The Neumann boundary condition for the $\infty$-Laplacian and the Monge-Kantorovich mass tranfer problem for measures supported on surfaces

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Benasque, 2005

## The $\infty$-Laplacian.

Let $\Delta_{p} u=\operatorname{div}\left(|D u|^{p-2} D u\right)$ be the usual $p$-laplacian. The limit operator $\lim _{p \rightarrow \infty} \Delta_{p}=\Delta_{\infty}$ is the $\infty$-Laplacian given by

$$
\Delta_{\infty} u=\sum_{i, j=1}^{N} \frac{\partial u}{\partial x_{j}} \frac{\partial^{2} u}{\partial x_{j} \partial x_{i}} \frac{\partial u}{\partial x_{i}}=D u\left(D^{2} u\right) D u .
$$

Indeed, formally,

$$
\operatorname{div}\left(|D u|^{p-2} D u\right)=|D u|^{p-2} \Delta u+(p-2)|D u|^{p-4} D u\left(D^{2} u\right) D u .
$$

Hence

$$
\begin{aligned}
& 0=\frac{|D u|^{2} \Delta u}{p-2}+\Delta_{\infty} u \\
& 0=0 \quad 1 \quad 0 \quad+\Delta_{\infty} u .
\end{aligned}
$$

The $\Delta_{\infty}$ appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function $f$, see
G. Aronsson, M.G. Crandall, P. Juutinen. (2004).

Notice that $\Delta_{\infty}$ is degenerate and is not in divergence form.

We want to study natural Neumann boundary conditions for the $\infty$-Laplacian. To this end consider the natural Neumann problem for the $p$-Iaplacian

$$
\begin{cases}-\Delta_{p} u=0 & \text { in } \Omega \\ |D u|^{p-2 \frac{\partial u}{\partial \nu}=g} & \text { on } \partial \Omega\end{cases}
$$

The boundary datum $g$ is continuous and verifies

$$
\int_{\partial \Omega} g=0
$$

If we impose the normalization

$$
\int_{\Omega} u_{p}=0
$$

then there exists a unique solution, $u_{p}$, that can be obtained by a variational principle,

$$
\max \left\{\int_{\partial \Omega} w g \quad: w \in W^{1, p}(\Omega), \int_{\Omega} w=0, \int_{\Omega}|D w|^{p} \leq 1\right\}
$$

Our first result states that there exist limit points of $u_{p}$ as $p \rightarrow \infty$ and that they are maximizers of a variational problem analogous to the one above.

For $q>N$ the set $\left\{u_{p}\right\}_{p>q}$ is bounded in $C^{\alpha}(\bar{\Omega})$. Hence, by compactness, we have subsequences that converge uniformly. Let $v_{\infty}$ be a uniform limit of a subsequence $\left\{u_{p_{i}}\right\}, p_{i} \rightarrow \infty$.

Theorem A limit function $v_{\infty}$ belongs to $W^{1, \infty}(\Omega)$ and is a solution to the maximization problem

$$
\max \left\{\int_{\partial \Omega} w g \quad: w \in W^{1, \infty}(\Omega), \int_{\Omega} w=0,\|D w\|_{L^{\infty}(\Omega)} \leq 1\right\}
$$

In particular

$$
\left\|D v_{\infty}\right\|_{L^{\infty}(\Omega)} \leq 1
$$

## Monge-Kantorovich mass transport problem

Recall Buttazzo's lecture.

Given two probability densities in $R^{N}, f_{1}, f_{2}$, there exists a map $T: R^{N} \rightarrow R^{N}$ such that

$$
\int_{A} f_{1}(x) d x=\int_{T(A)} f_{2}(y) d y
$$

and minimizes

$$
\inf _{T} \int|y-T(y)| f_{2}(y) d y . \quad\left(\inf _{T} \int c(y, T(y)) f_{2}(y) d y\right)
$$

This is a widely studied problem. Caffarelli, Ambrosio, Brenier, Buttazzo, Bouchitte, Ekeland, McCann, Kantorovich, Villani, Evans, Gangbo, Pratelli, De Pasquale, Trudinger, etc.

