Twisting *vs* bending in quantum waveguides

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Based on :

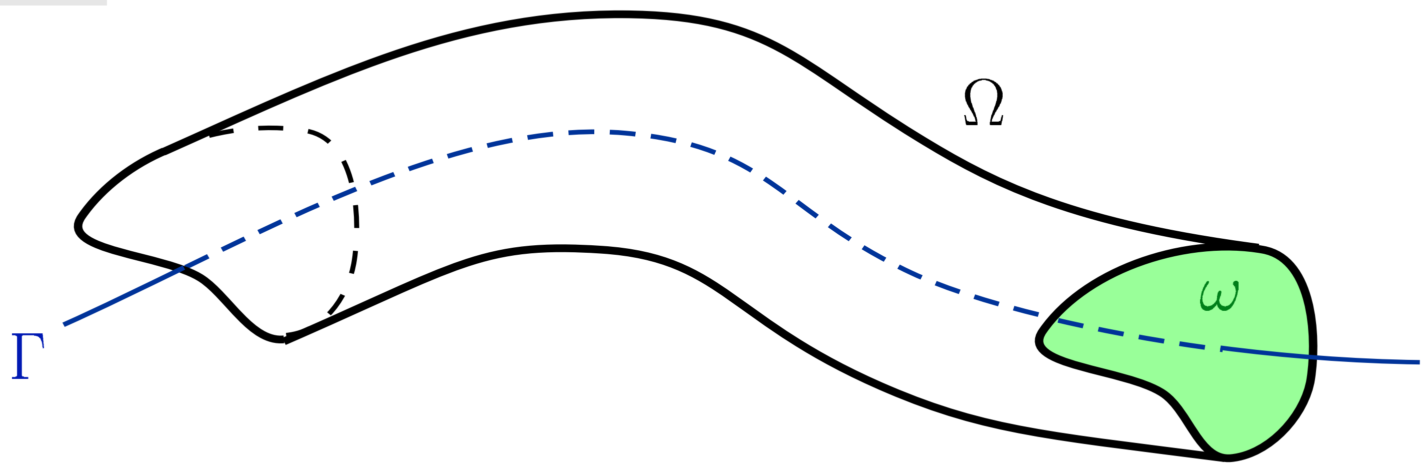
[B. Chenaud, P. Duclos, P. Freitas, D.K., Differential Geom. Appl. (2005)]

[T. Ekholm, H. Kovařík, D.K., submitted (2005)]

The Problem

Dirichlet Laplacian in a **tube** $\Omega \subset \mathbb{R}^3$ - of cross-section ω
- about an *infinite* curve Γ

$-\Delta_D^\Omega$



$$E_1 := \inf \sigma(-\Delta_D^\omega)$$

Straight geometry (*i.e.* $\Omega = \mathbb{R} \times \omega$) $\Longrightarrow$ $\sigma(-\Delta_D^\Omega) = \sigma_{\text{ac}}(-\Delta_D^\Omega) = [E_1, \infty)$

1. Which geometry preserves the essential spectrum $[E_1, \infty)$?
2. Which geometry produces a spectrum below E_1 ?

Motivations

Spectral Geometry

- relationship between geometry and spectral properties
- our *infinite* tubes are quasi-cylindrical domains !

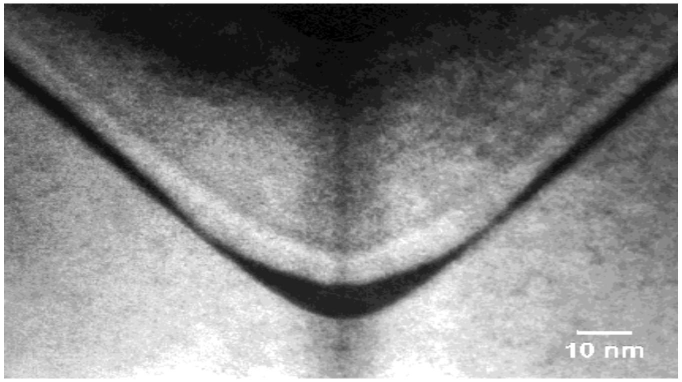
Quantum Mechanics (nanostructures)

