Dissipative Structure for Hyperbolic Systems

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1. Introduction

Linear PDE: $P(\partial_t, \partial_x)u = 0$.

Characteristic equation: $P(\lambda, i\xi) = 0$.

$\lambda = \lambda(i\xi)$: Dispersion relation

Dissipative structure:

- **Dissipativity:**

  $\mathrm{Re}\,\lambda(i\xi) \le 0 \quad$ for any ξ.

- **Strict dissipativity:**

  $\mathrm{Re}\,\lambda(i\xi) < 0 \quad$ for any $\xi \neq 0$.

Type (I): $\mathrm{Re}\,\lambda(i\xi) \le -\dfrac{c|\xi|^2}{1+|\xi|^2}$

Type (II): $\mathrm{Re}\,\lambda(i\xi) \le -\dfrac{c|\xi|^2}{(1+|\xi|^2)^2}$

Type (I): $\quad \mathrm{Re}\,\lambda(i\xi) \le -\dfrac{c|\xi|^2}{1+|\xi|^2}$

- **General framework:**

  □ Symmetric hyperbolic systems

  □ Symmetric hyperbolic-parabolic systems

  ∘ T. Umeda, S. K & Y. Shizuta (1984)

  ∘ Y. Shizuta & S.K (1985)

Type (II): $\quad \mathrm{Re}\,\lambda(i\xi) \le -\dfrac{c|\xi|^2}{(1+|\xi|^2)^2}$

For Type (II), $\lambda(i\xi)$ may approach the imaginary axis $\mathrm{Re}\,\lambda = 0$ for $|\xi| \to \infty$.

- **Decay at the consumption of regularity:**

  □ Dissipative Timoshenko system

  ∘ J.E.M. Rivera & R. Racke (2003)

  ∘ K. Haramoto & S. K (2005)

- **Difficulty in nonlinear problem:**

  ∘ T. Hosono & S. K (2005)

<u>Aim</u> :

- To survey the general theory for Type (I).

- To study the **dissipative Timoshenko system** as an example of type (II).

$$\begin{cases} w_{tt} - (w_x - \psi)_x = 0, \\ \psi_{tt} - a^2\,\psi_{xx} - (w_x - \psi) + \gamma\psi_t = 0, \end{cases}$$

where $a,\ \gamma > 0$ are constants.

- To study a **simple nonlinear model** system as an example of type (II).

$$\begin{cases} u_t + \left(u^2/2\right)_x + q_x = 0, \\ (\partial_x^4 - \partial_x^2 + 1)q + u_x = 0. \end{cases}$$

a modified radiating gas model

2. General theory for type (I)

Example 1. Dissipative wave equation (hyperbolic type):

$$\phi_{tt} - \phi_{xx} + \phi_t = 0,$$

which is equivalent to

$$\begin{cases} v_t - u_x = 0, \\ u_t - v_x + u = 0, \end{cases}$$

where $u = \phi_t$, $v = \phi_x$.

Example 2. Compressible Navier-Stokes equation (hyperbolic-parabolic type):

$$\begin{cases} v_t - u_x = 0, \\ u_t - v_x = u_{xx}, \end{cases}$$

which is equivalent to

$$\phi_{tt} - \phi_{xx} - \phi_{xxt} = 0,$$

where $u = \phi_t$, $v = \phi_x$.

Example 3. Radiating gas model (hyperbolic-elliptic type):

$$\begin{cases} u_t + q_x = 0, \\ -q_{xx} + q + u_x = 0, \end{cases}$$

which is equivalent to

$$\phi_t - \phi_{xx} - \phi_{xxt} = 0,$$

where $q = \phi_t$, $u = -\phi_x$.

Symmetric hyperbolic systems:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0, \qquad (1)$$

where $u = u(x,t)$: m-vector function of $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$ and $t > 0$.

(a) A^0 is symmetric and positive definite,

(b) A^j is symmetric for any j,

(c) L is symmetric and nonnegative definite.

Take the Fourier transform:

$$A^0 \widehat{u}_t + i|\xi| A(\omega)\widehat{u} + L\widehat{u} = 0, \qquad (2)$$

where

$$A(\omega) = \sum_{j=1}^{n} A^j \omega_j, \quad \omega = \xi/|\xi| \in S^{n-1}.$$

- Dispersion relation $\lambda = \lambda(i\xi)$:

$$\det(\lambda A^0 + A(i\xi) + L) = 0.$$

Dissipative structure:

Stability condition: Let $\mu \in \mathbf{R}$, $\varphi \in \mathbf{R}^m$, $\omega \in S^{n-1}$, and let $\mu A^0 \varphi + A(\omega)\varphi = 0$ and $L\varphi = 0$. Then $\varphi = 0$.

