Numerical methods for the approximation of the control of the wave equation

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Plan

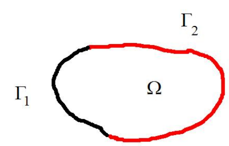
1. Introduction

Boundary controllability of the wave equation:

$$\begin{cases} u'' - \Delta u = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u(t,x) = 0 & \text{for } t > 0, \text{ and } x \in \Gamma_1 \\ u(t,x) = v(t,x) & \text{for } t > 0, \text{ and } x \in \Gamma_2 \\ u(0,x) = u^0(x) & \text{for } x \in \Omega \\ u'(0,x) = u^1(x) & \text{for } x \in \Omega \end{cases} \tag{1}$$

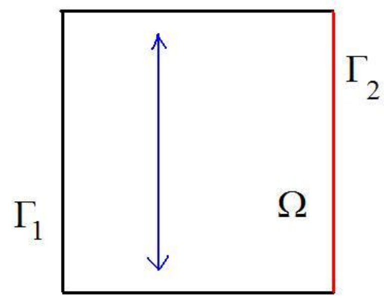
Problem: Given $T > 0$ and $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ find a control function $v \in L^2(\Gamma_1 \times (0,T))$ such that

$$u(T,\cdot) = u'(T,\cdot) = 0. \tag{2}$$



Two necessary conditions:

1. Time T_0 must be sufficiently large
2. Γ_2 must satisfy a geometric condition: no trapped rays



Theorem If the above conditions hold then the system is controllable

Problem Find a numerical approximation of the control.

Remark: In general, any discrete approximation of the wave equation produces numerical dispersion for the high frequencies.

The dispersion phenomenon may change the properties of the continuous model.

Example: Consider a uniform space mesh $0 < x_1 < x_2 < ... < x_{N+1} = 1$ and the following semi-discretization of the 1-d wave equation

$$u_j''(t) - \frac{u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)}{h^2} = 0$$

When considering solutions of the form $u_j = e^{i(\xi x_j - \omega t)}$ we obtain

$$\omega_d(\xi) = \pm \frac{2}{h} \sin\left(\xi h/2\right), \qquad \xi \in [-\pi/h, \pi/h]$$

Then, the group velocity

$$\frac{d}{d\xi}\omega_d(\xi) = \cos\left(\xi h/2\right) \rightarrow 0 \text{ when } \xi \rightarrow \pi/h$$

Thus, the velocity of propagation of some numerical waves becomes small as $h \rightarrow 0$.

2. HUM Method (J.-L. Lions)

$$\mathcal{J} : H_0^1(0,1) \times L^2(0,1) \to R$$

$$\mathcal{J}(w_T^0, w_T^1) = \frac{1}{2}\int_0^T \int_{\Gamma_2} |\partial_n w|^2 + \int_\Omega u^0(x) w'(0,x)dx - \left\langle u^1, w(0,\cdot)\right\rangle_{-1,1}$$

(w, w') being solution of the homogeneous backward equation

$$\begin{cases} w'' - \Delta w = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u(t,x) = 0 & \text{for } t > 0, \text{ and } x \in \partial\Omega \\ w(T,x) = w_T^0(x) & \text{for } x \in \Omega \\ w'(T,x) = w_T^1(x) & \text{for } x \in \Omega \end{cases} \tag{3}$$

Assume that $\mathcal{J}$ has a minimizer $(\widehat{w}_T^0, \widehat{w}_T^1)$. If $(\widehat{w}, \widehat{w}')$ is the corresponding solution of (3) with initial data $(\widehat{w}_T^0, \widehat{w}_T^1)$ then $v = \partial_n \, \widehat{w}_x|_{\Gamma_2}$ is the control of minimal L^2-norm (HUM control).