In our case, the maximization limit problem is obtained also by looking to a dual formulation of the mass transfer problem for the measures $\mu^{+}=g^{+} \mathcal{H}^{N-1} \mid \partial \Omega$ and $\mu^{-}=g^{-} \mathcal{H}^{N-1} \mid \partial \Omega$ that are supported on $\partial \Omega$.
L. Ambrosio, 2003.

The mass transfer compatibility condition $\mu^{+}(\partial \Omega)=\mu^{-}(\partial \Omega)$ holds since $g$ verifies $\int_{\partial \Omega} g=0$.

That mass transfer problems are related to the limits of the $p$-laplacian was first noticed in
L.C. Evans and W. Gangbo, (1999).

Concerning the equation satisfied by the limit we have

Theorem A limit $v_{\infty}$ is a viscosity solution of

$$
\begin{cases}\Delta_{\infty} u=0 & \text { in } \Omega \\ B(x, u, D u)=0, & \text { on } \partial \Omega\end{cases}
$$

Here

$$
B(x, u, D u) \equiv \begin{cases}\min \left\{|D u|-1, \frac{\partial u}{\partial \nu}\right\} & \text { if } g>0 \\ \max \left\{1-|D u|, \frac{\partial u}{\partial \nu}\right\} & \text { if } g<0, \\ H(|D u|) \frac{\partial u}{\partial \nu} & \text { if } g=0\end{cases}
$$

and $H(a)$ is given by

$$
H(a)= \begin{cases}1 & \text { if } a \geq 1 \\ 0 & \text { if } 0 \leq a<1\end{cases}
$$

Viscosity solutions with Neumann boundary conditions.
G. Barles, (1993). M.G. Crandall, H. Ishii and P.L. Lions, (1992).

Definition. Let

$$
\begin{cases}F\left(x, D u, D^{2} u\right)=0 & \text { in } \Omega \\ B(x, u, D u)=0 & \text { on } \partial \Omega\end{cases}
$$

A lower semi-continuous function $u$ is a viscosity supersolution if for every $\phi \in C^{2}(\bar{\Omega})$ such that $u-\phi$ has a strict minimum at the point $x_{0} \in \bar{\Omega}$ with $u\left(x_{0}\right)=\phi\left(x_{0}\right)$ we have: If $x_{0} \in \Omega$ then we require

$$
F\left(x_{0}, D \phi\left(x_{0}\right), D^{2} \phi\left(x_{0}\right)\right) \geq 0
$$

and if $x_{0} \in \partial \Omega$ the inequality

$$
\max \left\{B\left(x_{0}, \phi\left(x_{0}\right), D \phi\left(x_{0}\right)\right), F\left(x_{0}, D \phi\left(x_{0}\right), D^{2} \phi\left(x_{0}\right)\right)\right\} \geq 0
$$

Remark. If $u_{p}$ is the solution with boundary data $g$ and $\widehat{u}_{p}$ is the solution with boundary data $\widehat{g}=\lambda g, \lambda>0$, then


Therefore the limit $u_{\infty}$ is the same if we consider any positive multiple of $g$ as boundary data and the same subsequence.

As a consequence the limit problem must be invariant by scalar multiplication of the data $g$.

Two examples. An interval. $\Omega=(-L, L), g(L)=-g(-L)>0$.
From the equation

$$
\left(\left|u_{p}^{\prime}\right|^{p-2} u_{p}^{\prime}\right)^{\prime}=0,
$$

with $\int_{-L}^{L} u_{p}=0$ we obtain $u_{p}(x)=C x$. From the boundary condition, $\left|u_{p}^{\prime}\right|^{p-2} u_{p}^{\prime}(L)=g(L)$ we get $C=(g(L))^{1 /(p-1)}$. Hence,

$$
v_{\infty}(x)=\lim _{p \rightarrow \infty} u_{p}(x)=\lim _{p \rightarrow \infty}(g(L))^{1 /(p-1)} x=x
$$

If we reverse the sign of $g$ then $v_{\infty}(x)=-x$.
In this example the limit depends only on the sign of $g$. However the conjecture that the limit $v_{\infty}$ depends only on the sign of $g$ is not true.

