PDEs, Optimal Design and Numerics 2005
Benasque, September 2005

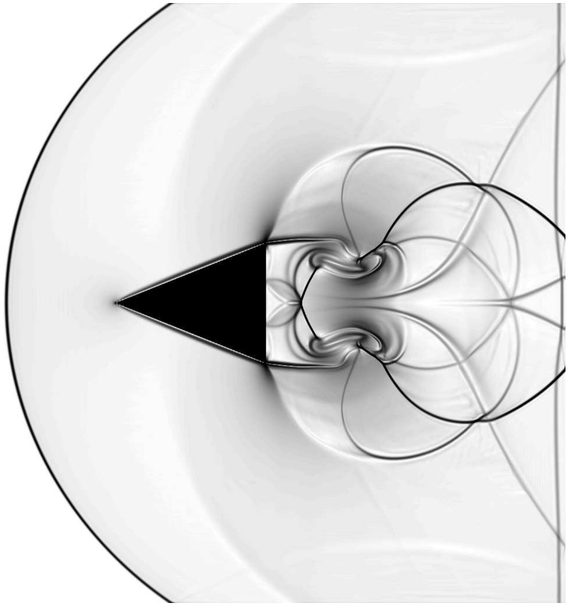
A Brinkman penalization method for fluid flow with obstacles

R. Donat
Departamento de Matematica Applicada
Universitat de València
València (Spain)

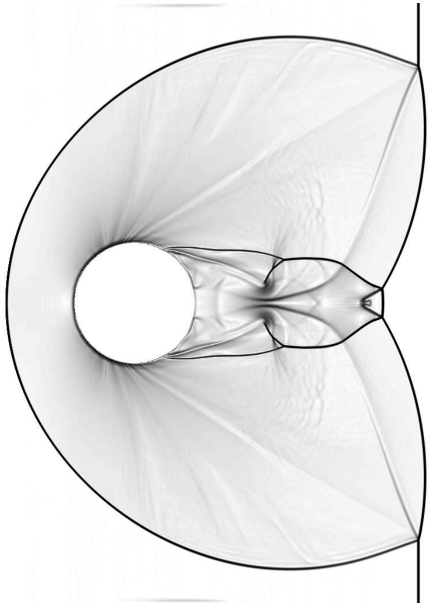
G. Chiavassa
EGIM-LATP
Technopôle de Chateau-Gombert
Marseille (France)

MOTIVATIONS

Mach 1.7



Mach 3



	Shock waves	High Order Shock Capturing Schemes
major difficulties	Obstacles in Cartesian Grids	
	BC on Obstacles	Penalization technique

Contents

I- Physical Problem and Governing Equations

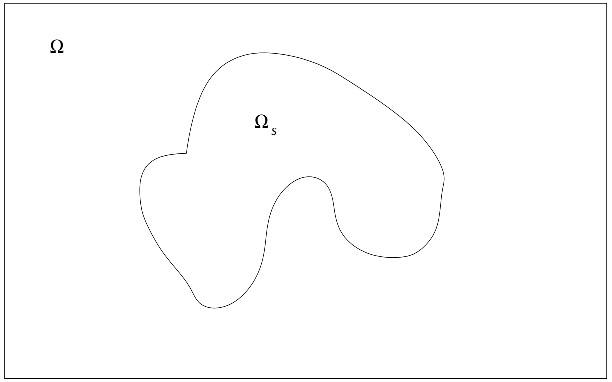
II- The Tools

1. Shock Waves: High Resolution Shock Capturing Schemes

2. Obstacles: A Penalization Method for compressible fluid flow

3. Computational Complexity: The multilevel scheme

Obstacles in compressible viscous flow



Ω : computational domain, Ω_s obstacle and $\Omega_f = \Omega \backslash \Omega_s$ compressible fluid domain

Governing Equations: Compressible N-S: $\overrightarrow{U} = (\rho, \rho u, \rho v, E)^T$,

$$\begin{aligned}
&\partial_t \overrightarrow{U} + f(\overrightarrow{U})_x + g(\overrightarrow{U})_y = f_V(\overrightarrow{U})_x + g_V(\overrightarrow{U})_y, && \text{in } \Omega_f \times \mathbb{R}^+ \\
&\overrightarrow{U}(x, y, 0) = \overrightarrow{U}_0(x, y) && \text{in } \Omega_f \\
&\text{no-slip BC} \quad u = 0; v = 0 && \text{on } \partial\Omega_s, \forall t
\end{aligned}$$

2D Compressible Navier-Stokes Equations: $\overrightarrow{U} = (\rho, \rho u, \rho v, E)^T$,

$$\partial_t \overrightarrow{U} + f(\overrightarrow{U})_x + g(\overrightarrow{U})_y \;=\; f_V(\overrightarrow{U})_x + g_V(\overrightarrow{U})_y,$$

Convective (Euler) fluxes:

$$f(\overrightarrow{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (E+p)u \end{pmatrix} \quad g(\overrightarrow{U}) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (E+p)v \end{pmatrix}.$$

Viscous fluxes:

$$f_V(\overrightarrow{U}) = \frac{1}{Re}\begin{pmatrix} 0 \\ \tau_{xx} = \lambda(u_x + v_y) + 2\mu u_x \\ \tau_{xy} = \mu(u_y + v_x) \\ u\tau_{xx} + v\tau_{xy} + \frac{\gamma\mu}{Pr} e_x \end{pmatrix} \quad g_V(\overrightarrow{U}) = \frac{1}{Re}\begin{pmatrix} 0 \\ \tau_{yx} = \tau_{xy} \\ \tau_{yy} = \lambda(u_x + v_y) + 2\mu v_y \\ u\tau_{yx} + v\tau_{yy} + \frac{\gamma\mu}{Pr} e_y \end{pmatrix}.$$

Equation of state for a polytropic gas

$$p = (\gamma - 1)\rho e.$$

Numerical Simulation: Finite diference (Finite volume) Discretization on a Cartesian Grid

- HRSC technology for convective fluxes
- Standard discretization of viscous fluxes

Time restrictions for explicit schemes: $\delta t_n = \min(\delta t_n^e, \delta t_n^V)$

$$\text{convective time step restriction}: \quad \delta t_n^e = C^e \, \frac{\min(\delta x, \delta y)}{S_{max}^n}$$

$$\text{viscous time step restriction}: \quad \delta t_n^V = C^V \, \min(\delta x, \delta y)^2 \, Re\frac{Pr}{\gamma\mu}.$$

High $Re\#$ flow allows explicit discretizations of viscous terms.

Finite diference discretization on a Cartesian grid (Method of lines):

$$\overrightarrow{U}_{ij} \simeq \overrightarrow{U}(x_i, y_j, t) \qquad x_i = x_{i-1} + \delta x,\ y_j = y_{j-1} + \delta y.$$

Convective fluxes: Conservative form Shu-Osher style

$$\left\{ f(\overrightarrow{U})_x + g(\overrightarrow{U})_y \right\}_{i,j} \approx \frac{\overrightarrow{F}_{i+1/2,j} - \overrightarrow{F}_{i-1/2,j}}{\delta x} + \frac{\overrightarrow{G}_{i,j+1/2} - \overrightarrow{G}_{i,j-1/2}}{\delta y} = D_{ij}(\overrightarrow{U})$$

Viscous fluxes: 4th order central finite differences

$$(u_x)_{i,j} \approx \frac{u_{i-2,j} - 8u_{i-1,j} + 8u_{i+1,j} - u_{i+2,j}}{12\delta x} + \mathcal{O}(\delta x^4),$$

$$(u_y)_{i,j} \approx \frac{u_{i,j-2} - 8u_{i,j-1} + 8u_{i,j+1} - u_{i,j+2}}{12\delta y} + \mathcal{O}(\delta y^4),$$

$$\longrightarrow \quad \left\{ f_V(\overrightarrow{U})_x + g_V(\overrightarrow{U})_y \right\}_{ij} \approx H_{ij}(\overrightarrow{U})$$

Global space approximation:

$$\partial_t \overrightarrow{U} + f(\overrightarrow{U})_x + g(\overrightarrow{U})_y \;=\; f_V(\overrightarrow{U})_x + g_V(\overrightarrow{U})_y,$$

$$\frac{d\overrightarrow{U_{ij}}}{dt} + B_{ij}(\overrightarrow{U}) = 0 \qquad \text{with} \qquad B_{ij}(\overrightarrow{U}) = D_{ij}(\overrightarrow{U}) - H_{ij}(\overrightarrow{U})$$

