Blow-up in finite time vs. globally defined solutions in reaction-diffusion equations with nonlinear boundary conditions

José M. Arrieta

Universidad Complutense
Madrid, Spain

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$$
\begin{cases}
u_t - \Delta u = \lambda(x) u^p & \text{in } \Omega \\
u = 0 & \text{on } \Gamma_D \\
\frac{\partial u}{\partial \vec{n}} = \alpha(x) u^q & \text{on } \Gamma_N \\
u(0,x) = u_0(x) \geq 0 & \text{in } \Omega
\end{cases}
\tag{1}
$$

i) $\Omega \subset \mathbb{R}^N$ bounded C^2 domain

ii) $p, q \geq 1$, $\lambda \in C^0(\bar{\Omega})$ and $\alpha \in C^0(\partial\Omega)$.

- If $p > 1$ and $\lambda(x) \geq \lambda_0 > 0$ in $B(x_0, R) \subset \Omega$, then there are solutions that blow up in finite time. For this, notice that the solutions of

$$\begin{cases} u_t - \Delta u = \lambda_0 u^p & \text{in } B(x_0, R) \\ u = 0 & \text{on } \partial B(x_0, R) \\ u(0, x) \geq 0 & \text{in } B(x_0, R) \end{cases}$$

  are subsolutions of the original problem and for this problem we have solutions that blow-up in finite time.

- If $\lambda(x) \leq 0$ and $\alpha(x) \leq 0$, that is, both, the interior reaction and the flux at the boundary are dissipative mechanisms, then we have that all solutions are globally defined and bounded. No blow-up is produced.

The most interesting case is when the interior reaction is dissipative ($\lambda(x) \leq 0$) and the boundary condition, puts energy into the system ($\alpha(x) \geq 0$). It is not clear whether these two competing mechanisms will produce blow-up or boundedness of solutions.

Hence, we denote by $\lambda(x) = -\beta(x) \leq 0$ and will assume that $\alpha(x) \equiv 1$:

$$\begin{cases} u_t - \Delta u = -\beta(x) u^p & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial \vec{n}} = u^q & \text{on } \Gamma_N \\ u(0,x) = u_0(x) \geq 0 & \text{in } \Omega \end{cases} \tag{2}$$

We are interested in determining the relative sizes of p, q and $\beta(x)$ that will garantee that some solutions will blow-up. Moreover we are interested in proving that the conditions are of a local nature in the sense that if the conditions hold in a small neighborhood of certain point of the boundary, then blow up is obtained near that point.

Proper minimal solutions. (Baras & Cohen '87, Galaktionov & Vazquez '97, 02)

Consider a sequence of smooth functions $g_n(u)$ which are globally Lipschitz and approach $g(u) = u^q$ monotonically from below. For instance, $g_n(u) = \mathrm{mín}\{u^q, n\}$. Denote by $u_n(t, x, u_0)$ the solution of

$$\begin{cases} u_t - \Delta u = -\beta(x)u^p & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial \vec{n}} = g_n(u) & \text{on } \Gamma_N \\ u(0, x) = u_0(x) \geq 0 & \text{in } \Omega \end{cases}$$

Then, u_n is defined for all $t \in [0, \infty)$ and for each fixed (t, x) it is monotone increasing in n. Hence, there exists a function $u(t, x, u_0)$ that maybe infitnity at some points (t, x) such that

$$u_n(t, x, u_0) \nearrow u(t, x, u_0)$$

M. Chipot, M. Fila, P. Quittner,“Stationary Solutions, blow up and convergence ...” Acta. Math. Univ. Comenianae Vol LX (1991)

$$(N = 1,\ \Omega = (0, l)) \begin{cases} u_t - u_{xx} = -\beta u^p & \Omega \\ \frac{\partial u}{\partial n} = u^q & \partial\Omega \\ u(0, x) \geq 0 & \Omega \end{cases}$$

If $p + 1 < 2q$ or $p + 1 = 2q$ and $\beta < q$, then $\exists$ blow up

If $p + 1 > 2q$ or $p + 1 = 2q$ and $\beta > q$, then all solutions globally bounded.

Balance: $p + 1$ vs. $2q$ and if $p + 1 = 2q$, β vs. q.

