac{d}{dh} K(h) \sqrt{I_S}\Big) \partial_x h = 0$$

Telegrapher's equation: linearization

$$\partial_{tt} h + (V_0^2 - h_0 g)\partial_{xx} h + 2V_0 \partial_{xt} h = gh\partial_x I_0$$

Network modelling: quasilinear form

$$\partial_t \begin{pmatrix} A_i \\ V_i \end{pmatrix} + \begin{pmatrix} V_i & A_i \\ \frac{g}{b(A_i)} & V_i \end{pmatrix} \partial_x \begin{pmatrix} A_i \\ V_i \end{pmatrix} = \begin{pmatrix} -Q_i^{\ell} \\ (I_{iS} - I_{iR}) \end{pmatrix}$$

at a multiple uncontrolled node we have:

$$h_i(v_J, t) + \frac{V_i^2(v_J,t)}{2g} = h_j(v_J, t) + \frac{V_j^2(v_J,t)}{2g} \quad \forall i, j \in \mathcal{I}_J$$
$$\sum_{i \in \mathcal{I}_J} \epsilon_{iJ} [A_i V_i]_i (v_J, t) = Q_J(t)$$

Boundary conditions

At a serial controlled node we have e.g.:

$$\boxed{Q(v_J,t) = f_J(t)\sqrt{2g(\epsilon_{iJ}h_i(v_J,t) + \epsilon_{jJ}h_j(v_J,t))}}$$

At a simple node we have e.g.: $Q_i(v_D,t) = 0$

We also have initial conditions.

Control of a single canal

We consider the perturbations $A = \overline{A} + a$ und $V = \overline{V} + v$, of a subcritical equilibrium. a and v satisfy:

$$\partial_t \begin{pmatrix} a \\ v \end{pmatrix} + \begin{pmatrix} \overline{V} + v & \overline{A} + a \\ g/b(\overline{A} + a) & \overline{V} + v \end{pmatrix} \partial_x \begin{pmatrix} a \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Boundary conditions:

$$v(0,t) = f^0(t, a(0,t)), \quad v(L,t) = f^L(t, a(L,t)), \quad t \in (0,T)$$

Initial conditions:

$$v(x,0) = v^0(x), \quad a(x,0) = a^0(x), \quad x \in (0,L)$$

Transparent boundary conditions

With

$$p(\alpha) = \sqrt{\frac{g}{[\overline{A} + \alpha]b(\overline{A} + \alpha)}}, \quad P(a) = \int_0^a p(\alpha)d\alpha$$

we can consider 'transparent boundary conditions':

$$v(0,t) = -P(a(0,t)), \quad v(L,t) = P(a(L,t)), \quad t \in (0,T)$$

Controllability

Thm(G. Schmidt and G.L. 02)
Let the St. Venant-system with initial- and boundary conditions

$$v(0,t) = f^0(a(0,t)) \quad \text{and} \quad v(L,t) = P(a(L,t))$$

be given. ($Df^0(0) \neq p(0)$). Let $T_\star$ satisfy

$$T_\star > \frac{L}{\overline{V} - \overline{\beta}} + \frac{L}{\overline{V} + \overline{\beta}}.$$

Then, for sufficiently small data $\| (a^0, v^0) \|_1$ there is a unique global solution satisfying

$$a(\cdot,t) \equiv 0 \quad \text{and} \quad v(\cdot,t) \equiv 0 \quad \text{for } t \geq T_\star.$$

Stabilizability for a star

Thm (G.Schmidt and G.L.02) Under the assumptions above, assume:

$$\| \nabla_{\xi\_} \mathbf{g}^{\mathbf{0}}(\mathbf{0}) \|_\infty \max_{\mathbf{1}\leq \mathbf{i} \leq \mathbf{n}} |\mathbf{D}\mathbf{g}_{\mathbf{i}}^{\mathbf{L}}(\mathbf{0})|$$

$$= \| \overline{\Lambda}_+^{-1} [G_+^{-1} G_-] \overline{\Lambda}_- \|_\infty \max_{1\leq i\leq n} \left| \frac{Df_i^L(0) + p_i(0)}{Df_i^L(0) - p_i(0)} \right| < 1. \quad (1)$$

Then, for $\| (\mathbf{a}^{\mathbf{0}}, \mathbf{v}^{\mathbf{0}}) \|_{\mathbf{1}}$ sufficiently small, there exists a unique continuously differentiable solution to the problem which is defined for all positive t and satisfies

$$\| (\mathbf{a}(\cdot, \mathbf{t}), \mathbf{v}(\cdot, \mathbf{t})) \|_{\mathbf{1}} < \mathbf{C}\mathbf{e}^{-\alpha \mathbf{t}} \| (\mathbf{a}^{\mathbf{0}}, \mathbf{v}^{\mathbf{0}}) \|,$$

where C and α are suitable positive constants.

Global controllability

Thm (Gugat and G.L. 03) From a constant subcritical state (ξ_-, ξ_+) (i.e. $|U| = |\xi_+ + \xi_-|/2 < c = |\xi_+ - \xi_-|/4$) the system can be steered to any other constant subcritical state by boundary controls in finite time with a continuously differentiable state.

A similar controllability result has been presented here by O. Glass
Even allowing for shocks

More on global controllability

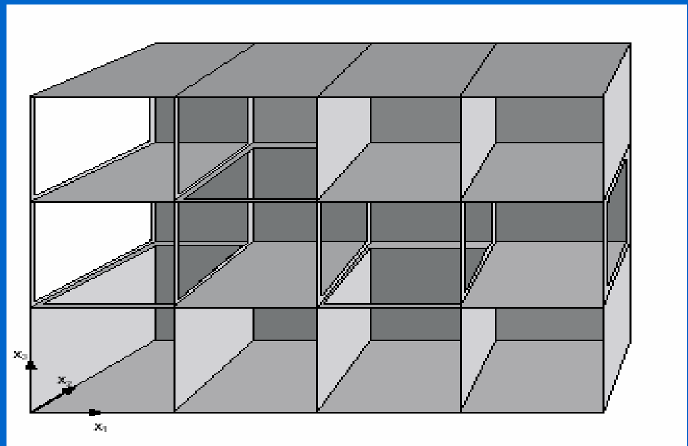
Corollary States close to a constant sub-critical state can be transfered free of shocks to states that are close to another subcritical state.

Has been extended to a tree-network even with sources and supercritical flows by
M. Gugat, G.Schmidt and G.L. 2004,
M. Gugat 2004 (also supercritical and subcritical modes)
see also Coron et. Al. 2003, Rao and Li Tatsien 03

Is it possible to achieve this kind of result even allowing for shocks by method presented by O. Glass?