Time discretization: Shu-Osher style 3rd order explicit Runge-Kutta ODE solver

$$\overrightarrow{U}_{ij}^{n} \simeq \overrightarrow{U}(x_i, y_j, t_n) \quad \longrightarrow \quad \left\{ \begin{array}{rcl} \overrightarrow{U}_{ij}^{*} & = & \overrightarrow{U}_{ij}^{n} - \delta t_n \; B_{ij}(\overrightarrow{U}^{n}) \\ \overrightarrow{U}_{ij}^{**} & = & \frac{3}{4}\overrightarrow{U}_{ij}^{n} + \frac{1}{4}\overrightarrow{U}_{ij}^{*} - \frac{\delta t_n}{4} \; B_{ij}(\overrightarrow{U}^{*}) \\ \overrightarrow{U}_{ij}^{n+1} & = & \frac{1}{3}\overrightarrow{U}_{ij}^{n} + \frac{2}{3}\overrightarrow{U}_{ij}^{**} - \frac{2\delta t_n}{3} \; B_{ij}(\overrightarrow{U}^{**}) \end{array} \right.$$

Stability condition of the algorithm : $\qquad \delta t_n = \min(\delta t_n^e, \delta t_n^V)$

$$\delta t_n = \delta t_n^e, \qquad \text{for } Re >> 1$$

Tools, I: Shu-Osher HRSC Schemes for homogeneous HCL

$$\partial_t \vec{U} + \vec{F}(\vec{U})_x + \vec{G}(\vec{U})_y = \vec{0}$$

$\vec{U} = \vec{U}(x, y, t) \qquad (x, y, t) \in \ \Omega \times]0, T[$ + initial and boundary conditions

- $\vec{U}_{ij} \approx \vec{U}(x_i, y_j, t)$.
- Method of lines (separate spatial and temporar accuracy)
- Dimension by dimension discretization in Multi-dimensions.

$$\partial_t \vec{U}_{ij} + \frac{\vec{F}_{i+1/2,j} - \vec{F}_{i-1/2,j}}{\Delta x} + \frac{\vec{G}_{i,j+1/2} - \vec{G}_{i,j-1/2}}{\Delta y} = 0$$

Key points: [Shu, Osher JCP 86]

- Dimension-by-Dimension discretization in multi-dimensions.
- Construction of 1-D numerical fluxes. Done as follows
  - Design first for scalar conservation laws
  - Extend to systems via a local characteristic approach

High accuracy in smooth regions+ Absence of oscillations at/around sharp profiles

- High order reconstruction of numerical fluxes for high order accuracy in space (ENO).
- Time discretization Runge-Kutta schemes $\vec{U}_{ij}^{n} \quad \Longrightarrow \quad \vec{U}_{ij}^{n+1}$. High order in time.
- Only Uniform grids [Merriman, J. Sci. Comput, 03]

$$u_t + f(u)_x = 0 \qquad \Rightarrow \qquad \partial_t U_i + \frac{F_{i+1/2} - F_{i-1/2}}{\Delta x} = 0$$

- First order: $F_{i+1/2} = F^{RF}(U_i^n, U_{i+1}^n)$

$$F^{RF}(u_l, u_r) = \begin{cases} f(u_l) & \text{if } f' > 0 \text{ in } [u_l, u_r] \\ f(u_r) & \text{if } f' < 0 \text{ in } [u_l, u_r] \\ \frac{1}{2}\left(f^+(u_l) + f^-(u_r)\right) & \text{else} \end{cases}$$

$$\begin{aligned} f^+(u) &= f(u) + \alpha u \\ f^-(u) &= f(u) - \alpha u \end{aligned} \qquad \alpha = \max_{u \in [u_l, u_r]} |f'(u)|$$

- Higher order: upwind-biased, ENO interpolation process

$$F_{i+1/2} = F^{RF}(U_i, U_{i+1}) + HOT_{i+1/2}$$

Numerical flux construction: only uses $f_j = f(u(x_j))$, $u_j = u(x_j)$

Extension to systems: Characteristic-based schemes

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 \qquad \text{1D–Hyperbolic System}$$

$J = J(u) = \dfrac{\partial F}{\partial u}$ is a diagonalizable matrix with real eigenvalues $\forall u$.

$$L(u)J(u)R(u) = \Lambda(u) = \mathrm{diag}(\lambda_i(u)), \qquad L(u)R(u) = I$$

Fix a state u_0, $L_0 := L(u_0)$, $R_0 := R(u_0)$, $J_0 := J(u_0)$, $\Lambda_0 := \Lambda(u_0)$

$$u_t + F(u)_x = 0 \quad \equiv \quad [L_0 u]_t + [L_0 F(u)]_x = 0$$

$$[L_0 u]_t + L_0 J(u) u_x = 0 \quad \equiv \quad [L_0 u]_t + L_0 J R_0 [L_0 F(u)]_x = 0$$

$L_0 J R_0 = \Lambda_0$ at $u = u_0$. Expected to be near-diagonal for $u \approx u_0$

At each interface $u_0 = u^* = u^*(u_l, u_r) \longrightarrow J_0 = J(u^*) \longrightarrow \Lambda_0 = \mathrm{diag}(\lambda_p^0)$

λ_p^0 wind directions at interface.

$L_0^p u$ characteristic variables for pth field:

$L_0^p F(u)$ characteristic fluxes for pth field

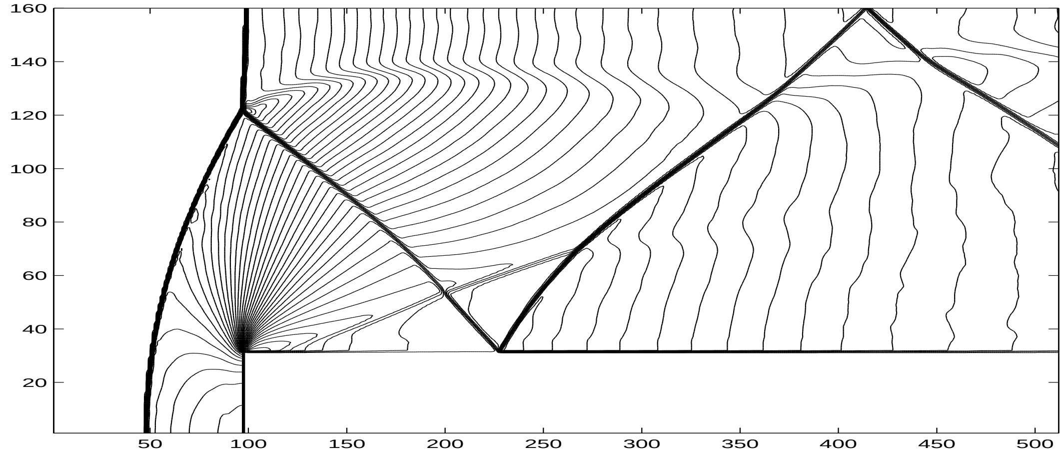
$$F_{i+1/2} = \sum_p F_{i+1/2}^p R_o^p$$

$F_{i+1/2}^p$ scalar numerical flux function for the p-th field. Constructed from upwind-biased (λ_p^0) ENO-RF interpolation process using

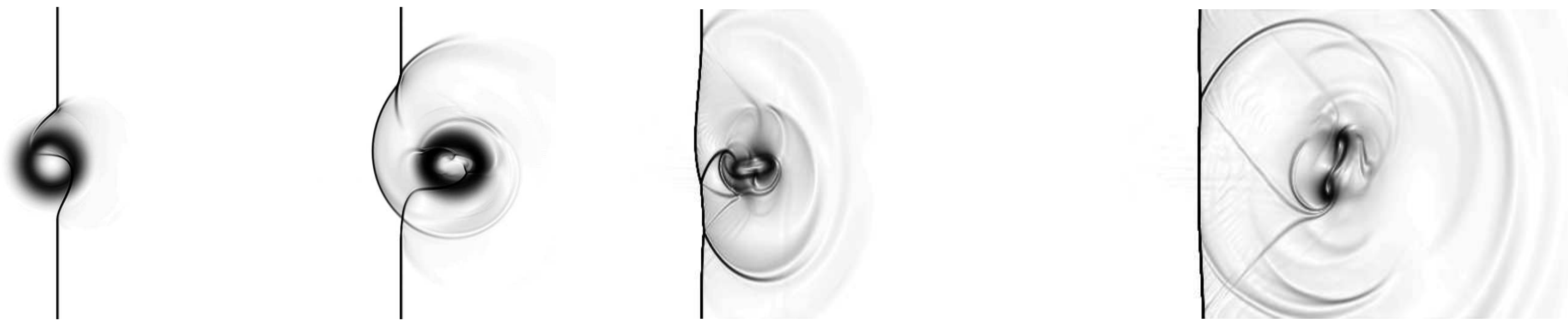
$F_j^p = L_0^p F(u_j)$, $u_j^p = L_0^p u_j$

Further refinements: 2-Jacobian Shu-Osher framework (Marquina's Flux Splitting Formula) [RD,Marquina JCP 96], [RD, Font, Ibañez, Marquina, JCP 99], [Marquina, Mulet JCP 2003], [Chiavassa, RD, SISC 01], [Rault, Chiavassa, RD JSC 03] ...