An Annulus. $\Omega=\left\{r_{1}<|x|<r_{2}\right\}$. Consider

$$
g_{0}(r)= \begin{cases}g_{1} & r=r_{1} \\ g_{2} & r=r_{2}\end{cases}
$$

$g_{1}>0, g_{2}<0$ two constants such that

$$
\int_{|x|=r_{1}} g_{1}+\int_{|x|=r_{2}} g_{2}=0
$$

Notice that the solutions $u_{p}$ are radial hence the limit $v_{\infty}$ must be a radial function. Direct integration shows that it must be a cone with gradient one,

$$
v_{\infty}(x)=C_{0}(x)=C-|x|
$$

In general this $v_{\infty}$ is not a maximizer for a different $g$ with $\operatorname{sign}(g)=\operatorname{sign}\left(g_{0}\right)$ and verifying the constraint

$$
\int_{\partial \Omega} g=0 .
$$

To see that, consider a displaced cone

$$
C_{x_{0}}(x)=C-\left|x-x_{0}\right| .
$$

$g$ could be modified, preserving the constraint and the sign, in order to have

$$
\int_{\partial \Omega} g(x) C_{0}(x) d x<\int_{\partial \Omega} g(x) C_{x_{0}}(x) d x .
$$

Hence, $C_{0}(x)$ is not a maximizer.

Therefore, there is no uniqueness for the limit PDE.

Uniqueness of the limit. Next, we deal with the uniqueness of the limit points of the family $\left\{u_{p}\right\}$ as $p \rightarrow \infty$.

We use that a limit is infinite harmonic in $\Omega$, a maximizer, and a geometric assumption involving $g$ and $\Omega$.

We need some geometric tools from Evans-Gangbo.
Let $\partial \Omega_{+}=\operatorname{supp}\left(g^{+}\right)$and $\partial \Omega_{-}=\operatorname{supp}\left(g^{-}\right)$.

Let $v_{\infty}$ a maximizer and define a transport ray as

$$
R_{x}=\left\{z ;\left|v_{\infty}(x)-v_{\infty}(z)\right|=|x-z|\right\} .
$$

Two transport rays cannot intersect in $\Omega$ unless they are identical. For every transport ray $R_{x}=\left[\begin{array}{ll}a & b\end{array}, a \in \partial \Omega_{+}\right.$and $b \in \partial \Omega_{-}$.

We define the transport set as
$\mathcal{T}\left(v_{\infty}\right)=\left\{\begin{array}{rl}z \in \bar{\Omega}: \exists x \in \partial \Omega_{+}, y \in \partial \Omega_{-}, & v_{\infty}(z)\end{array}=v_{\infty}(x)-|x-z|, ~\right.$.
Observe that this set $\mathcal{T}$ is closed. The union of the transport rays is the transport set $\mathcal{T}$ (Evans-Gangbo).

Proposition Suppose that $\Omega$ is a convex domain. Let $v_{\infty}$ be a maximizer with $\Delta_{\infty} v_{\infty}=0$, then

$$
\left|D v_{\infty}(x)\right|=1, \quad \text { for a.e. } x \in \mathcal{T}\left(v_{\infty}\right)
$$

Our geometric hypothesis for uniqueness is then

$$
\mathcal{T}\left(v_{\infty}\right)=\bar{\Omega} .
$$

Theorem There exists a unique infinite harmonic solution, $u_{\infty}$, that is a maximizer. Hence, the limit

$$
\lim _{p \rightarrow \infty} u_{p}=u_{\infty}, \quad \text { uniformly in } \Omega
$$

exists.
A uniqueness example. $\Omega=D=\{|(x, y)|<1\}$ a disk in $R^{2}$.
$g(x, y)>0$ for $x>0$ and $g(x, y)<0$ for $x<0$ with $\int_{\partial D} g=0$.
It is easy to see that $\mathcal{T}\left(v_{\infty}\right)=\bar{\Omega}$ and hence we obtain uniqueness of the limit.