A. Rodriguez-Bernal and A. Tajdine, “Nonlinear balance of reaction-diffusion ...” Journal of Diff. Eq. (2001) showed that for the problem

$$(N \geq 1) \begin{cases} u_t - \Delta u = -\beta u^p & \Omega \\ \frac{\partial u}{\partial n} = u^q & \partial\Omega \\ u(0,x) \geq 0 & \Omega \end{cases}$$

If $p+1 < 2q$ or $p+1 = 2q$ and $\beta < q$, then $\exists$ blow up

If $p+1 > 2q$ or $p+1 = 2q$ and β large enough, then all solutions globally bounded.

F. Andreu, J. Mazón, J. Toledo, J. Rossi, “Porous medium equation ...”, Nonlinear Analysis (2002)

They obtain for the N-dimensional problem the same balances as Chipot, Fila, Quittner

There are many more names associated to the problem of blow-up with nonlinear boundary conditions:

J. Filo, V. Galaktionov, J. Guo, B. Hu, G. Lieberman, J. López-Gómez, M. Marcus, Ph. Souplet, J. L. Vázquez, L. Véron, ...

and there are also many other interesting questions:

- rates, profiles, blow-up sets, continuation after blow-up, complete vs. incomplete blow-up, etc...

LOCALIZATION OF BLOW UP

We were able to prove the following result on blow-up:

Proposition 1. *(A. Rodríguez-Bernal, J.A.(2004))*

Let $p \geq 1$, $q > 1$ *and* $x_0 \in \Gamma_N$. *If one of the two following conditions hold*

i) $p + 1 < 2q$ *or*

ii) $p + 1 = 2q$ *and* $\beta(x_0) < q$,

then, there exists an initial condition ϕ_0 *with support in a small neighborhood of* x_0, *whose solution blows up in finite time.*

To be more precise, there exists a smooth function $v(t, x)$, *defined for* $t \in [0, T) \times \big(\Omega \cap B(x_0, \rho)\big)$ *monotone increasing in time such that*

$$v(t,x) \nearrow v(T,x) = \frac{C}{\textit{dist}(x,\partial\Omega)^{2/(p-1)}}, \qquad \textit{for } x \in \Omega \cap B(x_0,\rho).$$

$$v(t,x) \leq u(t,x,u_0), \quad \textit{for} \quad (t,x) \in [0,T) \times \Big(\Omega \cap B(x_0,\rho)\Big)$$

and there exists $\tau > T$*, such that*

$$v(T,x) \leq u(t,x,u_0), \quad \textit{for any} \quad T \leq t \leq \tau$$

A. Rodríguez-Bernal, J.A. “Localization on the boundary of blow-up for reaction-diffusion equations with nonlinear boundary conditions”, Comm. in PDE’s 29 (2004)

Remark 2. *i) The time T does not need to be the classical blow-up time T_∞, that is, the time for which the solution $u(t,x,u_0)$ satisfies*

$$u(t,x,u_0) < +\infty, \quad 0 \le t < T_\infty$$

$$\|u(t,x,u_0)\|_{L^\infty(\Omega)} \to +\infty, \text{ as } t \nearrow T_\infty$$

We always have $T_\infty \le T$.

ii) Local nature of the result: the blow-up phenomena is localized around x_0, independently of the behavior of $\beta(x)$ at other points of the boundary.

iii) If $\beta(x) > 0$ in Ω and $p > 1$ then there is no blow up in the interior of the domain

iv) If the nonlinear boundary condition is $\frac{\partial u}{\partial n} = \alpha(x)u^q$ with $\alpha(x_0) > 0$, then, the condition for blow up in case ii) is $\frac{\beta(x_0)}{\alpha^2(x_0)} < q$.

Proof. We will consider the case $p+1=2q$, $\beta(x_0)<q$.

Let us consider the one dimensional case $\Omega=(-1,0)$, $x_0=0$ and assume $\beta(x)\leq\beta_0<q$ in $(-b,0)$ for some b small.

If $v(t,x)$ is a smooth positive function in $(x,t)\in(-b,0)\times(0,\tau)$ such that

$$\begin{cases} v_t - v_{xx} \leq -\beta_0 v^p & \text{in } (-b,0) \\ v(-b) \leq u(t,-b) & \\ \frac{\partial v}{\partial \vec{n}} \leq v^q & \text{on } x=0 \\ v(0,x) \leq u_0(x) & \text{in } (-b,0) \end{cases}$$

then, $v(t,x)\leq u(t,x,u_0)$, $x\in(-b,0)$, $0<t<\tau$.