Emery's step test



Shock/vortex Interaction



Tools, II: Penalization methods for obstacles in Incompressible Fluid flow

Goal: Avoid body-fitted unstructured methods in order to use fast/effective spectral, finite-differences or finite-volume approximations on cartesian meshes.

How: Modify the system of PDEs by adding a penalized velocity term in the momentum equation

Observation: The penalization has to be extended to the volume of the body to give the correct physical solutions at high Re.

- Peskin [JCP 77]
- Arquis, Caltagirone [Compt. Rend. A.S.P. 84] Brinkman models with variable permeability for porous media.
- Angot, Bruneau, Fabrie [Num. Math. 99]

Angot et al.: Replace the IBVP

$$\begin{array}{ll}
\partial_t u - \frac{1}{Re}\Delta u + u \cdot \nabla u + \nabla p = f & \text{in } \mathbb{R}^+ \times \Omega_f \\
\text{div } u = 0 & \text{in } \mathbb{R}^+ \times \Omega_f \\
u(\cdot, 0) = u_0 & \text{in } \Omega_f \\
u = 0 & \text{on } \partial\Omega_f
\end{array}$$

by

$$\begin{array}{ll}
\partial_t u_\eta - \frac{1}{Re}\Delta u_\eta + u_\eta \cdot \nabla u_\eta + \frac{1}{\eta}\chi_{\Omega_s} u_\eta + \nabla p_\eta = f & \text{in } \mathbb{R}^+ \times \Omega \\
\text{div } u_\eta = 0 & \text{in } \mathbb{R}^+ \times \Omega \\
u_\eta(\cdot, 0) = u_0 & \text{in } \Omega \\
u_\eta = 0 & \text{on } \partial\Omega
\end{array}$$

and solve it on the whole domain Ω.

$\eta << 1$ penalization parameter, χ_{Ω_s} characteristic function of the obstacle.

Setting $u_\eta = u + \eta\tilde{u}, \qquad p_\eta = p + \eta\tilde{p} \qquad \longrightarrow \qquad \chi_{\Omega_s} u = 0$

Theoretical result (Angot et al):

$$u_\eta = u + \eta^{1/4} v_\eta, \qquad \text{in } \Omega_f$$

u, solution of the incompressible N-S equations in Ω_f. Moreover, $u|_{\Omega_s} = 0$

A Penalization technique for Compressible N-S:

• Enforce the boundary conditions on the obstacle : $u = v = 0$ and $T = T_0$ by adding a penalized term in the momentum and energy equations

Penalized system: for $\overrightarrow{U} = (\rho, \rho u, \rho v, E)^T$,

$$\partial_t \overrightarrow{U} + f(\overrightarrow{U})_x + g(\overrightarrow{U})_y + \frac{1}{\eta}\chi_{\Omega_s} \begin{pmatrix} 0 \\ \rho u \\ \rho v \\ E - E_0 \end{pmatrix} = f_v(\overrightarrow{U})_x + g_v(\overrightarrow{U})_y,$$

η penalization parameter, χ_{Ω_s} characteristic function of the obstacle and $E_0 = \rho C_v T_0$.

$$\text{Setting} \qquad \begin{array}{rcl} \rho_\eta &=& \rho + \eta\tilde{\rho} \\ (u,v)_\eta &=& (u,v) + \eta(\tilde{u},\tilde{v}) \\ E_\eta &=& E + \eta\tilde{E} \\ p_\eta &=& p + \eta\tilde{p} \end{array} \longrightarrow \chi_{\Omega_s} \begin{pmatrix} 0 \\ \rho u \\ \rho v \\ (E - E_{\Omega_s}) \end{pmatrix} = \vec{0}$$

Discretization of the Penalization term

Global spatial discretization:

$$\frac{d\overrightarrow{U_{ij}}}{dt} + B_{ij}(\overrightarrow{U}) + G_{ij}(\overrightarrow{U}) = 0 \qquad \text{with} \qquad G_{ij}(\overrightarrow{U}) = \frac{1}{\eta}\chi_{\Omega_s}\begin{pmatrix} 0 \\ (\rho u)_{ij} \\ (\rho v)_{ij} \\ E_{ij} - E_{0_{ij}} \end{pmatrix}$$

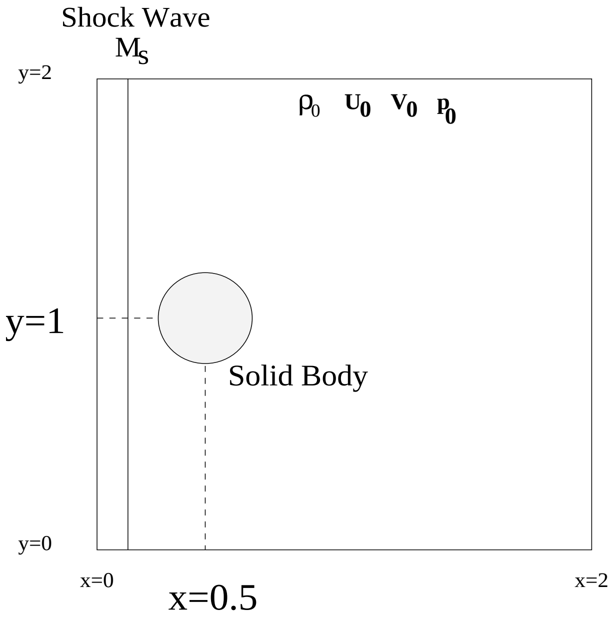
Time discretization: 3rd order explicit Runge-Kutta for the fluxes
implicit discretization of the penalization term ($\eta << 1$)

$$\left\{\begin{array}{rcll}
\overrightarrow{U_{ij}^{*}} & = & \overrightarrow{U_{ij}^{n}} - \delta t\ B_{ij}(\overrightarrow{U^{n}}) & - \delta t\ G_{ij}(\overrightarrow{U^{*}}) \\[2ex]
\overrightarrow{U_{ij}^{**}} & = & \frac{3}{4}\overrightarrow{U_{ij}^{n}} + \frac{1}{4}\overrightarrow{U_{ij}^{*}} - \frac{\delta t}{4}\ B_{ij}(\overrightarrow{U^{*}}) & - \frac{\delta t}{4}\ G_{ij}(\overrightarrow{U^{**}}) \\[2ex]
\overrightarrow{U_{ij}^{n+1}} & = & \frac{1}{3}\overrightarrow{U_{ij}^{n}} + \frac{2}{3}\overrightarrow{U_{ij}^{**}} - \frac{2\delta t}{3}\ H_{ij}(\overrightarrow{U^{**}}) & - \frac{2\delta t}{3}\ G_{ij}(\overrightarrow{U^{n+1}})
\end{array}\right.$$

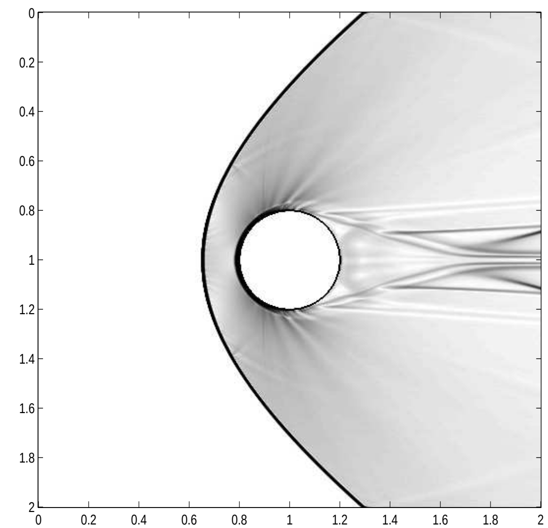
No extra stability condition.