Higher dimensional networks

(J.E. Lagnese and G.L. 2004)



2-D Membranes, Reissner-Mindlin and other plate models

$$W_i : \mathcal{P}_i \mapsto \mathbf{R}^m : x \mapsto W_i(x).$$

displacement

$$Q_{ie} W_i = Q_{je} W_j \qquad \text{on } e\ \Gamma_e = \Gamma_{je},\ e \in \mathcal{J}_M$$

$$a(W,\Phi) = \sum_{i\in I} \int_{\mathcal{P}_i} [A_i^{\alpha\beta}(W_{i,\beta} + B_i^{\beta} W_i)\cdot(\Phi_{i,\alpha} + B_i^{\alpha}\Phi_i) + C_i W_i \cdot \Phi_i]\, dx$$

$$a(W,\Phi) = \langle F, \Phi \rangle_{V'}, \qquad \forall \Phi \in \mathcal{V}$$

Questions

- Squeezing domains, singular perturbation lead to skeletons or grid- models? (Hale, Raugel, LeDret, Raoult, Cioranescu, Zhikov…..and more)
- Homogenization leads to simple substitute model ? (Tartar, Murat, Cioransecu, Papanicolau, Lions, St.-Jean-Paulin, Zhikov, Buttazzo, Bouchitte…….many pages)
- What does this all mean with respect to optimization, control and simulation ? (list gets smaller…..but still too large to display)
- Domain decomposition in space and time of optimal control problems? (Benamou, Lions, Turinici, Maday, Lagnese, G.L., Heinkenschloss et.al.
- Order of optimization, decomposition and discretization?

Optimal control problem Benamou and Brenier 2000

$$\min_{v,\rho} J(v,\rho) := \frac{1}{2}\int_0^T \int_{\mathbf{R}^d} \rho |v|^2 dx dt + \frac{k}{2}\|\rho(T,\cdot) - \rho_1\|^2$$

subject to

$$\partial_t \rho + \nabla(v\rho) = 0$$

$$\rho(\cdot,0) = \rho_0,\ \rho(\cdot,T) = \rho_1$$

MKP and fluid analogue: related work

$$\partial_t \rho + \nabla(v\rho) = 0$$

$$\partial_t \phi + v \cdot \nabla \phi = \frac{1}{2}|v|^2$$

$$v = \nabla \phi$$

$$\partial_t \phi + \frac{1}{2}\|\nabla \phi\|^2 = 0$$

Hamilton-Jacobi-eqn

$$\rho(\cdot, 0) = \rho_0, \; \phi(T) = -k(\rho(T) - \rho_T)$$

Chartrand, Vixie and Wohlberg 2005, Benamou, Brenier 2000-2003, K. Ito 2005, Cafarelli, Feldman and McCann 2001--, Gangbo et.al. 1996--, Haker, Tannenbaum 2005, Buttazzo and group…. See the working session

Remarks on a 1-D setting

$$\partial_t \phi + v \cdot \partial_x \phi = \frac{1}{2} v^2, \; v = \partial_x \phi$$

$$\partial_t \phi + \frac{1}{2}[\partial_x \phi]^2 = 0$$

$\Leftrightarrow$ under extra smoothness $+\epsilon g \rho$

$$\partial_t v + \partial_x [\frac{1}{2}(v^2)] = 0$$

St. Venant !

viscosity solutions of the HJB-eqn. are then equivalent to entropy solutions of the conservation law! (Dafermos, Conway...)

Monge-Kantorovich on a grid

$$\min_{\rho,v} \frac{1}{2} \int_0^T \sum_{i\in\mathcal{I}} \int_0^{\ell_i} \rho_i(t,x)|v_i(t,x)|^2 dx dt$$

s.t.

$$\partial_t \rho_i + \partial_x(v_i \rho_i) = 0$$

$$\rho_i(x_J) = \rho_J,\ i \in \mathcal{I}_J^{\rightarrow}$$

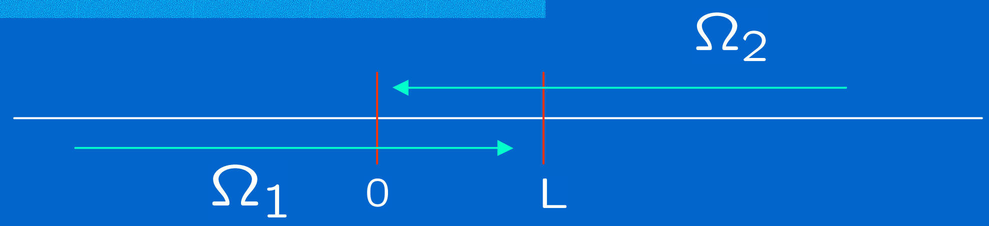
$$\sum_{\mathrm{l}\in\mathcal{I}_J^{\rightarrow}} d_{iJ}(\rho_i v_i)(x_J) = - \sum_{k\in\mathcal{I}_J^{\leftarrow}} d_{kJ}(v_i \rho_i)(x_J)$$

$$\rho_i(0,\cdot) = \rho_0,\ \rho_i(T,\cdot) = \rho_1$$

Wellposedness, controllability, stabilizability,

- Linear equations: J.E. Lagnese, E.J.P.G. Schmidt and G.L since 1989, S. Nicaise 1990ies, M. Belishev 1998-04, S. Avdonin 1989, 1998, E.Zuazua and G.L. 1998, E. Zuazua and R. Dager 2005……
- Nonlinear equations: E.J.P.G. Schmidt and G.L. 2002, M. Gugat and G.L. 2003-2005, M.Gugat, E.J.P.G. Schmidt and G.L. 2004, J.-M. Coron, Bastin, d'Andrea-Novel 2004
- Homogenization: R. Hoppe and I.Petrova 2004, P. Kogut and G.L. 2005, Y. Maday 2004
- Singular perturbations: T. Fischer and G.L. 2002 , A. Quarteroni et. al. 2003, Canic 2005, Canic and Mikelic 2005…………..

Overlapping domain decomposition



$$\partial_t u_1^n + \partial_x f(u_1^n) = 0 \qquad \text{in } \Omega_1 \times (0,T)$$
$$u_1^n(\cdot,0) = u_0 \qquad \text{in } \Omega_1$$
$$u_1^n(L,\cdot) = u_2^{n-1}(L,\cdot) \qquad \text{on } [0,T]$$

$$\partial_t u_2^n + \partial_x f(u_2^n) = 0 \qquad \text{in } \Omega_2 \times (0,T)$$
$$u_2^n(\cdot,0) = u_0 \qquad \text{in } \Omega_1$$
$$u_2^n(0,\cdot) = u_1^{n-1}(0,\cdot) \qquad \text{on } [0,T]$$

Theorem: Gander and Rhode 03

Let $\underline{u} = \text{essinf}_{x\in\mathbf{R}} u_0(x), \bar{u} = \text{esssup}_{x\in\mathbf{R}} u_0(x)$.
For $n \in \mathbf{N}$ let $T^* = T^*(n, L) = \frac{nL}{\lambda}$ with

$$\lambda = \sup_{\underline{u}\leq v\leq\bar{u}} \{|f'(v)|\}.$$

Then for $t \in [0, T] \cap [0, T^*(n, L)]$

$$\|e_1^{n+1}\|^2_{L^1(\infty,L]\times[0,t])} = \|e_2^{n+1}\|^2_{L^1([0,\infty)\times[0,t])} = 0$$

The algorithm terminates in a finite number of steps!