We will construct a function v such that $v(t,0)\to+\infty$ as $t\to\tau^-$.

Consider the solution of the ODE

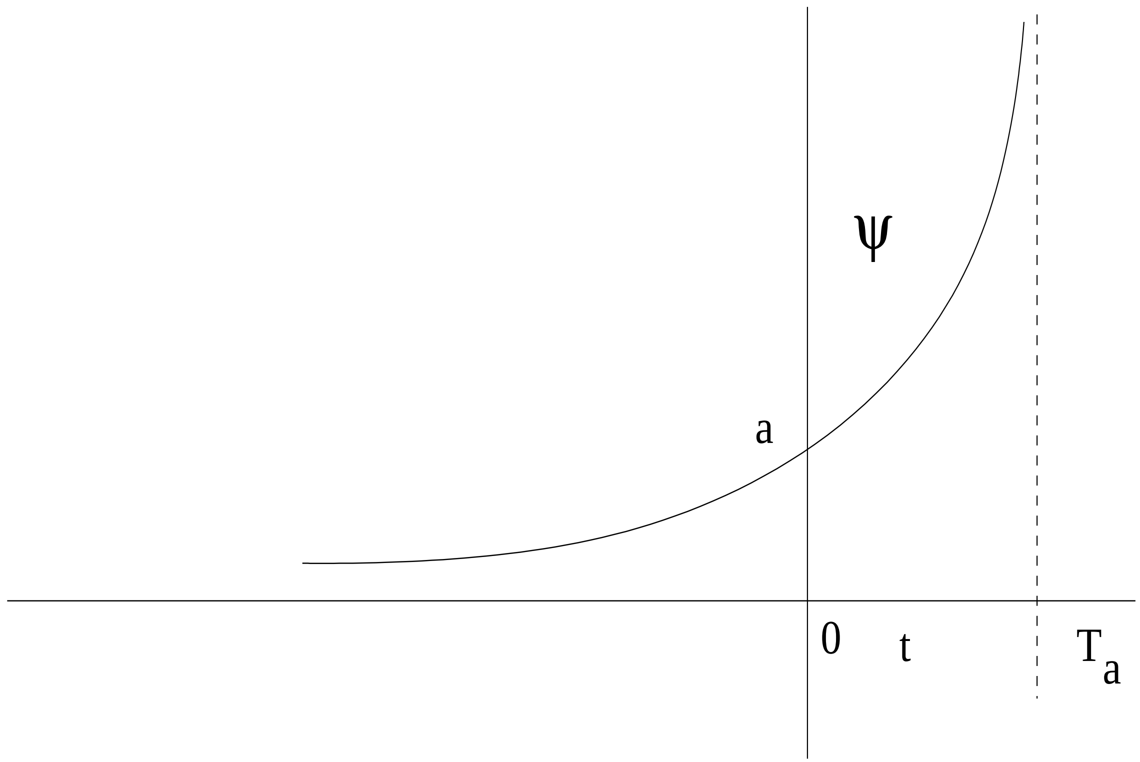
$$\begin{cases} \psi'(t) = \psi^q(t) \\ \psi(0) = a \geq 1 \end{cases}$$