Numerical simulations

Initial Data



Time $t = 0.45$



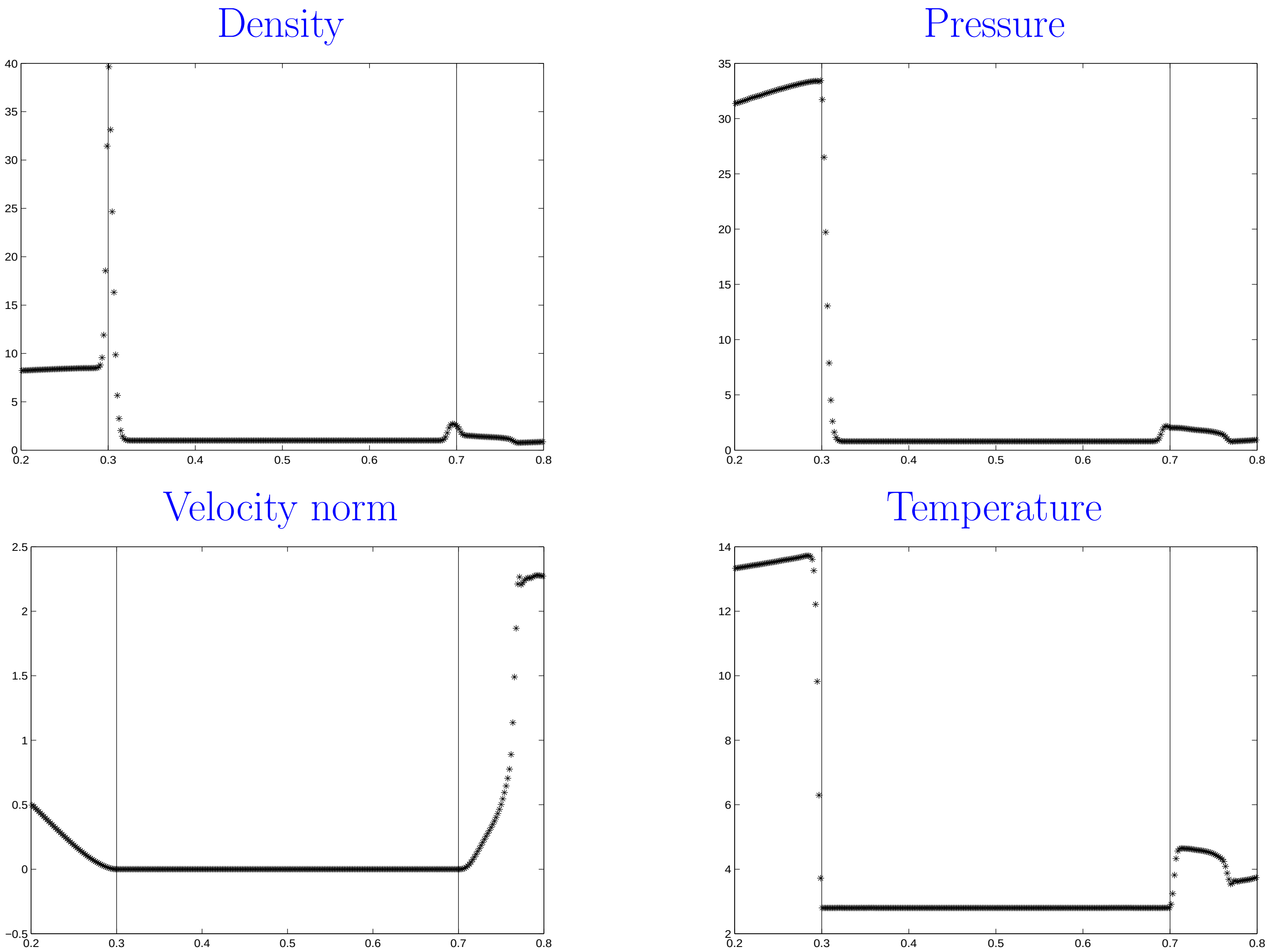
computational domain: $[0;2] \times [0;2]$

shock wave initially fixed at $x = 0.1$

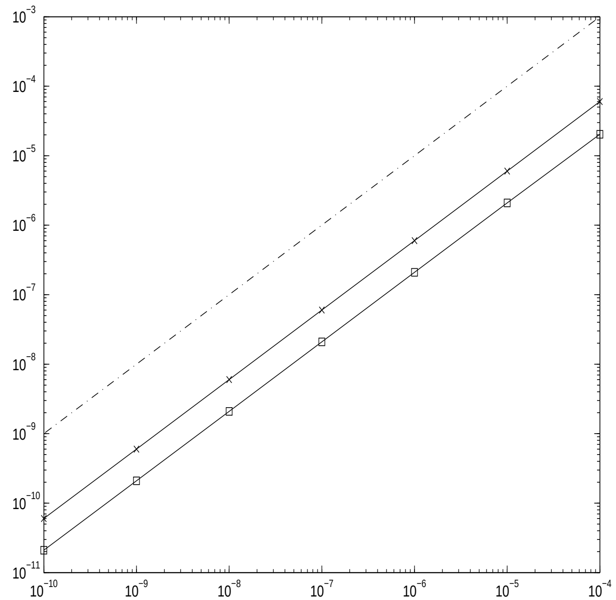
non-reflecting conditions

inlet supersonic flow, $M_S = 3$

Variables on the stagnation line

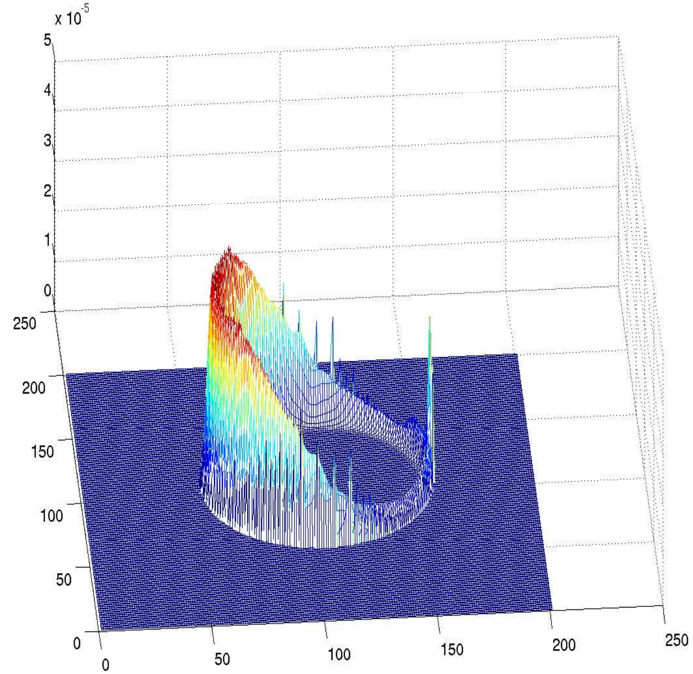


Velocity and Energy inside the obstacle



$\|(u,v)\|_{\Omega_S},\ \|E-E_0\|_{\Omega_S}$

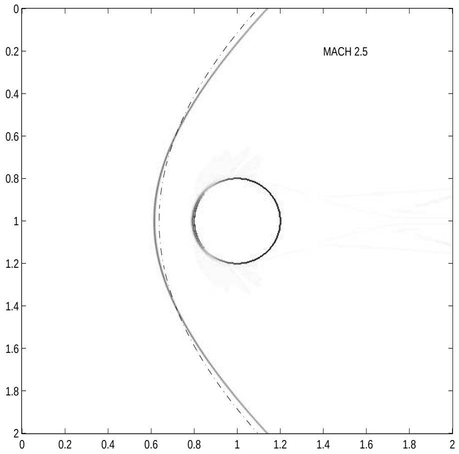
versus η



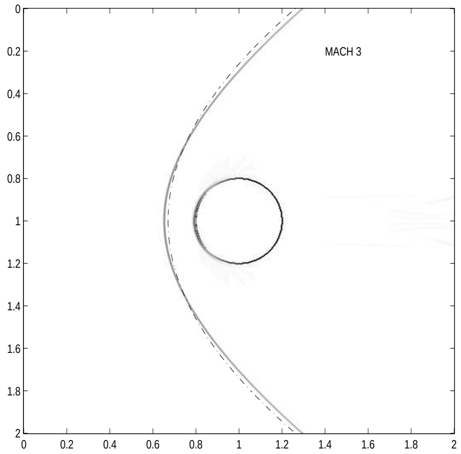
$\|(u,v)\|_{\Omega_S}$ for $\eta = 10^{-5}$

Comparison with Theoretical flow parameters ([Chiavassa, RD, Piquemal 05], submitted to *Computers and Fluids*.