- quantum waveguides

[Duclos, Exner, RMP (1995)]

[Londergan, Carini, Murdock, LNP (1999)]

GaAs/AlGaAs crescent shaped quantum wire



Classical Physics

- electromagnetic waveguides

setup AN/APS-134 X-band radar using Tallguide TG-134

Geometry of curved tubes

$\Gamma : \mathbb{R} \to \mathbb{R}^3$ unit-speed curve with curvatures κ_1, κ_2
- possessing an *appropriate* C^1-smooth Frenet frame $\{e_1, e_2, e_3\}$

$$\Rightarrow \text{Serret-Frenet formulae:} \quad \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & \kappa_2 \\ 0 & -\kappa_2 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$\omega \in \mathbb{R}^2$ open connected bounded set, $a := \sup_{t \in \omega} |t|$

$\mathcal{R}^\theta : \mathbb{R} \to \mathrm{SO}(2)$ family of rotation matrices: $\mathcal{R}^\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
- angle function $\theta \in C^1(\mathbb{R})$

$\Omega := \mathcal{L}(\mathbb{R} \times \omega)$ tube of cross-section ω:

$$\mathcal{L} : \mathbb{R} \times \omega \to \mathbb{R}^3 : \left\{ (s,t) \mapsto \Gamma(s) + t_\mu \mathcal{R}^\theta_{\mu\nu}(s)\, e_\nu(s) \right\} \qquad (\mu, \nu = 2, 3)$$

Assumption. $\|\kappa_1\|_\infty a < 1$ and Ω does not overlap itself

The Laplacian

$$-\Delta_D^\Omega \qquad \leftrightarrows \qquad Q_D^\Omega : W_0^{1,2}(\Omega) \longrightarrow L^2(\Omega) : \left\{u \longmapsto \|\nabla u\|^2\right\}$$

Strategy: $\mathcal{L} : \mathbb{R} \times \omega \to \Omega$ is a diffeomorphism $\Longrightarrow$ $\Omega \simeq \big(\mathbb{R} \times \omega, G\big)$

$$G = \begin{pmatrix} h^2 + h_\mu h_\mu & h_2 & h_3 \\ h_2 & 1 & 0 \\ h_3 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} h(s,t) &:= 1 - [t_2 \cos\theta(s) + t_3 \sin\theta(s)]\,\kappa_1(s) \\ h_2(s,t) &:= -t_3\,[\kappa_2(s) - \dot\theta(s)] \\ h_3(s,t) &:= t_2\,[\kappa_2(s) - \dot\theta(s)] \end{aligned}$$

1. $-\Delta_D^\Omega \;\simeq\; H \overset{\text{w}}{:=} -|G|^{-1/2}\,\partial_i\,|G|^{1/2}\,G^{ij}\,\partial_j$ on $L^2\big(\mathbb{R} \times \omega, d\mathrm{vol}\big)$

$$|G| := \det(G) = h^2, \qquad (G^{ij}) := G^{-1}, \qquad d\mathrm{vol} := h(s,t)\,ds\,dt$$

2. $H \;\simeq\; \tilde{H} \overset{\text{w}}{:=} -\partial_i\,G^{ij}\,\partial_j + V$ on $L^2\big(\mathbb{R} \times \omega\big)$ if κ_1 differentiable

$$V \overset{\text{w}}{:=} \partial_i(G^{ij}\,\partial_j F) + (\partial_i F)G^{ij}(\partial_j F), \qquad F := \log h^{1/2}$$

Stability of essential spectrum

Theorem. $$\lim_{|s|\to\infty}\big(|\kappa_1(s)| + |\kappa_2(s) - \dot\theta(s)|\big) = 0 \quad\Longrightarrow\quad \sigma_{\text{ess}}(-\Delta_D^\Omega) = [E_1, \infty)$$

Proof. Weyl's criterion for quadratic forms due to Iftimie. q.e.d.

Classical Weyl's criterion requires to impose additional conditions on derivatives !