which is given explicitely by

$$\psi_a(t) = \frac{E}{(T_a - t)^{\frac{1}{q-1}}}, \qquad -\infty < t < T_a$$

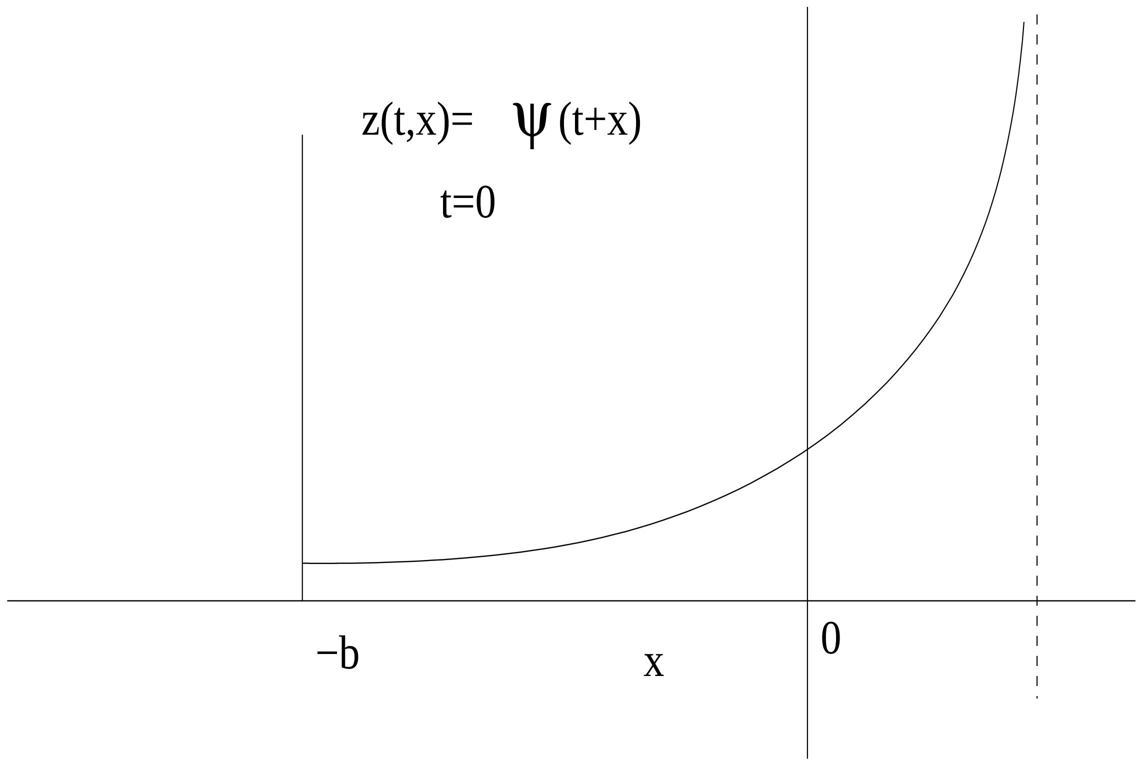
with

$$E = \frac{1}{((q-1))^{\frac{1}{q-1}}}, \quad \text{and} \quad T_a = \frac{1}{(q-1)a^{q-1}}$$