Condition (K): There exists $K(\omega)$ with the following properties:

(i) $K(\omega)A^0$ is skew-symmetric.

(ii) $\big[K(\omega)A(\omega)\big]' + L$ is positive definite,

where $[X]'$ is the symmetric part of X.

Theorem 1. (Shizuta & S. K. (1985))
The following four conditions are equivalent.

(a) Stability condition.

(b) Condition (K).

(c) $\mathrm{Re}\,\lambda(i\xi) \le -\dfrac{c|\xi|^2}{1+|\xi|^2}$ for any $\xi \in \mathbf{R}^n$.

(d) $\mathrm{Re}\,\lambda(i\xi) < 0$ for any $\xi \neq 0$.

Lyapunov function:

$$E[\,\widehat{u}\,] = (A^0\widehat{u},\,\widehat{u}) - \frac{\alpha|\xi|}{1+|\xi|^2}(iK(\omega)A^0\widehat{u},\,\widehat{u}),$$

where $\alpha > 0$ is a small constant.

$$\frac{\partial}{\partial t}E[\,\widehat{u}\,] + \frac{c|\xi|^2}{1+|\xi|^2}\,|\widehat{u}|^2 + c|(I-P)\widehat{u}|^2 \le 0,$$

where P is the orthogonal projection onto the null space $\mathcal{N}(L)$.

Energy estimate:

$$\|u(t)\|^2_{H^s} + \int_0^t \|\partial_x u(\tau)\|^2_{H^{s-1}} + \\ + \|(I-P)u(\tau)\|^2_{H^s}\,d\tau \le C\|u_0\|^2_{H^s},$$

where $s \ge 1$.

- Dissipative estimate without loss of regularity

Decay estimate:

$$\frac{\partial}{\partial t}E[\,\widehat{u}\,] + c\eta(\xi)E[\,\widehat{u}\,] \le 0,$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)$.

Theorem 2. (Umeda, S. K. & Shizuta (1984))
Under the stability condition,

$$|\widehat{u}(\xi,t)| \le Ce^{-c\eta(\xi)t}\,|\widehat{u}_0(\xi)|, \tag{3}$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)$.

Corollary. (Umeda, S. K. & Shizuta (1984))
Under the stability condition,

$$\|\partial_x^k u(t)\|_{L^2} \le Ce^{-ct}\,\|\partial_x^k u_0\|_{L^2} + \\ + C(1+t)^{-n/4-k/2}\|u_0\|_{L^1}, \tag{4}$$

where $k \ge 0$.

- Decay estimate without loss of regularity

3. Dissipative Timoshenko system

Dissipative Timoshenko system:

$$\begin{cases} w_{tt} - (w_x - \psi)_x = 0, \\ \psi_{tt} - a^2 \psi_{xx} - (w_x - \psi) + \gamma \psi_t = 0, \end{cases} \tag{5}$$

where $a > 0$, $\gamma > 0$ are constants.

The equivalent 1st order system is

$$\begin{cases} u_t - v_x = 0, \\ z_t - y_x = 0, \\ v_t - u_x + y = 0, \\ y_t - a^2 z_x - v + \gamma y = 0, \end{cases} \tag{6}$$

where

$$u = w_t,\ v = w_x - \psi,\ y = \psi_t,\ z = \psi_x.$$

The system is written as

$$A^0 U_t + A U_x + L U = 0, \tag{6'}$$

where

$$U = \begin{pmatrix} u \\ z \\ v \\ y \end{pmatrix}, \quad A^0 = \begin{pmatrix} 1 & & & \\ & a^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$A = -\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^2 \\ 1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \end{pmatrix}, \; L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & \gamma \end{pmatrix}.$$

Claim. L is not symmetric but satisfies the stability condition: If $\mu A^0\varphi + A\varphi = 0$ and $L\varphi = 0$, then $\varphi = 0$.

Dissipative structure:

- If $a = 1$, $\;\mathrm{Re}\,\lambda(i\xi) \le -\dfrac{c\xi^2}{1+\xi^2}$.

