RELAXED STATE EQUATIONS IN 2-D THROUGH SET-VALUED MAPS

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STARTING POINT: typical optimal design problem in conductivity

$$\text{Minimize in } u \in \mathcal{A}: \quad I(u),$$
$$\mathcal{A} = \left\{ u \in H_0^1(\Omega) : \text{div}[(\alpha\chi + \beta(1-\chi))\nabla u] = f, \chi \text{ characteristic function in } \Omega \right\},$$
$$\Omega \subset \mathbf{R}^2, \quad f \in L^2(\Omega), \quad 0 < \alpha < \beta.$$

Minimizing sequences $\{u_j\}$ are equi-bounded in $H^1(\Omega)$:

$$\chi_j \iff u_j, \quad u_j \rightharpoonup u \text{ in } H^1(\Omega),$$
$$I : H^1(\Omega) \to \mathbf{R} \text{ continuous} \Longrightarrow I(u), \text{minimum value},$$
$$\text{if } u \in \mathcal{A}: \quad \text{can find } \chi : \chi \iff u, \chi \text{ optimal}.$$

Often: don't know if $u \in \mathcal{A}$, even if it were, or how it can be recovered.
TASK: Study the family of state equations

$$\text{div}[(\alpha\chi + \beta(1-\chi))\nabla u] = f, \quad u \in H_0^1(\Omega)$$

and decide if weak limits of solutions are again a solution within the same family.

If THIS IS NOT SO,
IDEA: enlarge that family of state equations in a minimal way so as to incoporate sufficient state equations in such a way that all weak limits of solutions of the original class of state equations can be found as solutions of the new class, and moreover this new class enjoys this same property.

CONTRIBUTORS: Can hardly be covered ...(mathematicians, mechanicians, engineers, etc.)

MORE GENERAL FRAMEWORK (set-valued maps):

$$F : \mathbf{R}^2 \to 2^{\mathbf{R}^2}, \quad c\,|\lambda|^2 \le \rho \cdot \lambda \le C\,|\lambda|^2\,, \rho \in F(\lambda), \text{(ellipticity)}$$

$$\mathrm{div}\,[F(\nabla u(x))] = 0, \quad u = u_0 \text{ on } \partial\Omega,$$

$$\mathrm{div}\,[f(x, \nabla u(x))] = 0 \text{ in } \Omega, \quad u = u_0 \text{ on } \partial\Omega, f(x,\lambda) \in F(\lambda) \quad \text{a.e. } x \in \Omega,$$

$$(F(\lambda) = \{\alpha\lambda, \beta\lambda\}\,, \quad \text{MAIN MOTIVATION: non-linear cases}).$$

Want to define a (global) operation on F: $F \mapsto \overline{F}$ so that the (relaxed) state equation

$$\mathrm{div}\,[\overline{F}(\nabla u(x))] = 0, \quad u = u_0 \text{ on } \partial\Omega,$$

corresponding to $\overline{F}$ enjoys the stability property: weak limits of solutions are also solutions,

$$\mathrm{div}\,[f_j(x, \nabla u_j(x))] = 0, \text{ in } \Omega, \quad u_j = u_0 \text{ on } \partial\Omega, \quad f_j(x,\lambda) \in \overline{F}(\lambda), u_j \rightharpoonup u \text{ in } H^1(\Omega),$$

$$\mathrm{div}\,[f(x, \nabla u(x))] = 0, \text{ in } \Omega, \quad u = u_0 \text{ on } \partial\Omega, \quad f(x,\lambda) \in \overline{F}(\lambda).$$

and, at the same time, it is the smallest extension of F with this property.

OPTIMAL CONTROL PARALLELISM (much simpler situation)

$$\text{Minimize in } y \in \mathcal{B}: \quad I(u),$$

$$\mathcal{B} = \{y \in L^\infty(0,T) : y \in K, u' = g(u,y) \text{ in } (0,T), u(0) = u_0\},$$

$$F(\lambda) = \{g(\lambda, y) : y \in K\}, \mathcal{A} = \{u \text{ a.c.} : u' \in F(u) \text{ in } (0,T), u(0) = u_0\},$$

$$u_j' \in F(u_j), u_j' \rightharpoonup u' \Longrightarrow u' \in \text{co}F(u),$$

$$\overline{F}(\lambda) = \text{co}F(\lambda) \quad \text{(local transformation in } \lambda\text{).}$$

Job: do the same for elliptic PDE.

DEFINITION FOR PDE (in 2-d):

$$F : \mathbf{R}^2 \to 2^{\mathbf{R}^2}, \text{ elliptic, and quadratic.}$$

Step 1:

$$A = \left\{ X \in \mathbf{M}^{2\times 2} : \begin{pmatrix} \lambda \\ T\rho \end{pmatrix}, \lambda \in \mathbf{R}^2, \rho \in F(\lambda) \right\};$$

STEP 2: quasiconvexify $A \longrightarrow QA$;

Step 3: write QA back in the form

$$QA = \left\{ X \in \mathbf{M}^{2\times 2} : \begin{pmatrix} \lambda \\ T\rho \end{pmatrix}, \lambda \in \mathbf{R}^2, \rho \in \overline{F}(\lambda) \right\}.$$

$$\overline{F} : \mathbf{R}^2 \to 2^{\mathbf{R}^2}, \text{ elliptic, and quadratic.}$$

QUASICONVEXIFICATION OF SETS OF MATRICES:

$$A \subset \mathbf{M}^{2\times 2} \mapsto QA \subset \mathbf{M}^{2\times 2}:$$

QA is the smallest set containing A enjoying the weak stability property for GRADIENTS:

$$\nabla U_j \in A \text{ a.e. } x \in \Omega, \quad \nabla U_j \rightharpoonup \nabla U \Longrightarrow \nabla U \in QA \text{ a.e. } x \in \Omega.$$

Note: co(A) is the smallest set containing A enjoying the weak stability property (no gradient restriction)

$$F_j \in A \text{ a.e. } x \in \Omega, \quad F_j \rightharpoonup F \Longrightarrow F \in \text{co}A \text{ a.e. } x \in \Omega.$$

MAIN EXAMPLE:

$$F(\lambda) = \{\alpha\lambda, \beta\lambda\},$$

$$\overline{F}(\lambda) = \bigcup_{t\in[0,1]} B(a(t)\lambda, r(t)\,|\lambda|),$$

$$a(t) = \frac{t(1-t)(\beta-\alpha)^2 + 2\alpha\beta}{2(\beta(1-t)+\alpha t)}, \quad r(t) = \frac{t(1-t)(\beta-\alpha)^2}{2(\beta(1-t)+\alpha t)}.$$

Relaxed class of state equations:

$$\mathrm{div}\,[f(x,\nabla u(x))] = 0 \text{ in } \Omega, \quad u = u_0 \text{ on } \partial\Omega,$$

$$|f(x,\lambda) - a(t(x))\lambda| \leq r(t(x))\,|\lambda|, \quad t(x) \in [0,1].$$

Notice

1. $t = \chi$ recover initial state equations;
2. All weak limits of solutions for F are solutions for $\overline{F}$;
3. Any solution for $\overline{F}$ is a weak limit of a sequence of solutions for F.
4. The class of solutions for $\overline{F}$ is the SMALLEST class enjoying 2. and 3.

FURTHER WORK:

1. Many other examples. Non-linear cases: $F(\lambda) = \{\lambda, |\lambda|^p \, \lambda\}$.
2. 3-d situation;
3. Systems instead of equations (linear elasticity);
4. Relevance with respect to the optimal design problem.