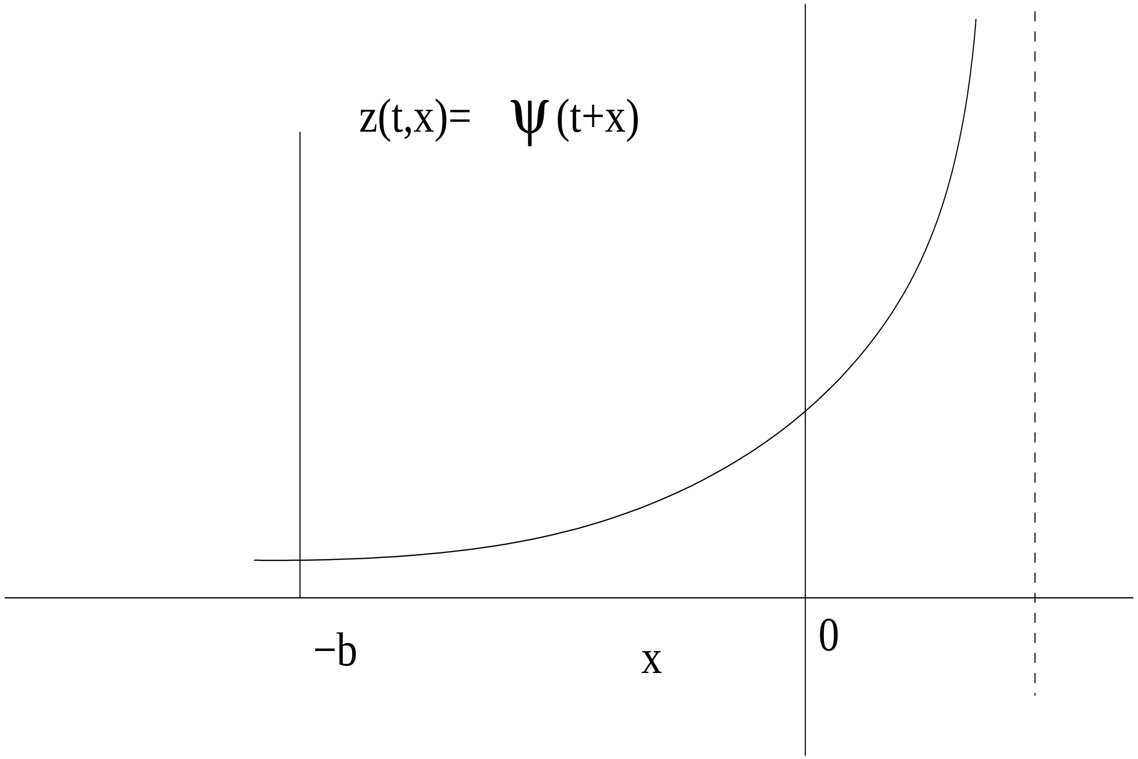

ψ
a
0
t
Ta

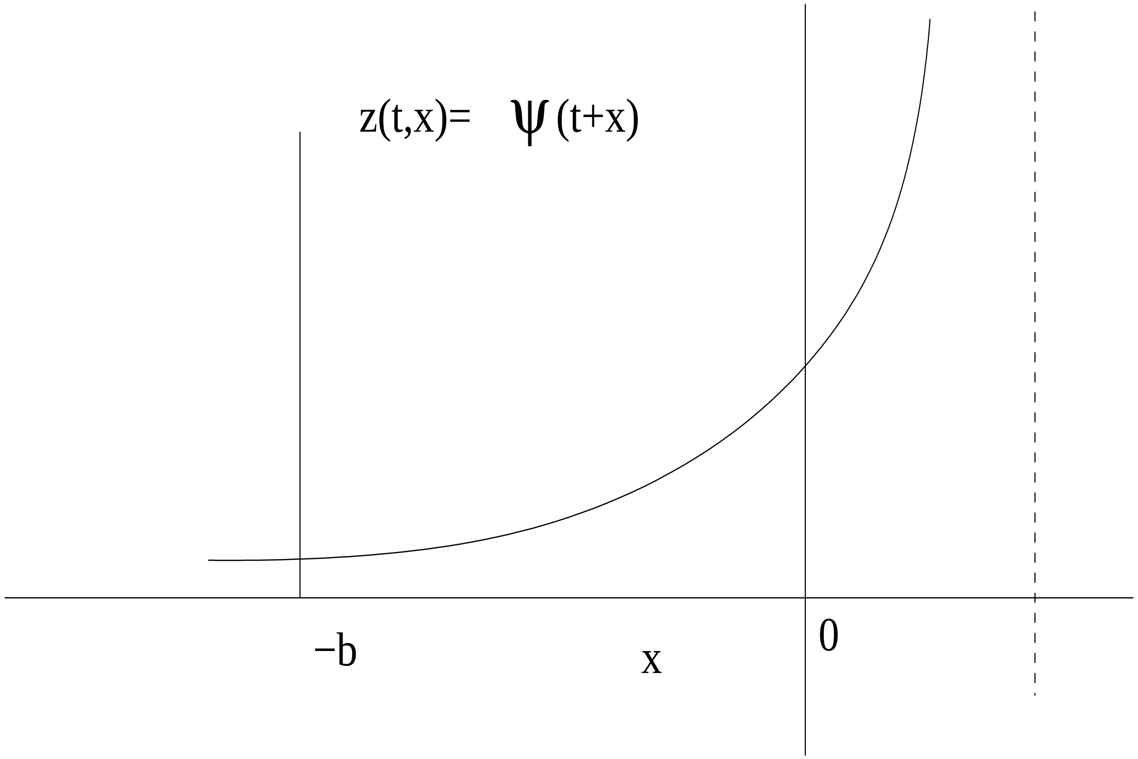
Let us define the function v as

$$v(t,x) = \psi(t+x), \quad 0 < t < T_a,\ -b < x < 0$$

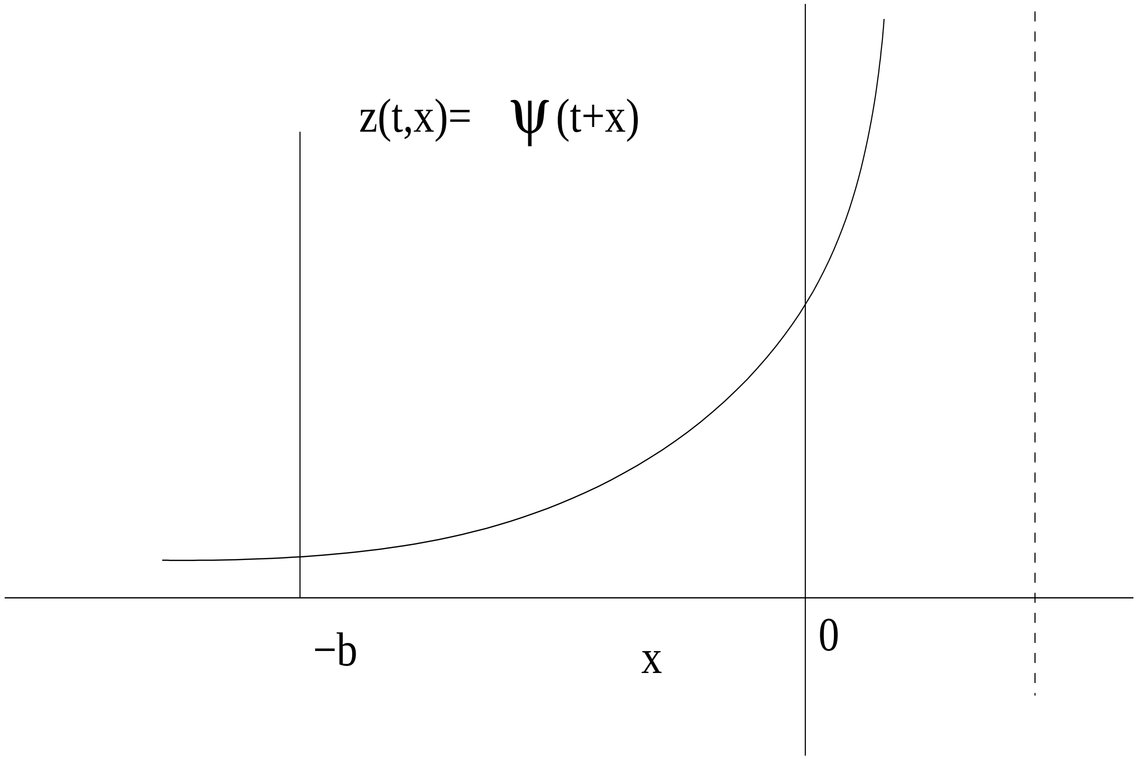


z(t,x)= ψ(t+x)