Domain decomposition for systems?

- No analysis seems to be available for nonlinear systems of conservation laws and those with source terms. See however A. Quarteroni‘s DD-book for methods…
- Linear hyperbolic systems: apply nonoverlapping Schwarz
- For the linearized system in 2-d with semidiscretization in time see e.g. S. Clerc 00
- Work in progress based on Riemann invariants…..

Schwarz ‚is Robin'....

- If applied to the transformed wave equation, the Schwarz iteration is equivalent to a Robin-type iteration for the original wave equation:

$$w_{1,x}^{k+1}(L,t)+\frac{1}{c}w_{1,t}^{k+1}(L,t) = w_{2,x}^{k}(0,t)+\frac{1}{c}w_{2,t}^{k}(0,t),$$

$$w_{2,x}^{k+1}(0,t)-\frac{1}{c}w_{2,t}^{k+1}(0,t) = w_{1,x}^{k}(L,t)-\frac{1}{c}w_{1,t}^{k}(L,t).$$

Thus nonoverlapping Schwarz leads to to transparent interfaces. Hence optimized Robin-DDM is adequate for wave propagation!

Domain decomposition: wave propagation on networks

$$\min \frac{1}{2}\sum_{i=1}^{n}\int_0^T\int_0^{\ell_i} |y_i - y_i^d|^2 dx dt + \frac{\alpha}{2}\sum_{J\in\mathcal{J}_S}\int_0^T |u(v_J)]|^2 dt$$
$$+\frac{k}{2}\sum_i \|(y_i(T,\cdot),\dot y_i(T,\cdot) - (y_T,\dot y_T)\|_{\mathcal{E}}^2$$

$$\ddot y_i - y_i'' + \sigma y_i = f_i, \qquad \text{in } (0,\ell_i)\times[0,T]$$
$$y_i'(v_N) = u_i, \qquad \forall i\in\mathcal{I}_N,\ t\in(0,T)$$
$$y_i(v_J) = y_j(v_J), \qquad i,j\in\mathcal{I}_J,\ t\in(0,T)$$
$$\sum_{i\in\mathcal{I}_J} d_{iJ} y_i'(v_J) = 0, \qquad J\in\mathcal{J}_M,\ t\in(o,T)$$
$$y_i(0,\cdot) = y_i^0,\ \dot y_i(0,\cdot) = y_i^1\ x\in[0,\ell_i]$$

Optimality system

$$\ddot{y}_i - y_i'' + \sigma y_i = f_i$$
$$y_i'(v_N) = u_i,\ N \in \mathcal{J}_N$$
$$y_i(v_J) = y_j(v_J),\ \forall i,j \in \mathcal{I}_J,\ J \in \mathcal{J}_M$$
$$\sum_{i\in\mathcal{I}_J} d_{iJ} y_i'(v_J) = 0,\ J \in \mathcal{J}_M$$
$$y_i(0) = y^0,\ \dot{y}_i(0) = y^1$$

forward

$$\alpha u_i - p_i(v_N) = 0\ N \in \mathcal{J}_N$$

optimality cond.

$$\ddot{p}_i - p_i'' + \sigma p_i = -(y_i - y_i^d)$$
$$p_i'(v_N) = 0,\ N \in \mathcal{J}_N$$
$$p_i(v_J) = p_j(v_J),\ \forall i,j \in \mathcal{I}_J,\ J \in \mathcal{J}_M$$
$$\sum_{i\in\mathcal{I}_J} d_{iJ} p_i'(v_J) = 0,\ J \in \mathcal{J}_M$$
$$p_i(T) = 0,\ \dot{p}_i(T) = 0$$

backward adjoint

Dirchlet, Neuman-type approach(es)

$$\min \frac{1}{2}\int_0^T\int_0^{\ell_i} |y_i - y_i^d|^2 dx dt + \frac{\alpha}{2}\int_0^T u_i^2 dt + \sum_J \int_0^T d_{iJ} y_i'(v_J) p_J dt$$

$$\ddot{y}_i - y_i'' + \sigma y_i = f_i$$
$$y_i'(v_N) = u_i,\; N \in \mathcal{J}_N$$
$$y_i(v_J) = y_J,\; \forall i,j \in \mathcal{I}_J,\; J \in \mathcal{J}_M$$
$$y_i(0) = y_i^0,\; \dot{y}_i(0) = y_i^1$$

$$\ddot{p}_i - p_i'' + \sigma p_i = -(y_i - y_i^d)$$
$$p_i'(v_N) = 0,\; N \in \mathcal{J}_N$$
$$p_i(v_J) = p_J,\; \forall i,j \in \mathcal{I}_J,\; J \in \mathcal{J}_M$$
$$p_i(T) = 0,\; \dot{p}_i(T) = 0$$
$$\alpha u_i - p_i(v_N) = 0\; N \in \mathcal{J}_N$$

‚Steklov operators‘

Given boundary data $(y_J, p_J),\ J \in \mathcal{J}_J$ solve

$$\begin{aligned} \ddot{y}_i - y_i'' + \sigma y_i &= f_i \\ y_i'(v_N) &= u_i,\ N \in \mathcal{J}_N \\ y_i(v_J) &= y_J,\ \forall i,j \in \mathcal{I}_J,\ J \in \mathcal{J}_M \\ y_i(0) &= y_i^0,\ \dot{y}_i(0) = y_i^1 \end{aligned}$$

$$\begin{aligned} \ddot{p}_i - p_i'' + \sigma p_i &= -(y_i - y_i^d) \\ p_i'(v_N) &= 0,\ N \in \mathcal{J}_N \\ p_i(v_J) &= p_J,\ \forall i,j \in \mathcal{I}_J,\ J \in \mathcal{J}_M \\ p_i(T) &= 0,\ \dot{p}_i(T) = 0 \end{aligned}$$

$$\alpha u_i - p_i(v_N) = 0,\ N \in \mathcal{J}_N$$

such that $y_i = \tilde{y}_i + \hat{y}_i\ p_i = \tilde{p}_i + \hat{p}_i$ with

$$(\tilde{y}_i, \tilde{p}_i) \leftrightarrow [f_i, y_i^d, y_i^0, y_i^1, y_J = p_J = 0]$$

$$(\hat{y}_i, \hat{p}_i) \leftrightarrow [f_i = y_i^d = y_i^0 = y_i^1 = 0, y_j, p_j]$$

Schur complement approach

Then restore the Kirchhoff condition at multiple joints

$$\sum_{i\in\mathcal{I}_J} d_{iJ} y_i'(v_J) = \sum_{i\in\mathcal{I}_J} d_{iJ}\hat{y}_i'(v_v) + \sum_{i\in\mathcal{I}_J} d_{iJ}\tilde{y}_i'(v_J) = 0 \; J \in \mathcal{J}_M$$

with $\mathcal{P}_i^y(y_J, p_J) := d_{iJ}\hat{y}_i'(v_J),\; J \in \mathcal{J}_M$ this reads as

$$\sum_{i\in\mathcal{I}_J} \mathcal{P}_i(y_J, p_J) + \sum_{i\in\mathcal{I}_J} d_{iJ}\tilde{y}_i'(v_J) = 0,\; J \in \mathcal{J}_M$$

accordingly for $\mathcal{P}_i^p(y_J, p_J)$.