Theorem Assume that $E_h(0) \leq C \int_0^T \int_{\Gamma_2} (\partial_n w)^2$ then $\mathcal{J}$ has a minimizer. Moreover

$$\|\partial_n \widehat{w}_x\|_{L^2} \leq C\|(u^0, u^1)\|_{L^2 \times H^{-1}}$$

Main question: How to discretize the controlled system, the adjoint system and the functional $\mathcal{J}$ to obtain:

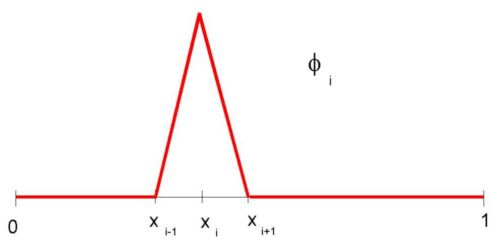
- Existence of the discrete control v_h.
- Boundedness of the sequence $(v_h)_{h>0}$ in $L^2(0,T)$.
- Convergence of the sequence $(v_h)_{h>0}$ to a control v of the wave equation.

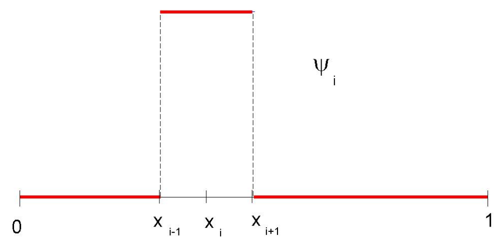
3. Mixed finite elements 1-D

Space semi-discretization: $N \in \mathbb{N}^*$, $h = \frac{1}{N+1}$, $x_j = jh$, $0 \leq j \leq N+1$.

Main idea:

$$u = \sum u_h^k \phi_k, \qquad u_t = \sum v_h^k \psi_k$$





$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h} & \text{if } x \in (x_{j-1}, x_j) \\ \frac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}) \\ 0 & \text{otherwise,} \end{cases}, \qquad \psi_j(x) = \begin{cases} \frac{1}{2} & \text{if } x \in (x_{j-1}, x_{j+1}), \\ 0 & \text{otherwise.} \end{cases}$$

Continuous system

$$\begin{cases} u''(t,x) - u_{xx}(t,x) = 0 & \text{for } x \in (0,1), \;\; t > 0 \\ u(t,0) = 0, \;\; u(t,1) = v(t), & \text{for } t > 0 \\ u(0,x) = u^0(x), \;\; u'(0,x) = u^1(x) & \text{for } x \in (0,1). \end{cases}$$

Finite difference semi-discrete system (FDS)

$$\begin{cases} h u_j''(t) - \dfrac{1}{h}\left[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\right] = 0 & \text{for } 1 \le j \le N, \;\; t > 0 \\ u_0(t) = 0, \;\; u_{N+1}(t) = v_h(t), & \text{for } t > 0 \\ u_j(0) = u_j^0, \;\; u_j'(0) = u_j^1 & \text{for } 1 \le j \le N. \end{cases}$$

Mixed finite element semi-discrete system (MFES)

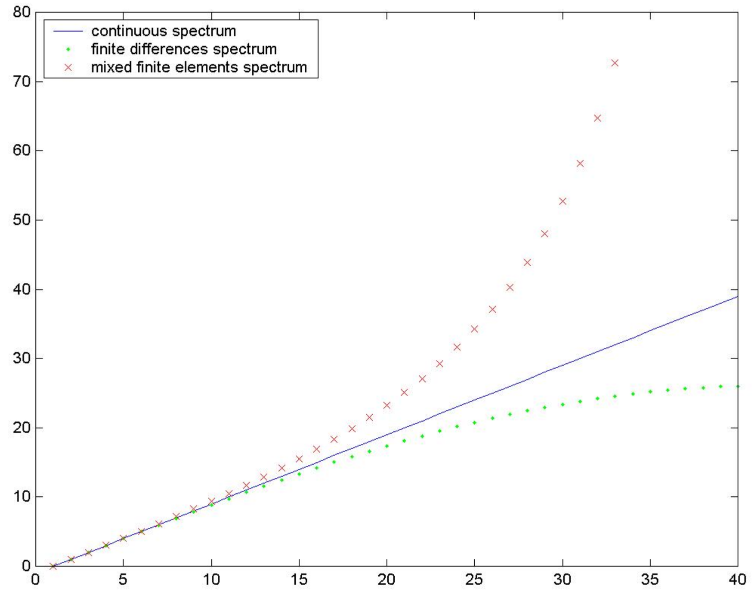
$$\begin{cases} \dfrac{h}{4}\left[2u_j''(t) + u_{j+1}''(t) + u_{j-1}''(t)\right] - \dfrac{1}{h}\left[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\right] = 0 & \text{for } 1 \le j \le N, \;\; t > 0 \\ u_0(t) = 0, \;\; u_{N+1}(t) = v_h(t), & \text{for } t > 0 \\ u_j(0) = u_j^0, \;\; u_j'(0) = u_j^1 & \text{for } 1 \le j \le N. \end{cases}$$