t=0
−b
x
0


z(t,x)= ψ(t+x)
−b
x
0



z(t,x)= ψ(t+x)
−b
x
0


z(t,x)= ψ (t+x)
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x
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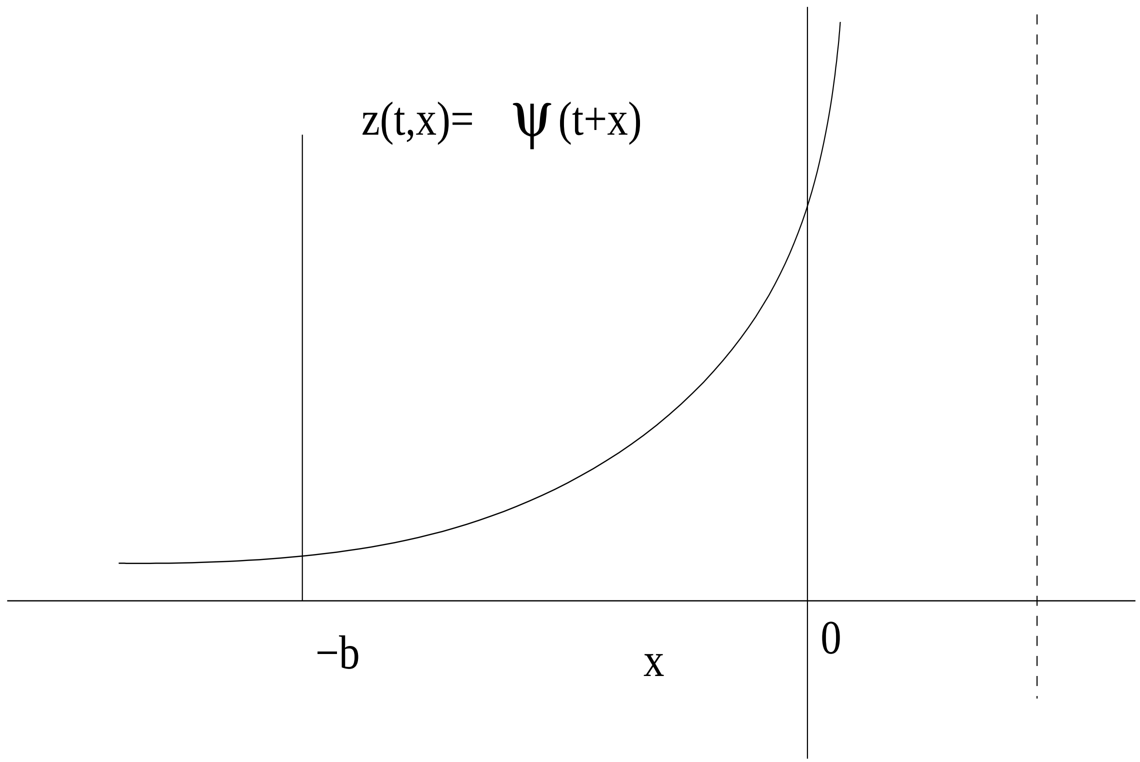