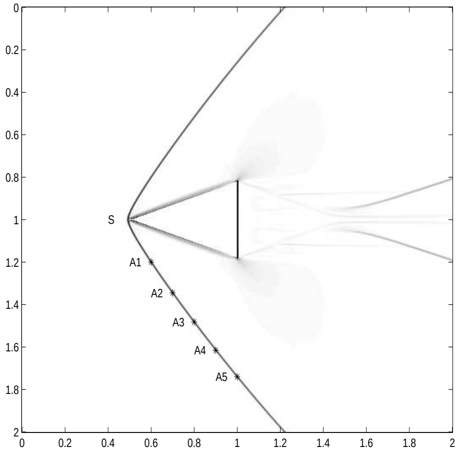
Density $M = 2.5$



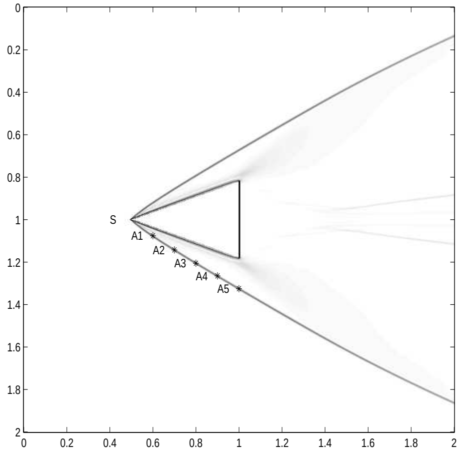
Density $M = 3$



Density $M = 2$



Density $M = 4.9$



Tools, III: Reducing the Computational cost: Multilevel Schemes for HCL

Basic idea:

- Reduce the cost asociated to a HRSC scheme by reducing the number of costly numerical flux evaluations

- HRSC- Numerical fluxes only needed at/around discontinuities (existing or in formation)

- Analyze the data at each time step of the simulation to determine smoothness zones

HOW?

- Use an appropriate Multiresolution Analysis of the avaliable data.
  - Finite Volume formulations: Discrete data are cell-averages
  - Shu-Osher Framework: Discrete data are interpreted as point-values

MR decompositions of discrete data sets

A multiresolution (MR) decomposition of a discrete data set is an equivalent representation that encodes the information as a coarse realization of the given data set plus a sequence of detail coefficients of ascending resolution.

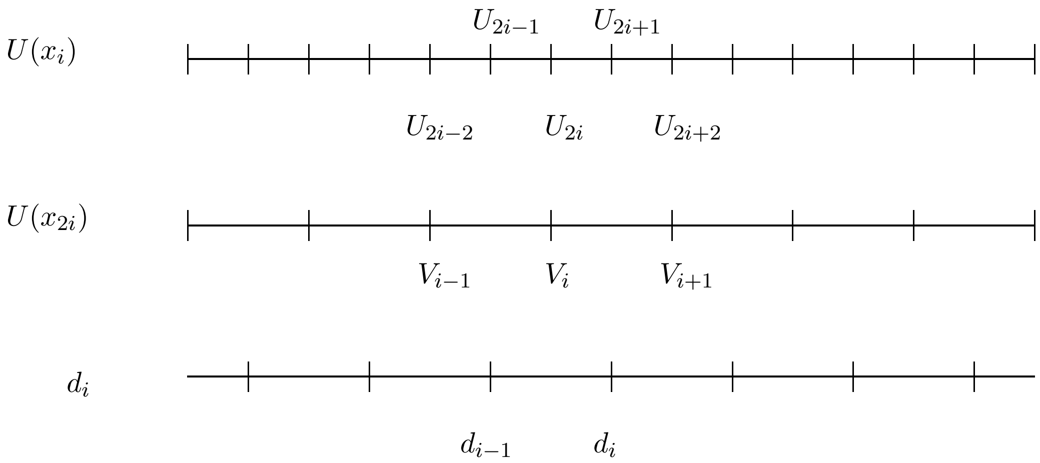
$$
\begin{array}{ccccccccc}
u^L & \longrightarrow & u^{L-1} & \longrightarrow & u^{L-2} & \longrightarrow & \ldots & \longrightarrow & u^0 \\
 & \searrow & d^{L-1} & \searrow & d^{L-2} & \searrow & \ldots & \searrow & d^0
\end{array}
$$

$$u^L \equiv M u^L = (u^0, d^1, d^2, \ldots, d^L)$$

- levels of resolution: Hierarchy of nested computational meshes
- detail coefficients: difference in information between consecutive levels
- direct relation between the detail coefficients and local regularity

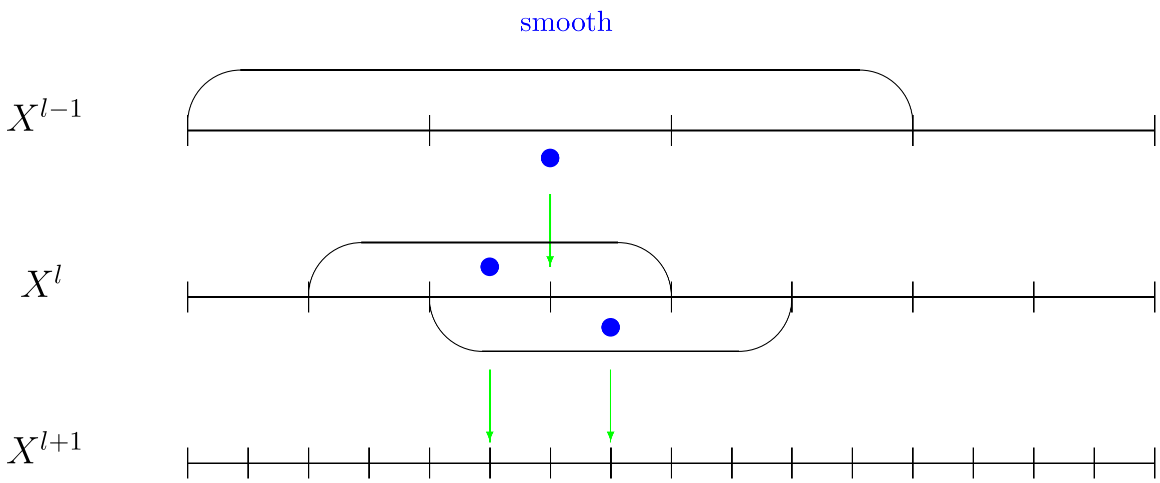
Compression properties: Small detail coefficients can be truncated with little loss of global information contents (Image/signal compression etc..).

Interpolatory MR Transform in 1D



$U_i = U(x_i); \quad V_i = U(x_{2i}); \quad \mathcal{I}(x, V)$ piecewise polynomial interpolation procedure

$$\left\{ \begin{array}{lcl} V_i & = & U_{2i} \\ d_i & = & U_{2i+1} - \mathcal{I}(x_{2i+1}, V) \end{array} \right\} \quad \leftrightarrow \quad \left\{ \begin{array}{rcl} U_{2i} & = & V_i \\ U_{2i+1} & = & \mathcal{I}(x_{2i+1}, V) + d_i \end{array} \right\}$$



$$\tilde{u}^l_{2k} = \mathcal{I}(x^l_{2k}, u^{l-1}) = \mathcal{I}(x^{l-1}_k, u^{l-1}) = u^{l-1}_k = u^l_{2k} \text{ (even values are exactly recovered)}$$

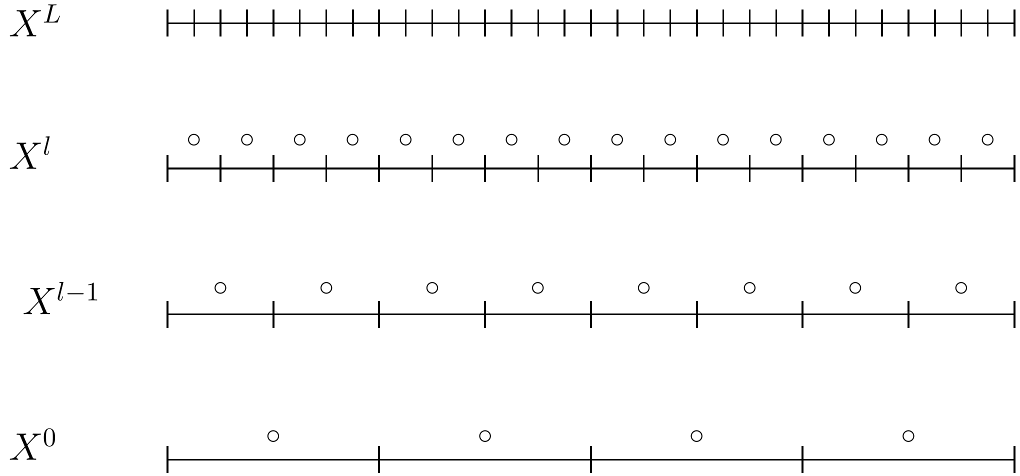
$$\tilde{u}^l_{2k+1} = \mathcal{I}(x^l_{2k+1}, v) \neq u^l_{2k+1} \qquad \text{scale coefficients: } d^{l-1}_k = u^l_{2k+1} - \tilde{u}^l_{2k+1}$$