Neumann-Neumann preconditioner

One defines the resiude of the global equation

$$r_J^y := \sum_{i\in\mathcal{I}_J} \mathcal{P}_i^y(y_J, p_J) + \sum_{i\in\mathcal{I}_J} d_{iJ}\tilde{y}_i'(v_J),\ J \in \mathcal{J}_M$$

accordingly for r_J^p,

and uses the local solves $(\mathcal{P}_i^y)^{-1}[-d_{iJ}\tilde{y}_i'(v_J)]$

$$\ddot{y}_i - y_i'' + \sigma y_i = 0$$
$$y_i(v_N) = 0,\ N \in \mathcal{J}_N \quad d_{iJ}y_i'(v_J) = r_J^y,\ J \in \mathcal{J}_M$$
$$\ddot{p}_i - p_i'' + \sigma p_i = 0$$
$$p_i(v_N) = 0,\ N \in \mathcal{J}_N, \quad d_{iJ}p_i'(v_J) = r_J^p\ J \in \mathcal{J}_M$$

+ initial and final data

as preconditioners for the global equation.

Robin-Robin-approach

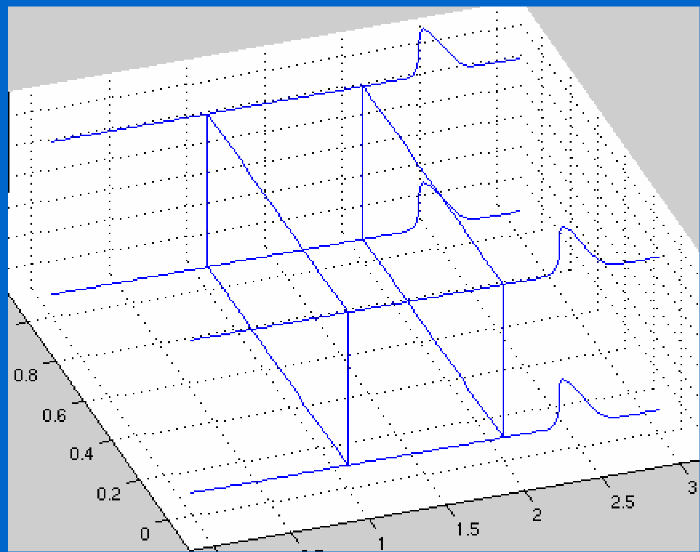
$$
\begin{gathered}
\ddot{y}_i - y_i'' + \sigma y_i = f_i \\
d_{iN} y_i'(v_N) + \alpha \dot{y}_i(v_N) + \dot{p}_i(v_N) = 0,\ N \in \mathcal{J}_N \\
d_{iJ} y_i'(v_J) + \beta \dot{y}_i(v_J) + \gamma_J \dot{p}_i(v_J) = \lambda_{iJ},\ \forall i,j \in \mathcal{I}_J,\ J \in \mathcal{J}_M \\
y_i(0) = y_i^0,\ \dot{y}_i(0) = y_i^1
\end{gathered}
$$

$$
\begin{gathered}
\ddot{p}_i - p_i'' + \sigma p_i = -(y_i - y_i^d) \\
d_{iN} p_i'(v_N) - \alpha \dot{p}_i(v_N) = 0,\ N \in \mathcal{J}_N \\
d_{iJ} p_i'(v_J) - \beta_J \dot{p}_i(v_J) + \gamma \dot{y}_i(v_J) = \mu_{iJ},\ \forall i,j \in \mathcal{I}_J,\ J \in \mathcal{J}_M \\
p_i(T) = 0,\ \dot{p}_i(T) = 0
\end{gathered}
$$

Update at transmission boundaries

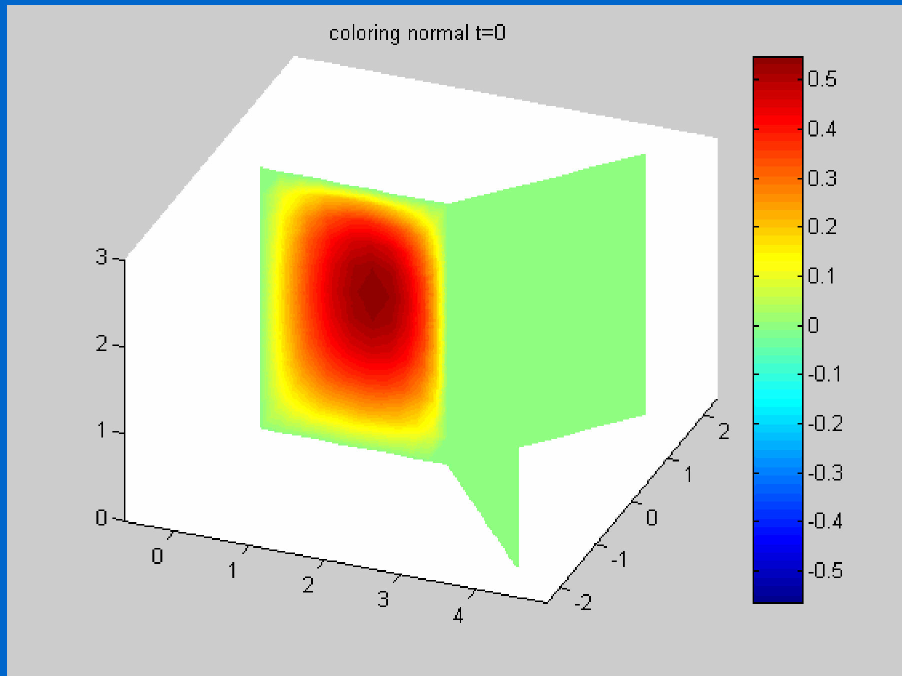
$$\lambda_{iJ}^k := \gamma_J \left(\frac{2}{d_J} \sum_{j \in \mathcal{I}_J} \dot{p}_j^k(v_J) - \dot{p}_i^k(v_J) \right)$$
$$+ \beta_J \left(\frac{2}{d_J} \sum_{j \in \mathcal{I}_J} \dot{y}_j^k(v_J) - \dot{y}_i^k(v_J) \right)$$
$$\left(\frac{2}{d_J} \sum_{j \in \mathcal{I}_J} d_{jJ} (y_j^k)'(v_J) - d_{iJ} (y_i^k)'(v_J) \right)$$