Dispertion analysis:

$$\omega_e(\xi) = \frac{2}{h} \tan(\xi h/2), \qquad \xi \in [-\pi/h, \pi/h].$$

Group velocity:

$$\frac{d}{d\xi}\omega_e(\xi) = 1 + \tan^2(\xi h/2), \qquad \xi \in [-\pi/h, \pi/h]$$



Observability inequality: Given $T > 2$, there exists C, independent of h, such that

$$E_h(0) \leq C \int_0^T \left[\left(\frac{w_N}{h}(t) \right)^2 + \frac{h^2}{4} (w_N'(t))^2 \right]. \tag{4}$$

where

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[\left| \frac{w_{j+1}'(t) + w_j'(t)}{2} \right|^2 + \left| \frac{w_{j+1}(t) - w_j(t)}{h} \right|^2 \right].$$

Theorem (A. E. Ingham, 1936) Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma > 0$ be such that

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \quad \forall n \in \mathbb{Z}. \tag{5}$$

For any real T with

$$T > \frac{2\pi}{\gamma} \tag{6}$$

there exists a positive constant $C_1(T,\gamma) > 0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}}$,

$$C_1 \sum_{n\in\mathbb{Z}} | a_n |^2 \leq \int_0^T \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt. \tag{7}$$

- In the mixed finite elements case

$$\nu_{n+1}(h) - \nu_n(h) = \frac{\frac{2}{h}\sin\left(\frac{\pi h}{2}\right)}{\cos\left(\frac{(n+1)\pi h}{2}\right)\cos\left(\frac{n\pi h}{2}\right)} > C, \quad \forall n.$$

- In the finite differences case

$$\lambda_{n+1}(h) - \lambda_n(h) = \frac{2}{h}\sin\left(\frac{\pi h}{2}\right)\cos\left(\frac{(2n+1)\pi h}{2}\right) \sim h \text{ as } n \sim h^{-1}.$$

$$(u^0, u^1) = \sum_{n\neq 0} a_0^n \Phi^n, \qquad (U_h^0, U_h^1) = \sum_{1\leq |n| \leq N} a_{0h}^n \varphi_n(h). \tag{8}$$

Theorem (S. Micu and CC) If the sequence $(U_h^0, U_h^1)_{h>0}$ of discretizations of the initial data (u^0, u^1) verifies

$$\left(\frac{a_{0h}^n}{\lambda_h^n}\right)_n \rightharpoonup \left(\frac{a_0^n}{n\pi i}\right)_n \text{ when } h \to 0 \text{ in } \ell^2 \tag{9}$$

then the sequences $(v_h)_{h>0}$ and $(hv'_h)_{h>0}$ are uniformly bounded in $L^2(0,T)$ and there exists a subsequence (denoted in the same way) and $v \in L^2(0,T)$ such that

$$\begin{array}{c} v_h \rightharpoonup v \text{ in } L^2(0,T) \\ hv'_h \rightharpoonup 0 \text{ in } L^2(0,T). \end{array} \tag{10}$$

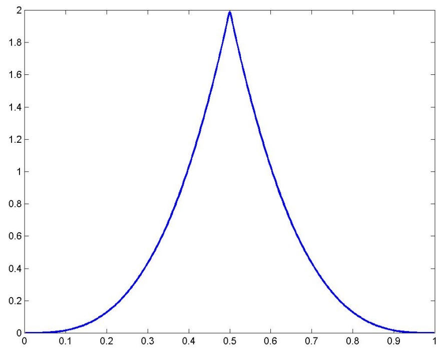
Moreover, the limit v is the HUM control of the continuous equation.