- If $a \neq 1$, $\;\mathrm{Re}\,\lambda(i\xi) \le -\dfrac{c\xi^2}{(1+\xi^2)^2}$.

When $a \neq 1$, the asymptotic expansion of $\lambda(i\xi)$ for $|\xi| \to \infty$ is:

$$\lambda(i\xi) = \pm\, i\xi \pm \frac{1}{2(a^2-1)}(i\xi)^{-1} + \\ + \frac{\gamma}{(a^2-1)^2}(i\xi)^{-2} + O(|\xi|^{-3}),$$

$$\lambda(i\xi) = \pm\, ai\xi - \frac{\gamma}{2} + O(|\xi|^{-1})$$

Lyapunov function: When $a \neq 1$,

$$E[\widehat{U}\,] = (A^0\widehat{U}, \widehat{U}) + \frac{\alpha_1}{1+\xi^2}\Big\{ - \mathrm{Re}\big(\widehat{v}\overline{\widehat{y}} + a^2\widehat{u}\overline{\widehat{z}}\big) + \\ + \frac{\alpha_2\xi}{1+\xi^2}\,\mathrm{Re}\big(i\widehat{v}\overline{\widehat{u}} + i\widehat{y}\overline{\widehat{z}}\big)\Big\},$$

where $\alpha_1,\ \alpha_2 > 0$ are small constants.

$$\frac{\partial}{\partial t}E[\widehat{U}\,] + cF[\widehat{U}\,] \leq 0,$$

where

$$F[\widehat{U}\,] = \frac{\xi^2}{(1+\xi^2)^2}\big(|\widehat{u}|^2 + |\widehat{z}|^2\big) + \frac{1}{1+\xi^2}|\widehat{v}|^2 + |\widehat{y}|^2.$$

__**Energy estimate**__: When $a \neq 1$,

$$\|U(t)\|_{H^s}^2 + \int_0^t \|\partial_x(u,z)(\tau)\|_{H^{s-2}}^2 + \\ + \|v(\tau)\|_{H^{s-1}}^2 + \|y(\tau)\|_{H^s}^2 \, d\tau \leq C\|U_0\|_{H^s}^2,$$

where $s \geq 2$.

- Dissipative estimate with loss of regularity

__**Decay estimate**__: When $a \neq 1$,

$$\frac{\partial}{\partial t} E[\,\widehat{U}\,] + c\rho(\xi) E[\,\widehat{U}\,] \leq 0,$$

where $\rho(\xi) = \xi^2/(1+\xi^2)^2$.

Theorem 3. (Haramoto & S. K (2005))
When $a \neq 1$,

$$|\hat{U}(\xi,t)| \leq Ce^{-c\rho(\xi)t}|\hat{U}_0(\xi)|, \qquad (7)$$

where $\rho(\xi) = \xi^2/(1+\xi^2)^2$.

Corollary. (Haramoto & S. K (2005))
When $a \neq 1$,

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-\ell/2}\|\partial_x^{k+\ell} U_0\|_{L^2} + \\ + C(1+t)^{-1/4-k/2}\|U_0\|_{L^1}, \qquad (8)$$

where k, $\ell \geq 0$.

• Decay estimate only at the consumtion of regularity

Proof of Corollary. We have

$$\begin{aligned}\|\partial_x^k U(t)\|_{L^2}^2 &= \int |\xi|^{2k} |\widehat{U}(\xi,t)|^2 \, d\xi \\ &\leq C \int |\xi|^{2k} e^{-c\rho(\xi)t} \, |\widehat{U}_0(\xi)|^2 \, d\xi \\ &= \int_{|\xi|\leq 1} + \int_{|\xi|\geq 1} \\ &:= I_1 + I_2.\end{aligned}$$

Low frequency term I_1 is the same as before.

$$I_1 \leq C\int_{|\xi|\leq 1} |\xi|^{2k} e^{-c|\xi|^2 t} |\widehat{U}_0(\xi)|^2 \, d\xi$$

$$\leq C \sup_{|\xi|\leq 1} |\widehat{U}_0(\xi)|^2 \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c|\xi|^2 t} \, d\xi$$

$$\leq C(1+t)^{-1/2-k} \|U_0\|_{L^1}^2.$$

High frequency term I_2 can be estimated as

$$I_2 \leq C\int_{|\xi|\geq 1} |\xi|^{2k} e^{-ct/|\xi|^2} |\widehat{U}_0(\xi)|^2 \, d\xi$$

$$\leq C \sup_{|\xi|\geq 1} \frac{e^{-ct/|\xi|^2}}{|\xi|^{2\ell}} \int_{|\xi|\geq 1} |\xi|^{2(k+\ell)} |\widehat{U}_0(\xi)|^2 \, d\xi$$

$$\leq C(1+t)^{-\ell} \|\partial_x^{k+\ell} U_0\|_{L^2}^2.$$

This shows the desired estimate (8).