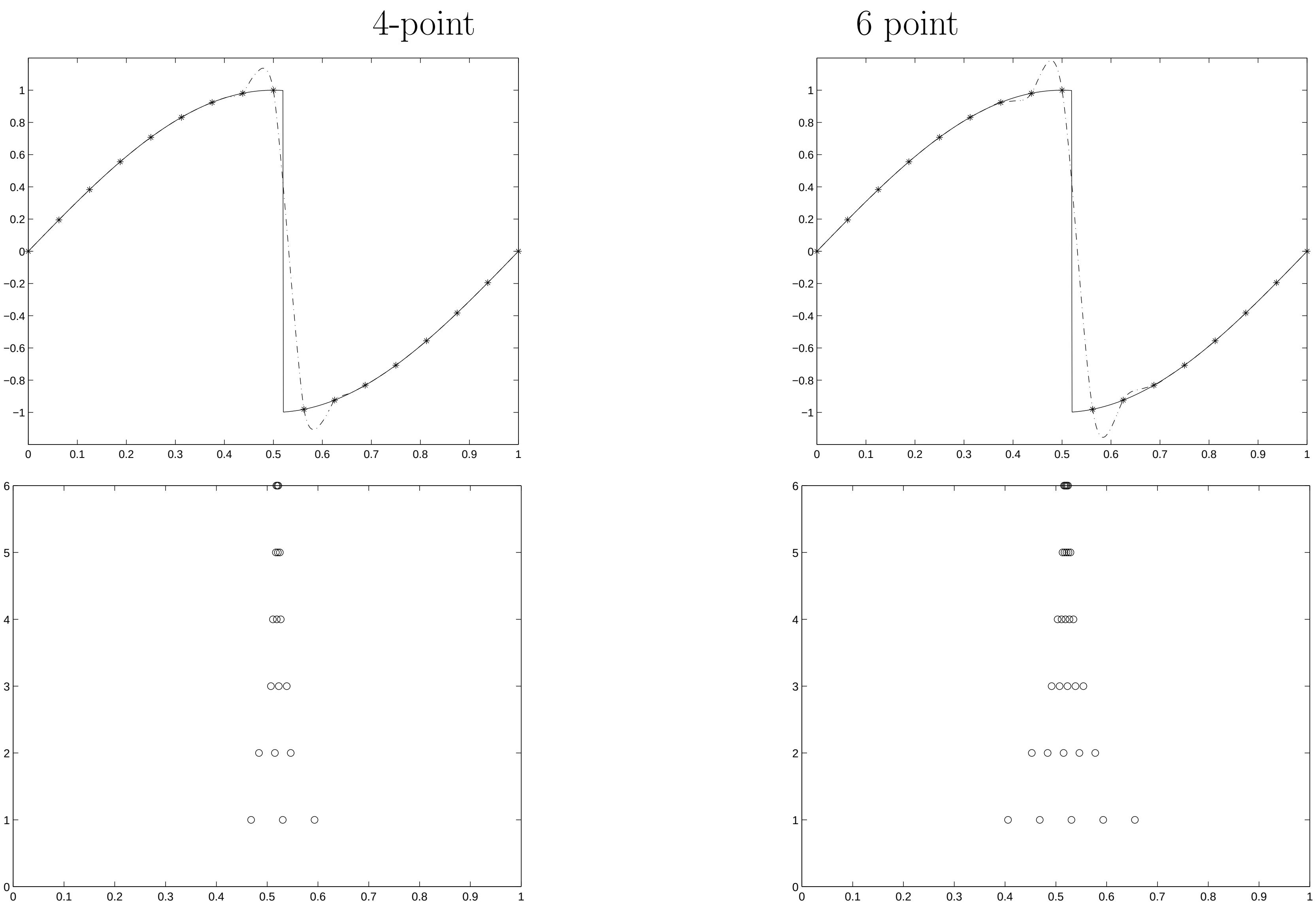
$$u^l \Leftrightarrow (u^{l-1}, d^{l-1}) \qquad \begin{array}{ccc} u^l & \longrightarrow & u^{l-1} \\ & \searrow & d^{l-1} \end{array}$$

MR transformation: Start at finest level X^L and repeat process for $l = L, \dots, 1$.

$$u^L \Leftrightarrow \{u^{L-1}, d^{L-1}\} \quad \cdots \Leftrightarrow \cdots \; \{u^0; d^0; d^1; \cdots d^{L-1}\} = M u^L$$

$$\begin{array}{ccccccc} u^L & \longrightarrow & u^{L-1} & \longrightarrow & u^{L-2} & \longrightarrow & \dots \longrightarrow u^0 \\ & \searrow & d^{L-1} & \searrow & d^{L-2} & \searrow & \dots \searrow d^0 \end{array}$$




4-point
6 point
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
6
5
4
3
2
1
0

Adaptive schemes for HCL within Harten's framework

Numerical values correspond to a particular discretization of the solution.

- Cell-Averages
  - Harten (1D)
  - Bihari-Harten (2D-tensor product)
  - Bihari (1D viscous, source terms, hexahedra)
  - Abgrall-Harten (2D-unstructured)
  - Dahmen, Gottschlich-Müller, Müller (2D curvilinear)
- Point-Values
  - Chiavassa-Donat (2D-tensor product)

Main idea: Exploit the relation between scale coefficents and local regularity of the function being discretized.

In the interpolatory and cell-average frameworks, the decay rate of the scale coefficients is related to the smoothness of the data and the order of the interpolation used in the prediction.

- Harten's original implementation [Comm. Pure and Appl. Math (1995)] seeked to *"Perform a uniform fine grid computation to a prescribed accuracy by reducing the number of arithmetic operations and computer memory requirements to the level of an adaptive grid computation"*.
- Cost-reduction implementation.
  - Bihari-Harten JCP (1996)—SISC (1997) (1D-2D/tensor-product –CA-MR) Bihari AIAA (2003) (CA-MR unstructured)
  - Abgrall-Harten SINUM (1996) (CA-MR unstructured)
  - Chiavassa-Donat SISC-(2001) (2D/tensor-product –PV-MR)
  - Dahmen, Gottschlich-Müller,Müller Num. Math (2000) (curvilinear meshes, Cell-Average Framework)
  - Cohen, Dyn, Kaber, Postel JCP (2000) (2D-unstructured)
- Fully-Adaptive Implementation.
  - Müller(2002); Cohen, Kaber, Müller, Postel (2003) CA linked to an adaptive grid that evolves in time. Data management required: tree structures (Roussel), hash-tables (Müller)

Cost-Reduction MR schemes for Shock Computations

MR-based smoothness analysis (1D)

$$U_i^{n+1} = U_i^n - \lambda\ D_i^n \qquad D_i^n = F_{i+1/2}^n - F_{i-1/2}^n, \qquad \lambda = \frac{\Delta t}{\Delta x}$$

$F_{i+1/2}^n = \hat{F}(U_{i-s}^n, \ldots, U_{i+s}^n)$ Numerical flux function.

- $U^n = U_L^n$ numerical solution on X^L, finest of $\{X^l\}_{l=0}^L$
- $D^n = D_L^n$ numerical divergence on X^L.
- M MR-transformation.

$$U_{i,L}^{n+1} = U_{i,L}^n - \lambda_L\ D_{i,L}^n \quad \Rightarrow \quad MU_L^{n+1} - MU_L^n = -\lambda_L M D_L^n$$

$d_{i,l}(U^{n+1})$ and $d_{i,l}(U^n)$ "small + correct decay rate" $\Rightarrow$ $(x_{2i+1}^{l+1}) \in$ smooth area.

Computation in smooth locations does not need HRSC numerical fluxes.