Theorem If the sequence $(U_h^0, U_h^1)_{h>0}$ of discretizations of the initial data (u^0, u^1) verifies

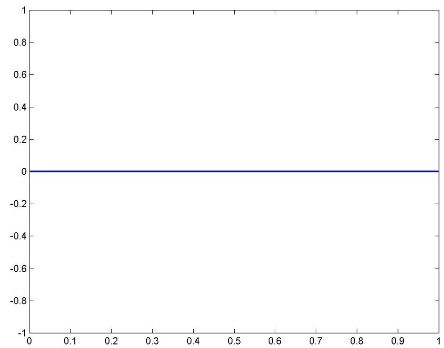
$$\left(\frac{a_{0h}^n}{\lambda_h^n}\right)_n \to \left(\frac{a_0^n}{n\pi i}\right)_n \text{ when } h \to 0 \text{ in } \ell^2 \tag{11}$$

then the sequence of controls $(v_h)_{h>0}$ converges strongly in $L^2(0,T)$ to v.

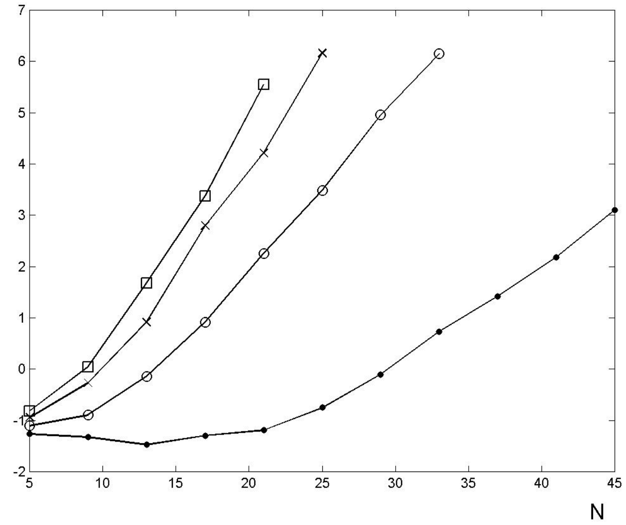
Experiment 1:



$u^0(x)$

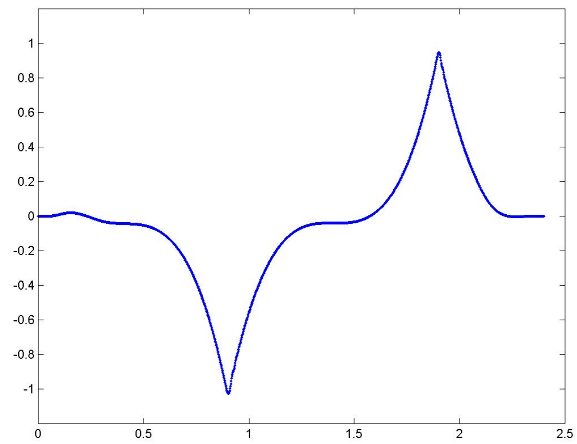


$u^1(x)$



FDS: N versus $\log(\|v_h\|_{L^2(0,T)})$. $\Delta t/h = 0.9$ (dots), 0.7 (circles), 0.5 (crosses), 0.3 (squares). $T = 2.4$.