4. Simple nonlinear model

Modified radiating gas model:

$$\begin{cases} u_t + \left(u^2/2\right)_x + q_x = 0, \\ (\partial_x^4 - \partial_x^2 + 1)q + u_x = 0. \end{cases} \tag{9}$$

Dissipative structure:

Linearization gives

$$\begin{cases} u_t + q_x = 0, \\ (\partial_x^4 - \partial_x^2 + 1)q + u_x = 0. \end{cases}$$

Take the Fourier transform and eliminate q:

$$\widehat{u}_t + \frac{\xi^2}{1 + \xi^2 + \xi^4}\,\widehat{u} = 0.$$

- Dispersion relation:

$$\lambda = -\rho(\xi),$$

where $\rho(\xi) = \xi^2/(1 + \xi^2 + \xi^4)$.

Decay estimate:

The associated semigroup:

$$e^{tA}\varphi = \mathcal{F}^{-1} e^{-\rho(\xi)t} \mathcal{F}\varphi.$$

Claim:

$$\|\partial_x^k e^{tA}\varphi\|_{L^2} \leq C(1+t)^{-\ell/2} \|\partial_x^{k+\ell}\varphi\|_{L^2} + \\ + C(1+t)^{-1/4-k/2}\|\varphi\|_{L^1}, \qquad (10)$$

where k, $\ell \geq 0$.

Difficulty in nonlinear problem:

- **Standard energy method** gives

$$\frac{d}{dt}\|\partial_x^k u\|_{L^2}^2 + 2\|\partial_x^k q\|_{H^2}^2 \leq C\|u_x\|_{L^\infty}\|\partial_x^k u\|_{L^2}^2, \qquad (11)$$

where $k \geq 0$ (when $k = 0$, we have the equality with zero right hand side).

This yields **energy estimate**

$$\|u(t)\|_{H^s}^2 + 2\int_0^t \|q(\tau)\|_{H^{s+2}}^2 \, d\tau$$
$$\leq \|u_0\|_{H^s}^2 + C\int_0^t \|u_x\|_{L^\infty} \|\partial_x u\|_{H^{s-1}}^2 \, d\tau.$$

• However, **dissipative estimate** (with loss of regularity)

$$\int_0^t \|\partial_x u\|_{H^{s-2}}^2 \, d\tau \leq C\int_0^t \|q\|_{H^{s+2}}^2 \, d\tau$$

CAN NOT control the nonlinearity.

• Also, **optimal decay**

$$\|u_x(t)\|_{L^\infty} \leq C(1+t)^{-1}$$

is *NOT* integrable in t, so that it is *NOT* sufficient to control the nonlinearity.

<u>Global existence</u>:

- Energy method with weight

$$(1+t)^{\alpha}, \quad \alpha = -1/2 \ (negative!)$$

combined with the optimal decay for $\|u_x\|_{L^\infty}$ can prove the global existence.

Theorem 4. (Hosono & S. K (2005)) If u_0 is small in $H^s \cap L^1$, where $s \geq 7$, then there exists unique global solution $u(t)$ suth that

$$\|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-1/4-k/2},$$

where $k = 0, 1, 2$.

A priori estimate:

$E(t)$: energy $\qquad$ $D(t)$: dissipation
$M(t)$: optimal decay

$$E(t)^2 = \sum_{j=0}^{[s/2]} \sup_{\tau \le t} (1+\tau)^{j-1/2} \|\partial_x^j u(\tau)\|_{H^{s-2j}}^2$$

$$D(t)^2 = \sum_{j=0}^{[s/2]} \int_0^t (1+\tau)^{j-3/2} \|\partial_x^j u(\tau)\|_{H^{s-2j}}^2 \, d\tau$$

$$M(t) = \sum_{j=0}^{2} \sup_{\tau \le t} (1+\tau)^{1/4+j/2} \|\partial_x^j u(\tau)\|_{L^2}$$

Proposition 1. Let $s \ge 0$.