The multiresolution algorithm

Harten's Idea: Expensive schemes for $D_{ij}(\overrightarrow{U})$ are necessary only near sharp gradients of $\overrightarrow{U}_{ij}$

$$\overrightarrow{U}_{ij}^{n+1} = \overrightarrow{U}_{ij}^{n} - \delta t \; D_{ij}(\overrightarrow{U}^{n})$$

Local smoothness analysis of $\overrightarrow{U}^{n}$

$\longrightarrow$ computation of $D(\overrightarrow{U}^{n})$

Evaluation of the smoothness of $\overrightarrow{U}^{n+1}$

$\overrightarrow{U}_{ij}^{n}$ $\longrightarrow$ Multiresolution Transform $\longrightarrow$ Preprocessing Tool $\longrightarrow$ Multilevel computation $D_{ij}(\overrightarrow{U}^{n})$

The preprocessing tool

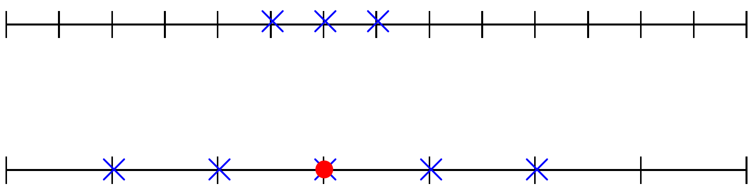
Goal: Local smoothness analysis of $\overrightarrow{U}_{ij}^{n}$ and $\overrightarrow{U}_{ij}^{n+1}$

Tolerance parameter ϵ $\quad + \quad$ Mask coefficients $M_{ij,l} = \{^0_1$

if $|d_{i,j,l}| \geq \varepsilon \quad \Longrightarrow \quad \left| \begin{array}{l} M_{i-k,j-m,l} = 1 \\ k,m = -2,..,2 \end{array} \right.$ and if $|d_{i,j,l}| \geq 2^r\varepsilon \quad \Longrightarrow \quad \left| \begin{array}{l} M_{2i-k,2j-m,l+1} = 1 \\ k,m = -1,0,1 \end{array} \right.$

$M_{2i,l+1}$

$M_{i,l}$



$\longrightarrow$ apparition of discontinuities

$\longrightarrow$ moving of discontinuities during δt

Multilevel computation of the numerical divergence

Goal: $D_{ij}(\overrightarrow{U}^n)$ on the finest grid $\mathcal{G}_L$

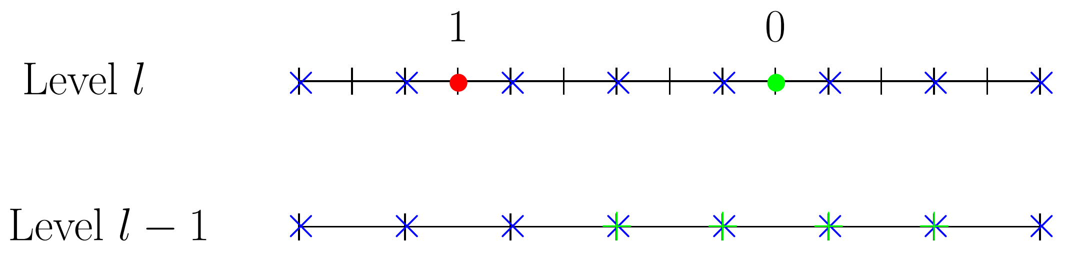
First step: Computation on the coarsest grid $\mathcal{G}_0$

$$D_{ij,0}(\overrightarrow{U}^n) \quad \text{using} \quad \overrightarrow{U}^n_{ij,L} \quad \Longrightarrow \quad \text{numerical values on finest grid}$$

Second step: Recursive Computation on finer levels checking mask coefficients

$$\text{if} \;\; M_{ij,l} \; = \; 1 \quad \Longrightarrow \quad \text{Expensive solver computation of } D_{ij,l}(\overrightarrow{U}^n)$$

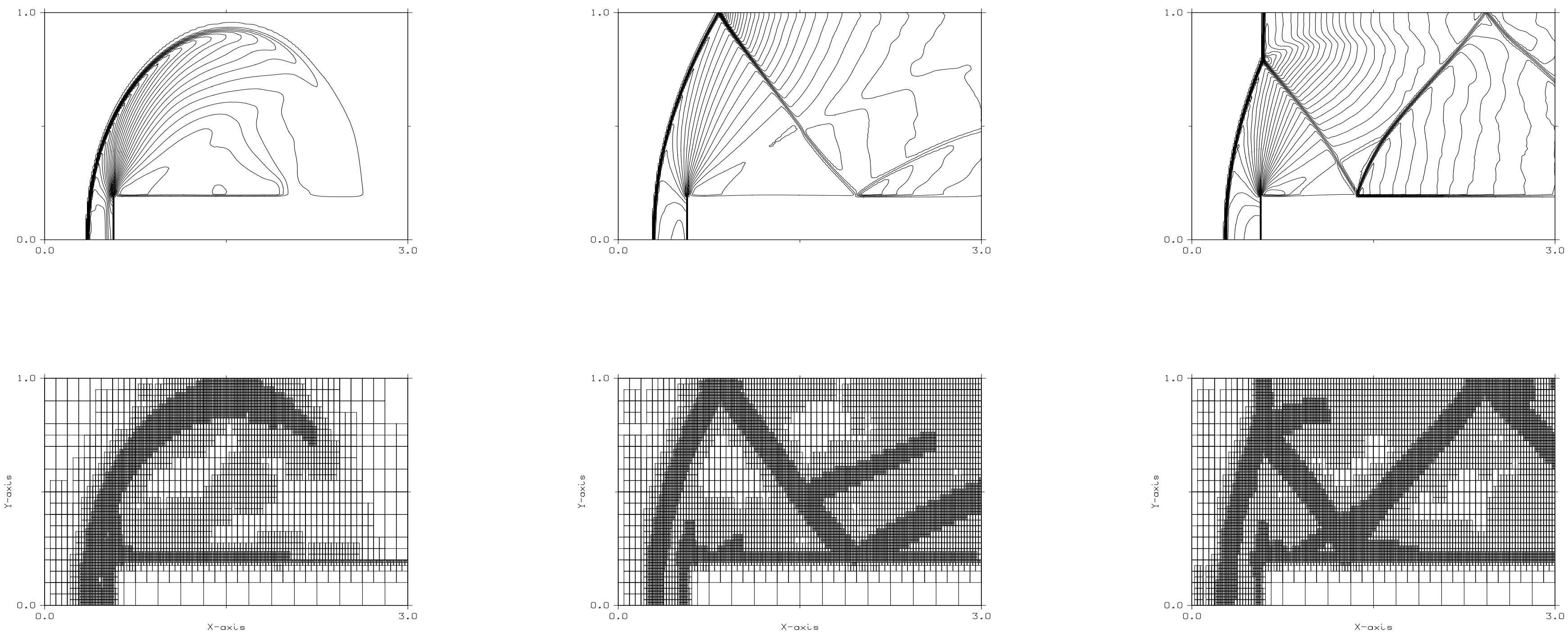
$$\text{if} \;\; M_{ij,l} \; = \; 0 \quad \Longrightarrow \quad \text{Cheap interpolation } D_{ij,l}(\overrightarrow{U}^n) = P\left[D_{km,l-1}(\overrightarrow{U}^n)\right]$$



Results for Euler equations [Chiavassa, RD SISC 00]

Efficiency: Percentage of numerical divergence computed with the solver: $\%D$

$$\theta = \frac{\text{CPU time for reference computation}}{\text{CPU time for multilevel computation}}$$



grid 256×80, 4 levels $\qquad$ $\%D = 4 \longrightarrow 45$ $\qquad$ $\theta = 1.5$

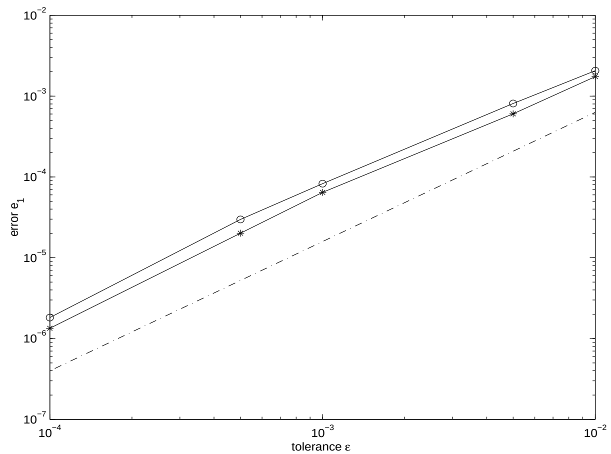
Error estimate

Theoretical bound *(Harten, A. Cohen, S.M Kaber, S. Muller, M. Postel)*

Scalar equation, contractive scheme:

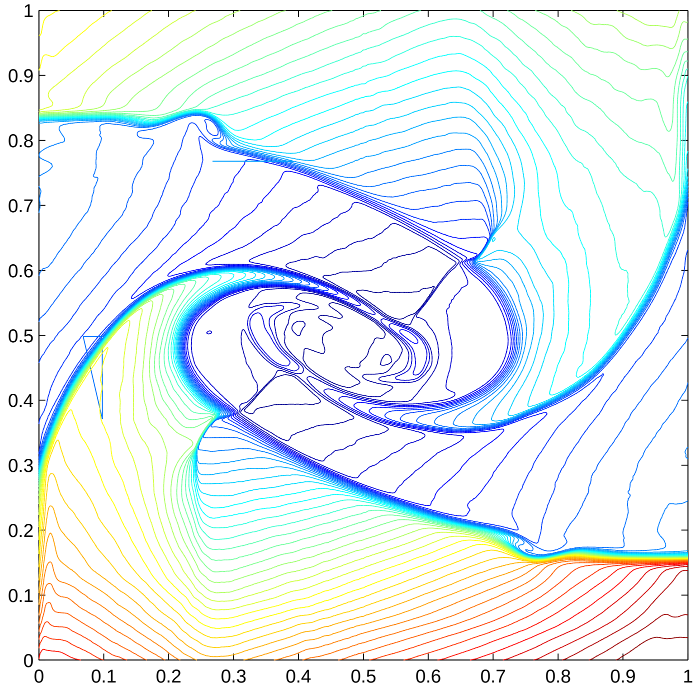
$$\|u^n_{ref} - u^n_\epsilon\|_1 \leq C\ \epsilon$$