Wave propagation in a network



Computations by R. Hundhammer 02

A simple 2-D networked wave problem



Computation by W. Rathmann 03

Remarks

- The Robin-type DDM orginated by P.L. Lions 89 been extended to 2-D/3-D elliptic and wave equations on networked domains (Benamou 1996, J.E. Lagnese and G.L. 1998-2004), monograph by J.L. and G.L. 2004
- Can be extended to parabolic equations on networked domains (not yet published)
- Can be used for time domain decomposition for wave equations, where parareal-decomposition is known to perform poorly.
- Important feature: DDM of optimality system leads to optimality system of local problem
- Convergence theory available (J.L. and G.L.)
- A posteriori error estimates available (Otto and Lube 2000, J.L. and G.L. 2000-2004)

Homogenization

Consider $\epsilon > 0$. Find $y^\epsilon \in V$ such that

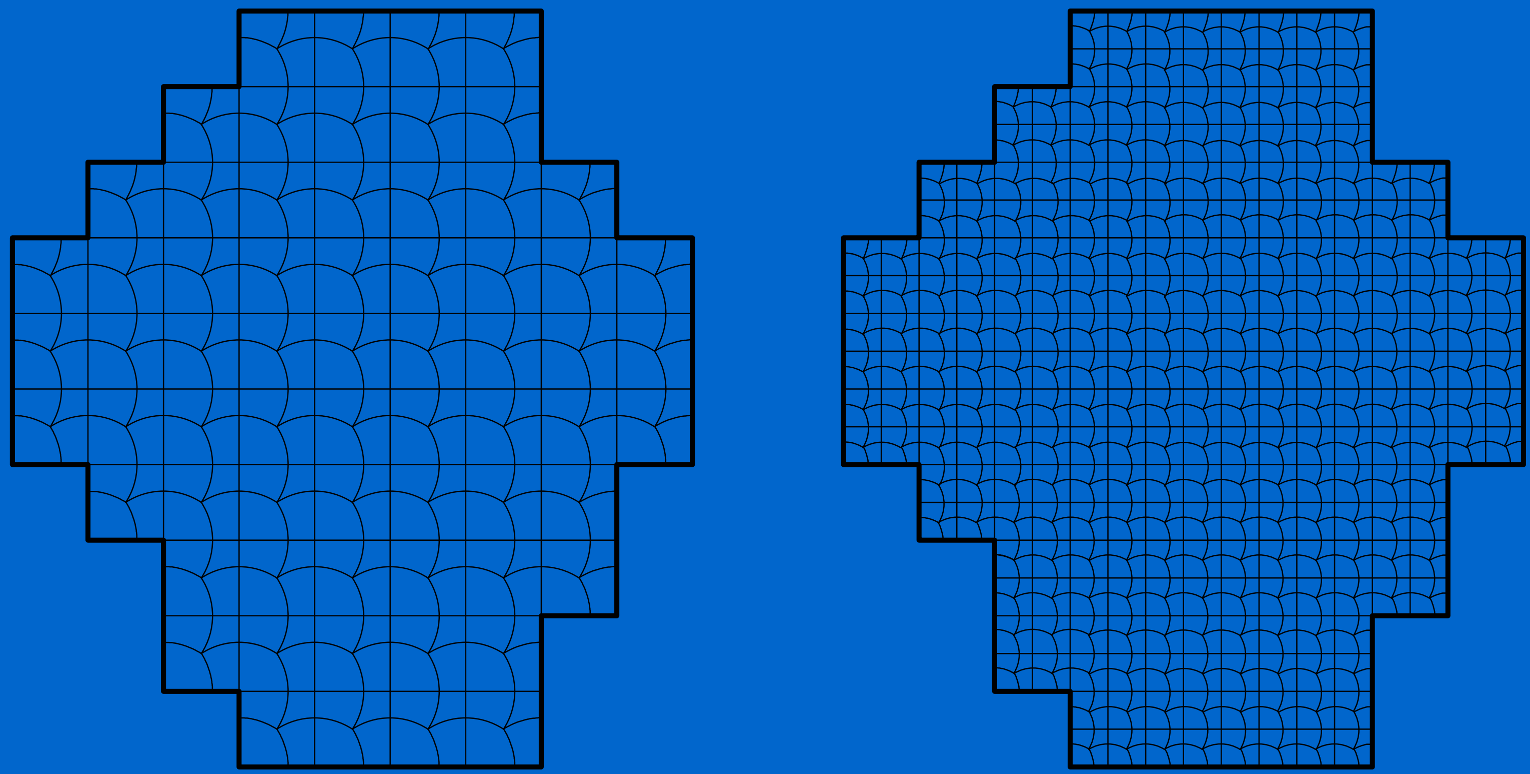
$$\sum_{i=1}^{d} \int_\Omega A^\epsilon(x) \frac{\partial y^\epsilon}{\partial x_i} \frac{\partial v}{\partial x_i} dx = \int_\Omega f^\epsilon v dx \quad \forall v \in V$$

$$\Leftrightarrow y^\epsilon := \text{ arg min }_{v \in V} \{ J^\epsilon(v) := \frac{1}{2} a^\epsilon(v,v) - (f^\epsilon, v) \}$$

In particular consider periodic coefficients

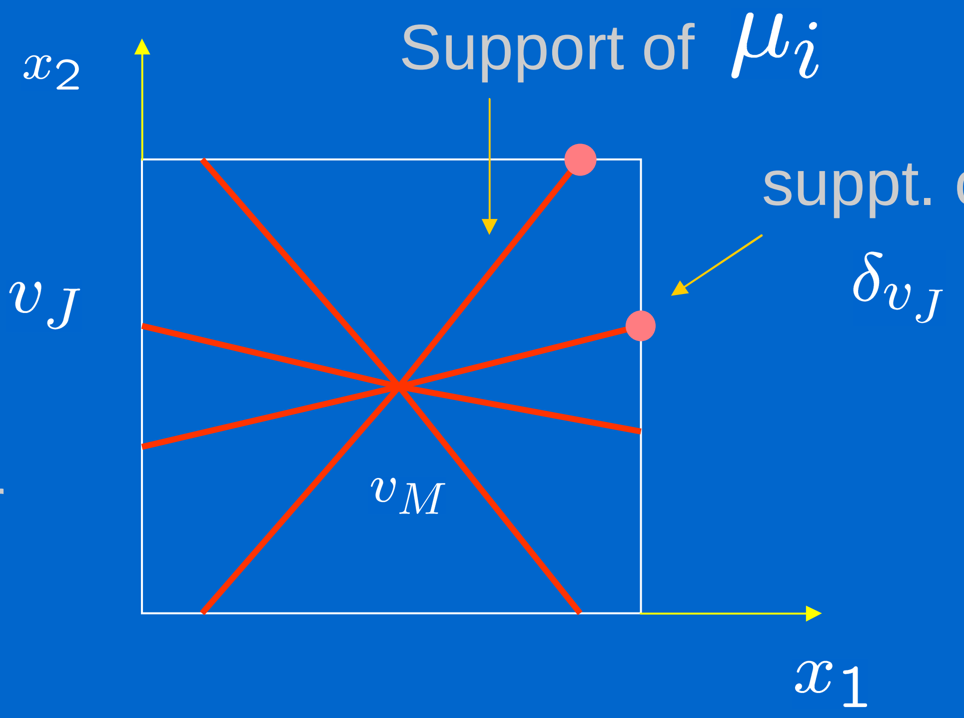
$$A^\epsilon(x) = A\left(\frac{x}{\epsilon}\right) \qquad \text{a.e. on } \mathbf{R}^d$$