History :

[Goldstone, Jaffe 1992] ... κ_1 of compact support & ω =disc

[Duclos, Exner 1995] ... additional vanishing of $\dot\kappa_1$ and $\ddot\kappa_1$ & ω =disc

[Dermenjian, Durand, Iftimie 1998] ... σ_{ess} of multistratified cylinders

[Chenaud, Duclos, Freitas, D.K. 2005] ... $\dot\theta = \kappa_2$ (ω arbitrary)

Geometrically induced spectrum

Theorem. $\kappa_1 \neq 0 \quad \& \quad \dot\theta = \kappa_2 \quad \Longrightarrow \quad \inf \sigma(-\Delta_D^\Omega) < E_1$

Proof. Trial function based on $\mathcal{J}_1$ ($\leftrightarrow$ E_1). q.e.d.

Corollary. $\sigma_{\text{disc}}(-\Delta_D^\Omega) \neq \varnothing$ if in addition $\lim_{|s|\to\infty} \kappa_1(s) = 0$

Remark. $\{e_1, \mathcal{R}_{2\mu}^\theta e_\mu, \mathcal{R}_{3\mu}^\theta e_\mu\}$ with $\dot\theta = \kappa_2$ is called Tang frame.

History :

[Goldstone, Jaffe 1992] ... κ_1 of compact support & $\omega =$disc

[Duclos, Exner 1995] ... additional conditions on $\dot\kappa_1$ and $\ddot\kappa_1$ & $\omega =$disc

[Chenaud, Duclos, Freitas, D.K. 2005] ... ω arbitrary

Why do we have the curvature-induced eigenvalues ?

N.B. $-\Delta_D^\Omega \quad \simeq \quad \tilde{H} = -\partial_i G^{ij} \partial_j + V$ on $L^2(\mathbb{R} \times \omega)$

In the limit $a \to 0$, $$\tilde{H} \sim \left(-\Delta^{\mathbb{R}} - \tfrac{1}{4}\kappa_1^2\right) \otimes 1 + 1 \otimes \left(-\Delta_D^\omega\right)$$

Here $-\frac{1}{4}\kappa_1^2$ represents an attractive interaction as long as $\begin{cases} \kappa_1 \neq 0 \\ \kappa_1 \xrightarrow{\infty} 0 \end{cases}$

It turns out that the discrete spectrum exists for any a *provided* $\dot{\theta} = \kappa_2$.
In particular, whenever ω is circular.

Is the choice $\dot{\theta} = \kappa_2$ just a technical hypothesis for non-circular ω ?

NO !

A lower bound to the spectral threshold

[Exner, Freitas, D.K. 2004]

Theorem. $\dot{\theta} = 0 \quad \Longrightarrow \quad \inf \sigma(-\Delta_D^\Omega) \geq \min \left\{ \lambda(\sup \kappa_1), \lambda(\inf \kappa_1) \right\}$

where $\lambda(\kappa)$ denotes the lowest eigenvalue of the Dirichlet Laplacian in the torus of cross-section ω about a circle of radius κ^{-1}.

Remark 1. The lower bound does not depend on torsion κ_2.

QMath9

Giens 2004

Conjecture [Weidl 2004]. $\exists$ Hardy inequality in twisted tubes

Remark 2. It is already known that there exists a Hardy inequality in curved *strips* in the presence of local magnetic field due to [Ekholm, Kovařík 2004].

A Hardy inequality in twisted tubes

[Ekholm, Kovařík, D.K. 2005]

twisted tube $:\Longleftrightarrow \begin{cases} (1) \ \exists \alpha \in (0, 2\pi), \ \left\{ \left(t_\mu \mathcal{R}_{\mu 2}^\alpha, t_\mu \mathcal{R}_{\mu 3}^\alpha \right) \mid (t_2, t_3) \in \omega \right\} \neq \omega \\ (2) \ \kappa_2 - \dot{\theta} \neq 0 \end{cases}$

angular-derivative operator : $\partial_\tau := t_3 \, \partial_3 - t_2 \, \partial_3$

Theorem. Assume (1). Let $\sigma \in C_0(\mathbb{R})$ with $\dot{\sigma} \in L^\infty(\mathbb{R})$ satisfy (2') $\sigma \neq 0$.

$$L_\sigma := \boxed{-(\partial_1 - \sigma \, \partial_\tau)^2 - \partial_2^2 - \partial_3^2 \ \geq \ E_1 + \frac{c}{1 + (s - s_0)^2}} \quad \text{on} \quad L^2(\mathbb{R} \times \omega).$$

Here $s_0 \in \mathbb{R}$ is such that $\sigma(s_0) \neq 0$ and $c = c(s_0, \sigma, \omega) > 0$.