Numerically

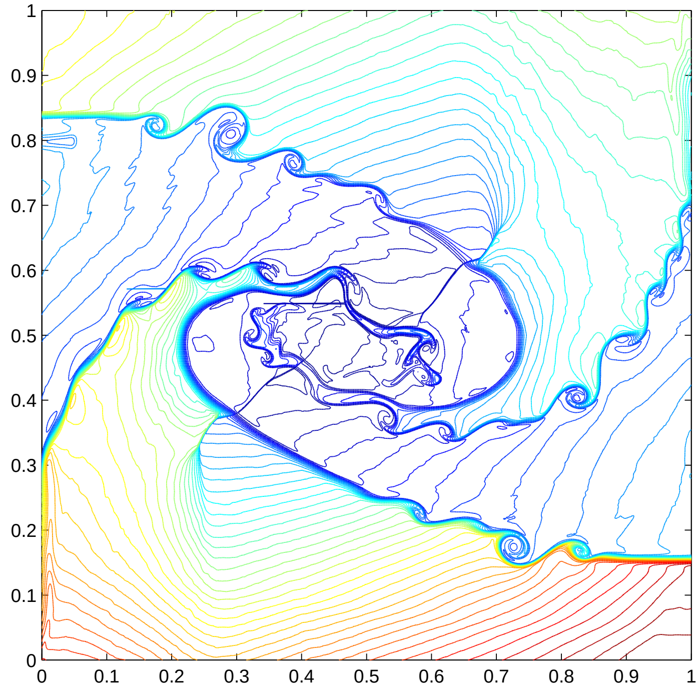


Discret L_1 norm : $\|\rho - \rho_{ref}\|_1$, $\|P - P_{ref}\|_1$ versus ϵ
dotted line $\epsilon^{1.6}$.

2D: Riemann-Problem; 4-contact Configuration with M&M-PHM

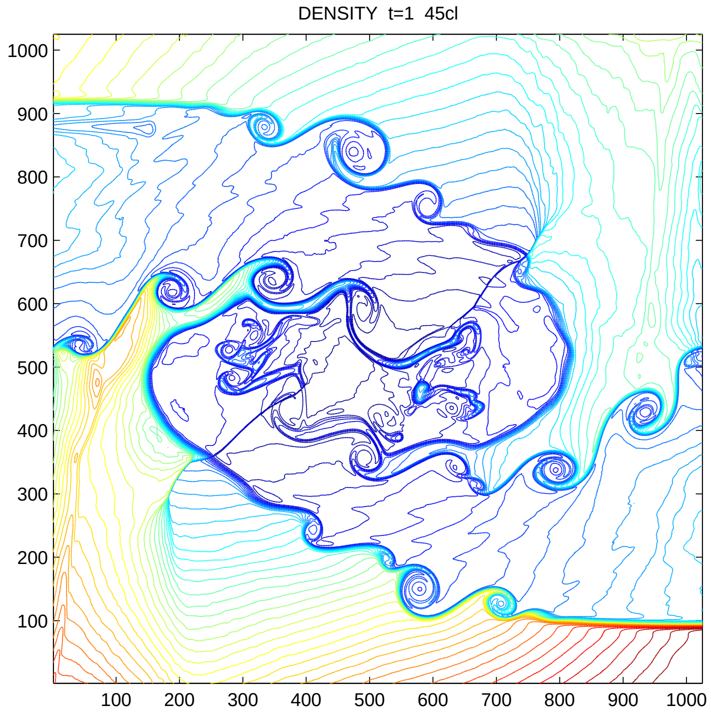


$(t = .8, \quad 512 \times 512)$



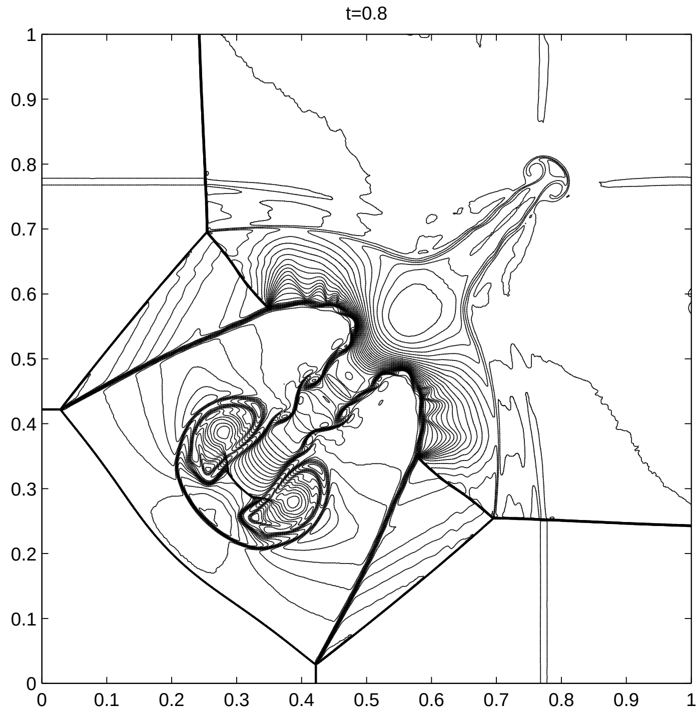
$(t = .8, \quad 1024 \times 1024)$

$\%D : 4 \rightarrow 28, \theta = 3.6$



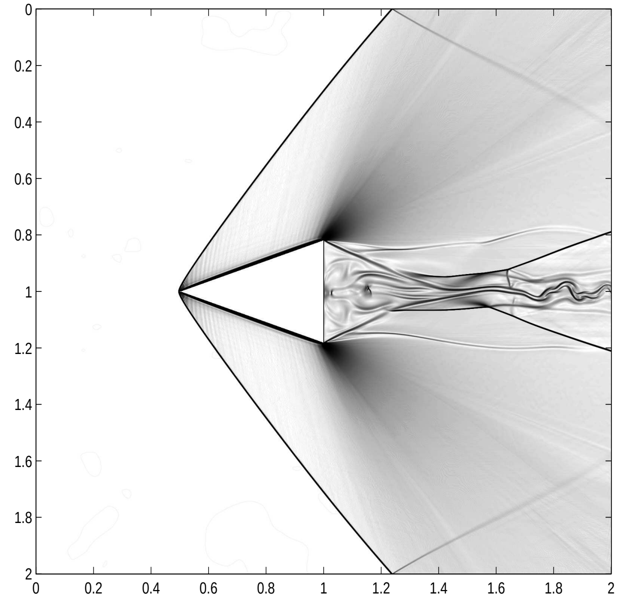
[Chiavassa, RD: SISC., 2001.]

$\%D : 3 \rightarrow 22, \theta = 4.2$



Conclusion

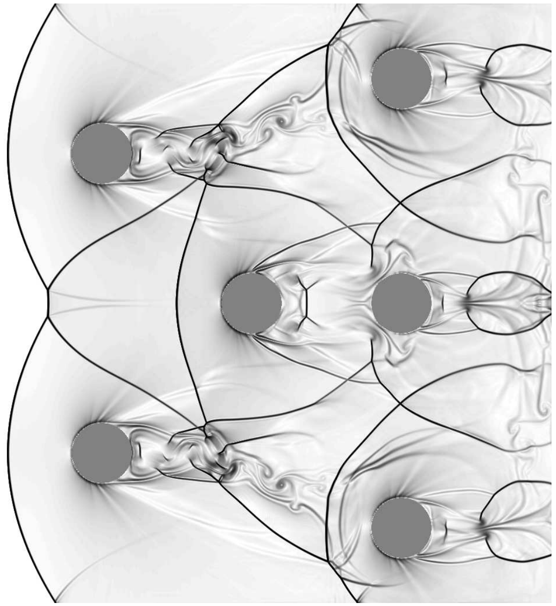
Grid 1536×1536



Mach 2, Density

CPU time: 2.5 days ($\%D_{max} = 10$, $\theta \sim 5$)

Complex Geometry



Mach 3, Density, 1024^2

CPU time: 1 day, same as for 1 cylinder

Future works: Drag, Lift, Wakes,... function of Reynolds number

Application to real industrial flows