MFES:

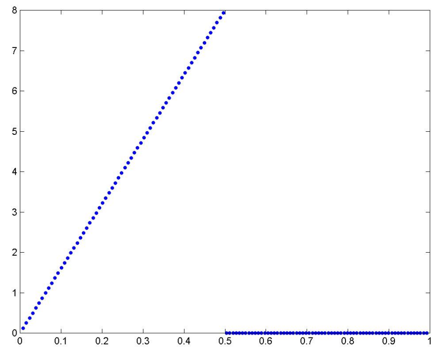


Control

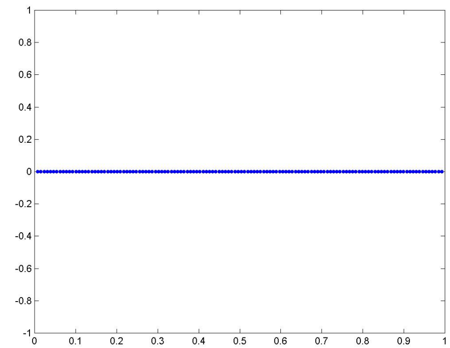
h	1/64	1/128	1/256	1/512
CG iter.	3	3	3	3
$\|\|v_h\|\|_{L^2}$	0.5274	0.5271	0.5273	0.5272

Numerical results with MFES and $\Delta t/h = 0.9$.

Experiment 2:

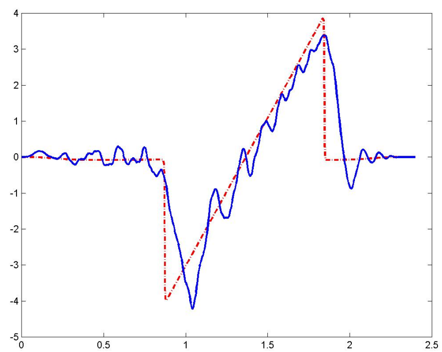


$u^0(x)$

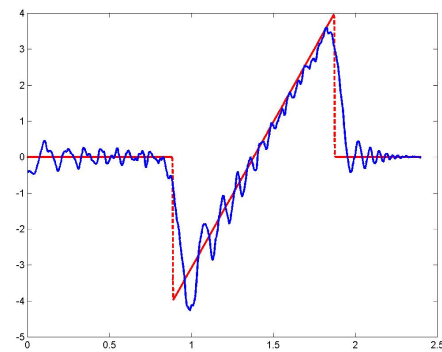


$u^1(x)$

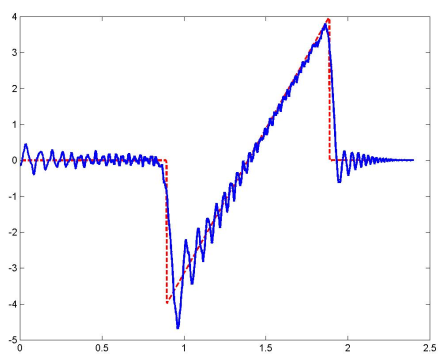
$h = 1/32$



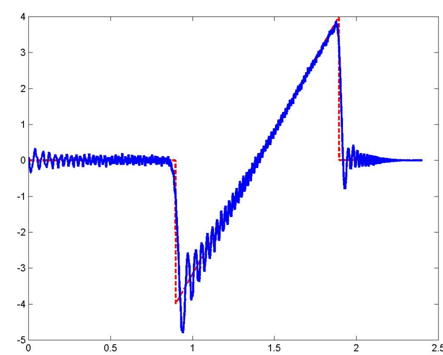
$h = 1/64$



$h = 1/128$



$h = 1/256$



Mixed finite elements 2-D (S. Micu, A. Munch and CC)

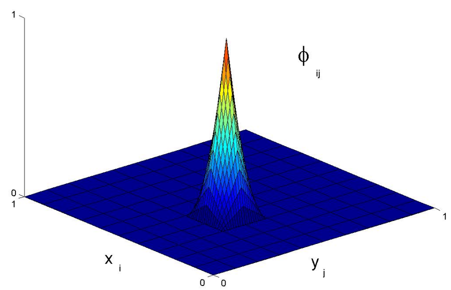
$$\begin{cases} u_{tt} - \Delta u = 0, & (x,y) \in (0,1)\times(0,1), \ \ t \in (0,T) \\ u(0,y;t) = u(x,0;t) = 0, & x \in (0,1), \ y \in (0,1) \quad t \in (0,T) \\ u(1,y;t) = f(t), & t \in (0,T) \\ u(x,1;t) = g(t), & t \in (0,T) \\ u(x,0) = u^0(x,y), \qquad u_t(x,0) = u^1(x,y), & (x,y) \in (0,1)\times(0,1) \end{cases}$$