Remark. $\sigma = \dot{\theta} \ \rightsquigarrow$ Dirichlet Laplacian in twisted straight tubes

Proof. Writing $\psi(s,t) = \mathcal{J}_1(t) \, \phi(s,t), \ \psi \in C_0^\infty(\mathbb{R} \times \omega)$,

$$\big(\psi, [L_\sigma - E_1] \, \psi\big) = \|\mathcal{J}_1 \partial_1 \phi\|^2 + \|\mathcal{J}_1 \partial_2 \phi\|^2 + \|\mathcal{J}_1 \partial_3 \phi\|^2$$

$$+ \|\sigma \, (\mathcal{J}_1 \partial_\tau \phi + \phi \, \partial_\tau \mathcal{J}_1)\|^2 + \text{mixed terms}$$

... q.e.d.

Twisted bent tubes

[Ekholm, Kovařík, D.K. 2005]

We restrict to curves characterised by:
$$\begin{cases} \kappa_1,\, \kappa_2 \in C^1(\mathbb{R})\,, & \\ \kappa_1 > 0 & \text{on} \quad I \text{ (bounded)}\,, \\ \kappa_1,\, \kappa_2 = 0 & \text{on} \quad \mathbb{R} \setminus I\,. \end{cases}$$

and rotations determined by: $\left\{ \theta \in C_0^1(\mathbb{R})\,, \quad \ddot{\theta} \in L^\infty(\mathbb{R})\,. \right.$

Theorem 1. Assume (1). If $\kappa_2 - \dot{\theta} \neq 0$ then there exists $\varepsilon > 0$ such that

$$\|\kappa_1\|_\infty + \|\dot{\kappa}_1\|_\infty \leq \varepsilon \quad \Longrightarrow \quad \sigma(-\Delta_D^\Omega) = [E_1, \infty)$$

Here $\varepsilon = \varepsilon(\kappa_2, \dot{\theta}, \omega)$.

Theorem 2. Assume (1). If $\dot{\theta} \neq 0$ then there exists $\varepsilon > 0$ such that

$$\|\kappa_1\|_\infty + \|\dot{\kappa}_1\|_\infty + \|\kappa_2\|_\infty \leq \varepsilon \quad \Longrightarrow \quad \sigma(-\Delta_D^\Omega) = [E_1, \infty)$$

Here $\varepsilon = \varepsilon(I, \dot{\theta}, \omega)$.

Remark. Theorem 1 contains a better lower bound than [Exner, Freitas, D.K. 2004].

Conclusions

Summary : Spectral analysis of the Dirichlet Laplacian in infinite curved tubes

- → stability of σ_{ess} if the bending and twisting vanish at infinity
- → instability of σ_{disc} due to bending (no twisting)
- → stability of σ_{disc} due to twisting (small bending)
- → Hardy inequality in twisted tubes

Possible extensions : (concerning the Hardy inequality)

- ◇ compact support of curvatures $\mapsto$ $\mathcal{O}(s^{-2})$ decay at infinity
- ◇ bending $\mapsto$ other perturbations (enlargement, potential-type, *etc*)

Open problems :

Hardy inequality :

- ¿ slowly decaying bending ?
- ¿ higher-dimensional generalisations ? (OK for rotations just in one hyperplane)
- ¿ effect of twisting on the essential spectrum ? (embedded eigenvalues, resonances)
- ¿ other boundary conditions ? (acoustic waveguides)

in general :

- ¿ detailed analysis of essential spectrum ? [D.K., Tiedra 2004]