$$E(t)^2 + D(t)^2 \le C\|u_0\|_{H^s}^2 + CM(t)D(t)^2.$$

Proposition 2. Let $s \ge 7$.

$$M(t) \le C(\|u_0\|_{L^1} + \|u_0\|_{H^5}) + CM(t)^2 + CM(t)E(t).$$

Proof of Proposition 1:

Multiply (11) by $(1+t)^\alpha$. Integrate in t.

$$(1+t)^\alpha \|\partial_x^k u\|_{L^2}^2 + 2\int_0^t (1+\tau)^\alpha \|\partial_x^k q\|_{H^2}^2\, d\tau$$

$$\leq \|\partial_x^k u_0\|_{L^2}^2 + \alpha \int_0^t (1+\tau)^{\alpha-1} \|\partial_x^k u\|_{L^2}^2\, d\tau$$

$$+ C\int_0^t (1+\tau)^\alpha \|u_x\|_{L^\infty} \|\partial_x^k u\|_{L^2}^2\, d\tau,$$

where $k \geq 0$, $\alpha \in \mathbf{R}$.

Step 1. Put $\alpha = -1/2$. Add for $0 \leq k \leq s$.

$$(1+t)^{-1/2}\|u\|_{H^s}^2 + \int_0^t (1+\tau)^{-1/2} \|q\|_{H^{s+2}}^2\, d\tau +$$

$$+ \int_0^t (1+\tau)^{-3/2} \|u\|_{H^s}^2\, d\tau \leq C\|u_0\|_{H^s}^2 +$$

$$+ \underbrace{C\int_0^t (1+\tau)^{-1/2} \|u_x\|_{L^\infty} \|u\|_{H^s}^2\, d\tau}_{\leq CM(t)D(t)^2}.$$

$$\int_0^t (1+\tau)^{-1/2} \|\partial_x u\|_{H^{s-2}}^2 \, d\tau$$
$$\leq C \int_0^t (1+\tau)^{-1/2} \|q\|_{H^{s+2}}^2 \, d\tau \leq \cdots ,$$

where $s \geq 2$.

Step 2. Put $\alpha = 1/2$. Add for $1 \leq k \leq s-1$.

$$(1+t)^{1/2} \|\partial_x u\|_{H^{s-2}}^2 + \int_0^t (1+\tau)^{1/2} \|\partial_x q\|_{H^s}^2 \, d\tau$$
$$\leq C \|\partial_x u_0\|_{H^{s-2}}^2 + C \int_0^t (1+\tau)^{-1/2} \|\partial_x u\|_{H^{s-2}}^2 \, d\tau$$
$$+ \underbrace{C \int_0^t (1+\tau)^{1/2} \|u_x\|_{L^\infty} \|\partial_x u\|_{H^{s-2}}^2 \, d\tau}_{\leq CM(t)D(t)^2} .$$

$$\int_0^t (1+\tau)^{1/2} \|\partial_x^2 u\|_{H^{s-4}}^2 \, d\tau$$
$$\leq C \int_0^t (1+\tau)^{1/2} \|\partial_x q\|_{H^s}^2 \, d\tau \leq \cdots ,$$

where $s \geq 4$.

Proof of Proposition 2:

Rewrite euation (9) as

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-\tau)A}\partial_x g(\tau)\,d\tau,$$

where $g = -u^2/2$. Apply (10).

$$\begin{aligned}\|\partial_x^k u(t)\|_{L^2}^2 \le\ & C(1+t)^{-1/4-k/2}\|u_0\|_{L^1} + \\ & + C(1+t)^{-(k+1)/2}\|\partial_x^{2k+1}u_0\|_{L^2} + \\ & + C\int_0^{t/2}(1+t-\tau)^{-3/4-k/2}\|g\|_{L^1}\,d\tau \\ & + C\int_0^{t/2}(1+t-\tau)^{-(k+1)/2}\|\partial_x^{2k+2}g\|_{L^2}\,d\tau \\ & + C\int_{t/2}^{t}(1+t-\tau)^{-3/4}\|\partial_x^k g\|_{L^1}\,d\tau \\ & + C\int_{t/2}^{t}(1+t-\tau)^{-1/2}\|\partial_x^{k+2}g\|_{L^2}\,d\tau.\end{aligned}$$

Optimal decay for $0 \le k \le [(s-1)/2] - 1$ $(2k+2 \le s-1)$. This requires $s \ge 7$.