Structure of periodic domains



The Cell Graph

Edges seen as support of measures, boundary as Sppt. of singular measures



$$\mathcal{F}_\epsilon := \epsilon\mathcal{F} = \{\epsilon x | x \in \mathcal{F}\}, \quad \Omega_\epsilon := \Omega \cap \mathcal{F}_\epsilon$$

$\mathcal{G}_\epsilon$ accordingly

Sobolev Space wrt. Measures (Zhikov 00)

y belongs to $V_{\Gamma_2}(\Omega, d\mu)$, if

$$\exists z \in L^2(\Omega, d\mu)^2,\ y_m \in C^\infty_{\Gamma_2}(\Omega), \quad \text{s.t.}$$

$$\lim_{m\to\infty} \int_\Omega (y_m - y)^2 d\mu = 0,$$

$$\lim_{m\to\infty} \int_\Omega (\nabla y_m - z)^2 d\mu = 0$$

Then z is called *a* gradient of y. The set of gradients of y is denoted

$$\Gamma(y) := \nabla y + \Gamma(0)$$

Proposition Zhikov 00

Let $\mathcal{F}$ be a $\square$-periodic graph, μ the $\square$-periodic Borel measure above, and $y \in V_{\Gamma_2}(\Omega, d\mu)$. Then

1. $y|_{I_i} \in H^1(I_i)$ for any edge e_i s.t. $I_i \in \Omega \cap \mathcal{F}$,

2. $y|_{\Omega \cap \mathcal{F}}$ is continuous.

Variational Format: relation to equations on the 1-d network

$y \in V_{\Gamma_2}(\Omega, d\mu_\epsilon)$ is a solution of the elliptic problem $\mathcal{P}_\epsilon$ on the graph, if

$$\int_\Omega (A_\epsilon(x)\nabla y, \nabla\phi)d\mu_\epsilon + \int_\Omega \lambda y\phi d\mu_\epsilon = \int_\Omega u\phi d\mu_\epsilon + \int_{\Gamma_1} h\phi d\mu_\epsilon^s, \;\; \forall \phi \in C^\infty_{\Gamma_2}(\Omega)$$

Variational convergence of constrained optimization problems on varying sets

We consider the sequence of constrained optimization problems

$$\inf_{(y,u,h)\in\Theta_\epsilon} J_\epsilon(y,u,h)$$

such that

i.) $\Theta_\epsilon \subset H^1(\Omega, d\mu_\epsilon)\times L^2(\Omega, d\mu_\epsilon)\times L^2(\Gamma_1, d\mu_\epsilon^s)$

ii.) $J_\epsilon : \Theta_\epsilon \longrightarrow \bar{\mathrm{R}}$

need: Γ-type convergence for J_ϵ and some set-convergence for Θ_ϵ!

Sequence of optimization problems

- $Graph(A_\epsilon) := \{(y,u,h) | y \text{ solves } \mathcal{P}_\epsilon\}$
- The admissible set Θ_ϵ is given by

$$\Theta_\epsilon := \left\{ (y,u,h) \,\middle|\, \begin{array}{l} (y,u,h) \in Graph(A_\epsilon), \\ |u| \le c_u\ \mu_\epsilon - a.e. \ \text{ in } \Omega \\ |h| \le c_h\ \mu_\epsilon^s - a.e. \ \text{ in } \Gamma_1 \end{array} \right\}$$

- For each ϵ the optimal control problem

$$\inf_{(y,u,h)\in\Theta_\epsilon} J_\epsilon(y,u,h),$$

$$J_\epsilon(y,u,h) := k_1 \int_\Omega (y-y_d)^2 d\mu_\epsilon + k_2 \int_\Omega u^2 d\mu_\epsilon + k_3 \int_{\Gamma_1} h^2 d\mu\epsilon^s$$

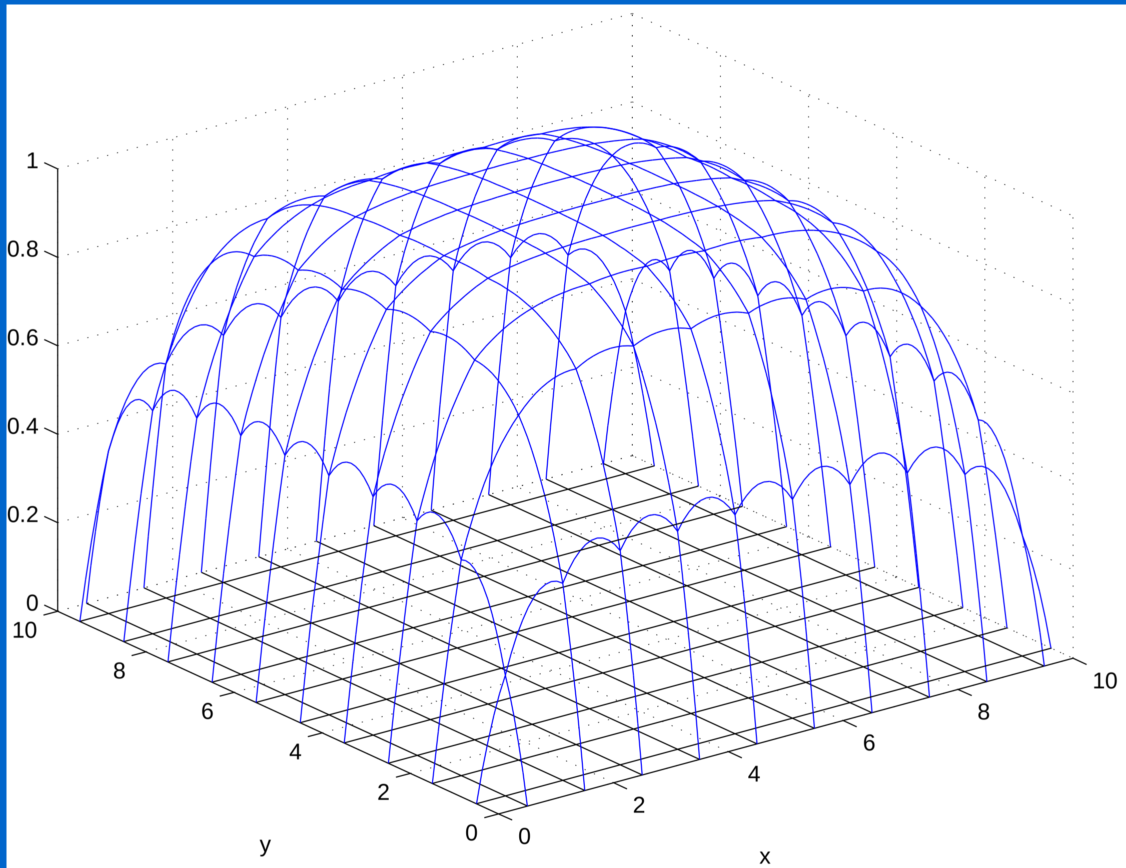
admits a unique solution.