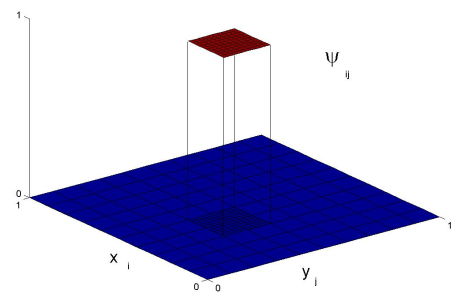
Given $T > 0$, find f, g such that

$$u(x,y;T) = u_t(x,y;T) = 0$$

Main idea:

$$u = \sum u_h^k \phi_k, \qquad u_t = \sum v_h^k \psi_k$$

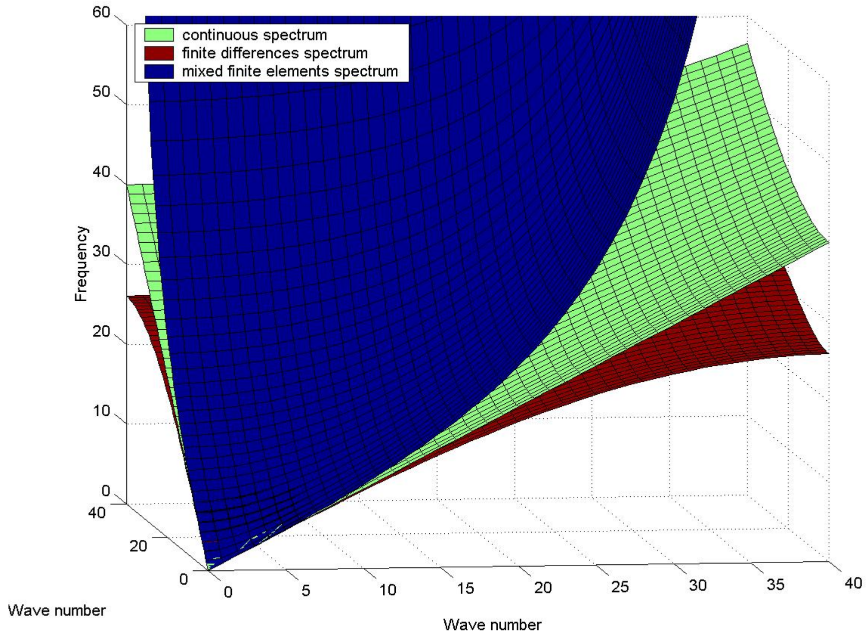




The corresponding homogeneous adjoint system is

$$\begin{cases} \dfrac{h^2}{16}\left(4w''_{ij} + 2w''_{i+1j} + 2w''_{i-1j} + 2w''_{ij+1} + 2w''_{ij-1} + w''_{i+1j+1} + w''_{i+1j-1} + w''_{i-1j+1} + w''_{i-1j-1}\right) \\ +\dfrac{1}{3}\left(8w_{ij} - w_{i+1j} - w_{i-1j} - w_{ij+1} - w_{ij-1} - w_{i+1j+1} - w_{i+1j-1} - w_{i-1j+1} - w_{i-1j-1}\right) = 0, \\ \\ w_{i0} = w_{iN+1} = 0, \quad i = 0, ..., N+1 \\ w_{0j} = w_{N+1j} = 0, \quad j = 0, ..., N+1 \\ w_{ij}(T) = w_{ij}^0, \quad w'_{ij}(T) = w_{ij}^1, \text{ for } 0 \le i, j \le N+1. \end{cases}$$

Dispersion:



Continuous: $\omega(\xi) = |\xi|, \quad \xi = (\zeta, \eta)$

Finite differences:

$$\omega_d(\xi) = \frac{2}{h}\sqrt{\sin^2(\zeta h/2) + \sin^2(\eta h/2)}, \qquad \xi \in [-\pi/h, \pi/h]^2,$$

Mixed finite elements

$$\omega_e(\xi) = \pm\frac{2}{h}\sqrt{\tan^2(\zeta h/2) + \tan^2(\eta h/2) + \frac{2}{3}\tan^2(\zeta h/2)\tan^2(\eta h/2)}, \qquad \xi \in [-\pi/h, \pi/h]^2$$

Observability

$$\begin{cases} \frac{h}{32}\int_0^T \left(\sum_{j=0}^{N} |w'_{Nj+1}(t) + w'_{Nj}(t)|^2 + \sum_{i=0}^{N} |w'_{i+1N}(t) + w'_{iN}(t)|^2 \right) dt + \\ +\frac{1}{6h}\int_0^T \left(\sum_{j=0}^{N} |w_{Nj+1}(t) + w_{Nj}(t)|^2 + \sum_{i=0}^{N} |w_{i+1N}(t) + w_{iN}(t)|^2 \right) dt - \\ -\frac{1}{6h}\int_0^T \left(\sum_{j=0}^{N} |w_{Nj}(t)|^2 + \sum_{i=0}^{N} |w_{iN}(t)|^2 \right) dt \geq CE_h(t). \end{cases}$$

where,

$$\begin{aligned} E_h(t) \; = \; & \frac{h^2}{2} \sum_{i,j=0}^{N} \left[\left| \frac{w'_{i,j} + w'_{i,j+1} + w'_{i+1,j} + w'_{i+1,j+1}}{4} \right|^2 \right. \\ & + \frac{1}{3} \left(\frac{w_{i,j+1}(t) - w_{i,j}(t)}{h} \right)^2 + \frac{1}{3} \left(\frac{w_{i+1,j}(t) - w_{i,j}(t)}{h} \right)^2 \\ & \left. + \frac{2}{3} \left(\left(\frac{w_{i+1,j+1}(t) - w_{i,j}(t)}{\sqrt{2}h} \right)^2 + \left(\frac{w_{i+1,j}(t) - w_{i,j+1}(t)}{\sqrt{2}h} \right)^2 \right) \right]. \end{aligned}$$

Time discretization:

$$Mu'' + Ku = 0$$

- Explicit:

$$\frac{Mu^{k+1} - 2Mu^k + Mu^{k-1}}{\Delta t^2} = Ku^k$$

  Stability under CFL

- Implicit (Newmark with $\gamma = 1/2$, $\beta = 1/4$):

$$\frac{Mu^{k+1} - 2Mu^k + Mu^{k-1}}{\Delta t^2} = K\frac{u^{k+1} + 2u^k + u^{k-1}}{4}$$

  Inconditionally stable.

1-D wave equation with mixed finite elements

$$\omega(\xi) = \frac{2}{h} \tan(\frac{\xi h}{2})$$

- Explicit: CFL implies

$$\frac{\Delta t}{h} \leq Ch$$

- Implicit:

$$\omega(\xi) = \frac{2}{\Delta t} \arcsin\left(\frac{\Delta t}{2} \sqrt{\frac{\omega_{sd}^2(\xi)}{1 + \frac{\Delta t^2}{4}\omega_s d(\xi)}} \right)$$

$$\|\nabla_\xi \omega(\xi)\| > C \text{ if } \Delta t \leq ch$$

2-D wave equation with mixed finite elements

$$\omega(\xi) = \frac{2}{h}\sqrt{\frac{\sin^2(\frac{\xi_1 h}{2}) + \sin^2(\frac{\xi_2 h}{2})}{\cos^2(\frac{\xi_1 h}{2})\cos^2(\frac{\xi_2 h}{2})}}$$

- Explicit: CFL implies

$$\frac{\Delta t}{h} \leq Ch^2$$

- Implicit:

$$\|\nabla_\xi \omega(\xi)\| > C \text{ if } \Delta t \leq ch^{3/2}$$