Homogenized problem (Kogut and G.L. 2005)

There exists a **unique** homogenized optimal control problem which has the following representation:

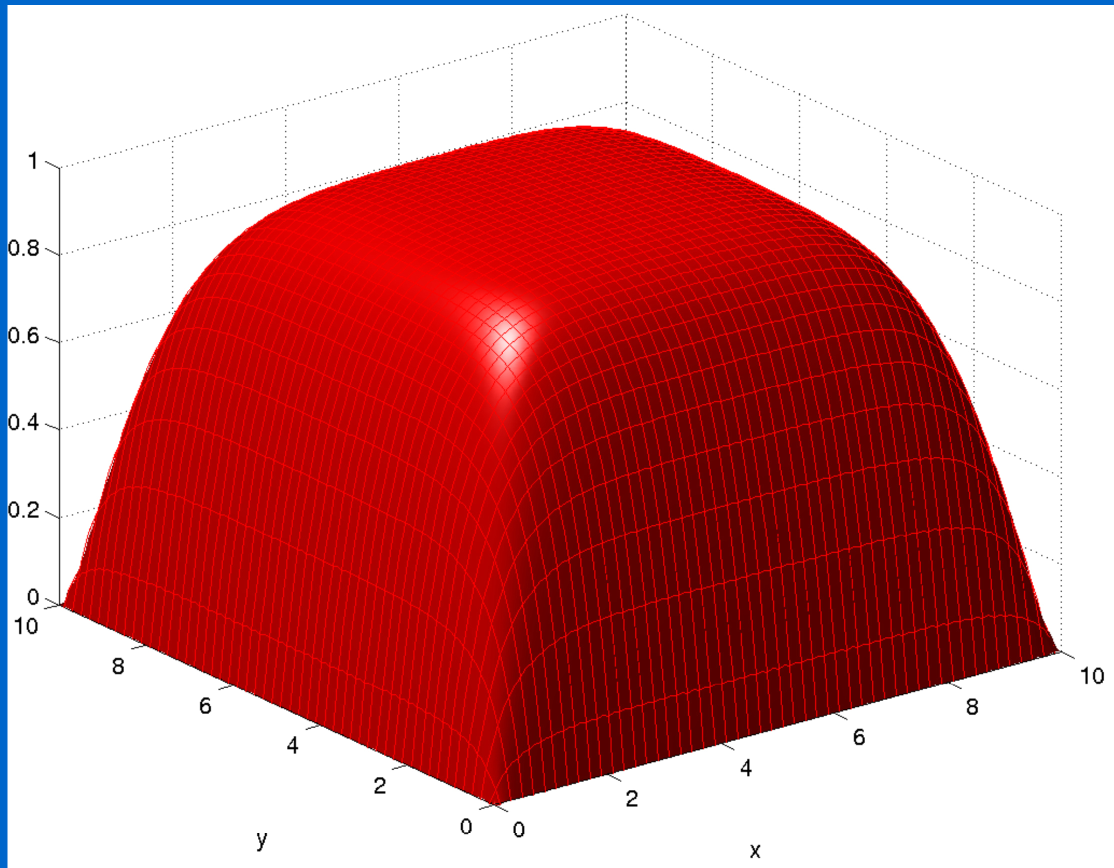
$$-\text{div}(A^{hom}(x)\nabla y) + \gamma y = u \quad \text{in } \Omega$$
$$y = 0 \quad \text{on } \Gamma_2$$
$$\frac{\partial}{\partial \nu_{A^{hom}}} y = h \quad \text{on } \Gamma_1$$
$$|u| \leq c_u, \quad |h| \leq c_h \quad \text{a.e.}$$
$$J_0(y,u,h) = k_1 \int_\Omega (y - z_d)^2 dx + k_2 \int_\Omega u^2 dx + k_3 \int_{\Gamma_1} h^2 d\gamma$$

Numerical simulations (Kropat 2005)

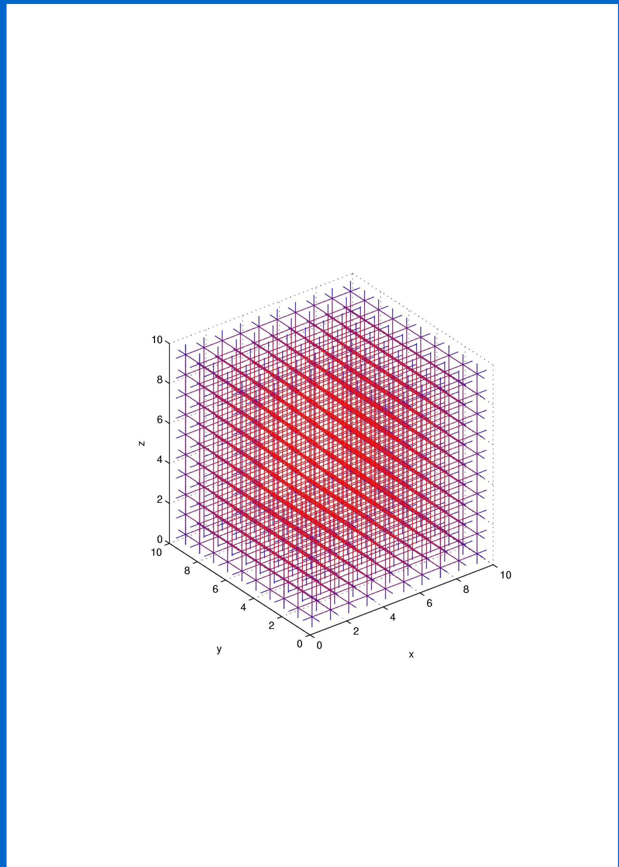


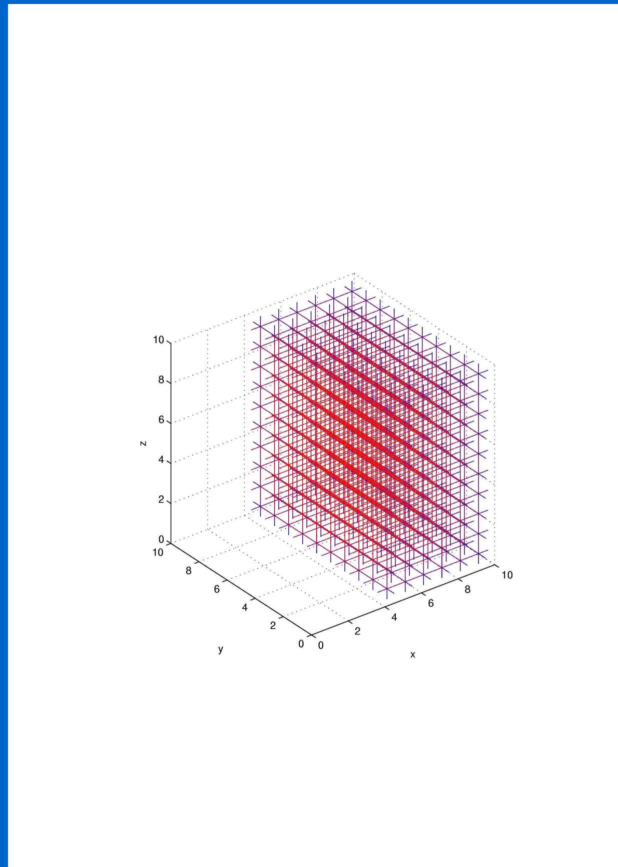
Computations by E. Kropat 05

Homogenized problem



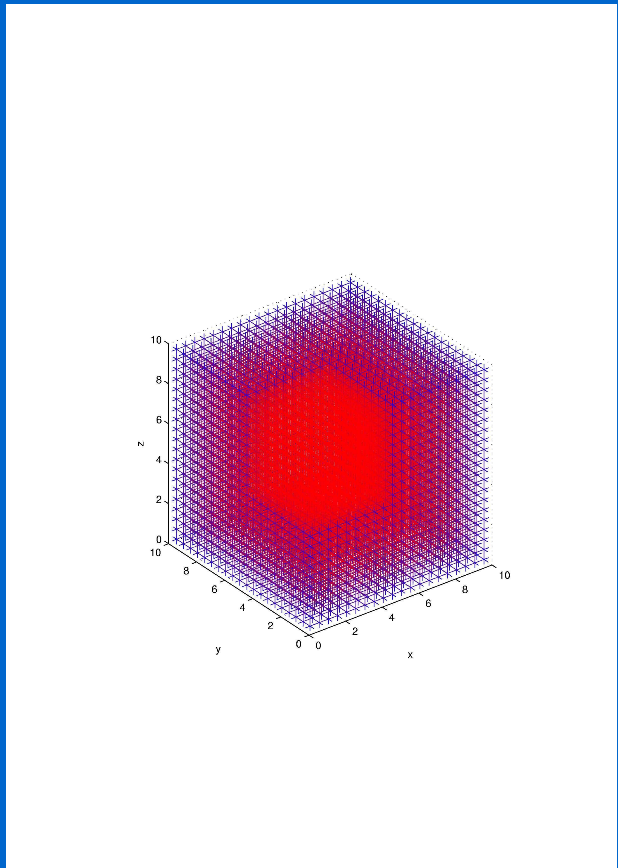
3-D Diffusion-reaction problem

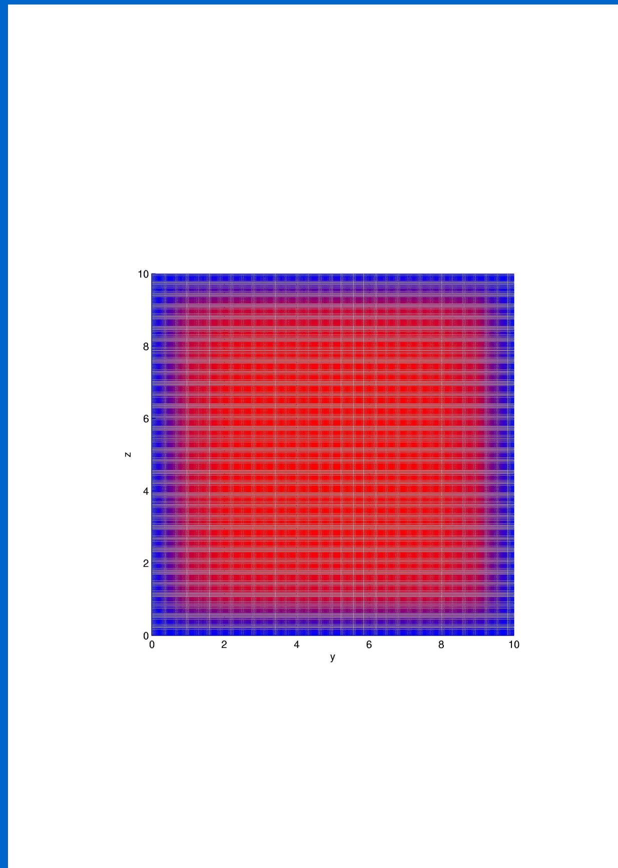




6000 edges, $\epsilon = 1$

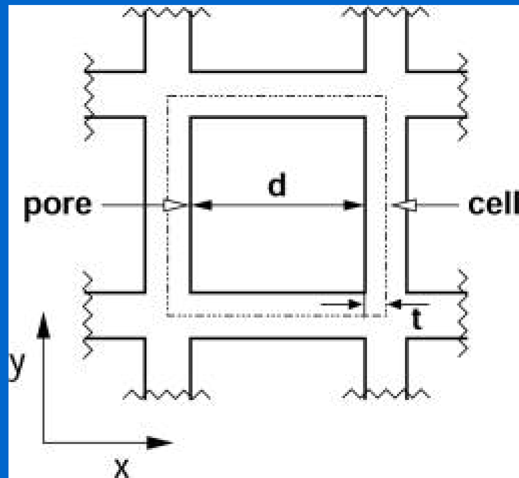
3-D Diffusion-reaction problem





48000 edges, $\epsilon = .5$ slice

Fattenened Graphs



$$A_{\epsilon,\delta}, y_{\epsilon,\delta}, u_{\epsilon,\delta}, h_{\epsilon,\delta} \text{ etc.}$$

Commutativity

$$
\begin{array}{ccc}
A_{\epsilon,\delta} & \xrightarrow{\delta \to 0} & A_{\epsilon}^{sing} \\
\Big\downarrow \epsilon \to 0 & & \Big\downarrow \epsilon \to 0 \\
A_{\delta}^{hom} & \xrightarrow{\delta \to 0} & A^{hom}
\end{array}
$$

…. As a resume on this topic

- Grid-strucutures via singular measures
- Variational formulation of grid problems in Sobolev spaces on varying ‚manifolds‘
- Constrained optimal control problem on each scale
- Variational two-scale limits of OCP
- Domain decomposition of such problems
- 2-D and 3-D Honeycombstrucutres
- Shape and sizing problems in this context

Conclusion

- Network modelling very important: many new applications
- Source of open problems for the mathematical analysis and numerics
- Many interesting problems for homogenization and hierarchical numerical schemes
- Open problems in optimal control, in particular control of coefficients
- Optimal shape, optimal material, optimal topology
- Hybrid mixed integer and continuous problems
- Multilevel optimization