z(t,x)= ψ (t+x)
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x
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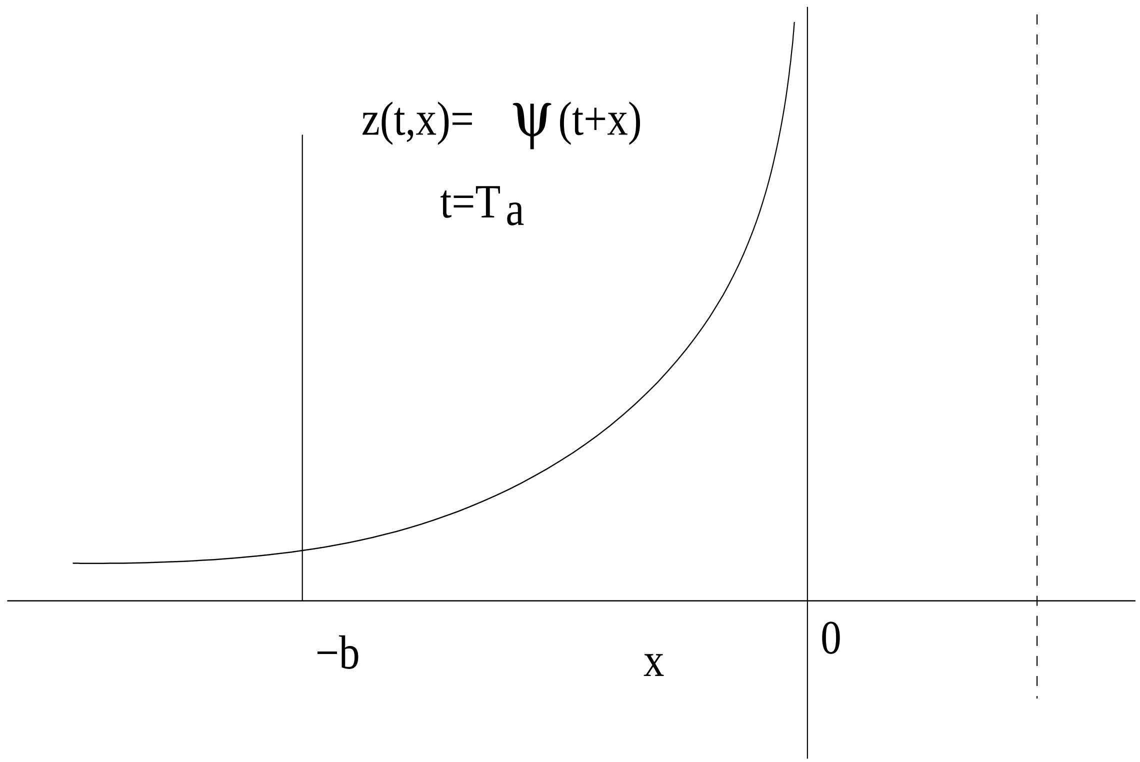

z(t,x)= ψ (t+x)
t=T a
−b
x
0

If $v(t,x) = \psi(t+x)$ and $\psi' = \psi^q$, we have

$$v_t = v_x = \psi' = v^q$$

$$v_{xx} = qv^{q-1}v_x = qv^{2q-1} = qv^p$$

Hence,

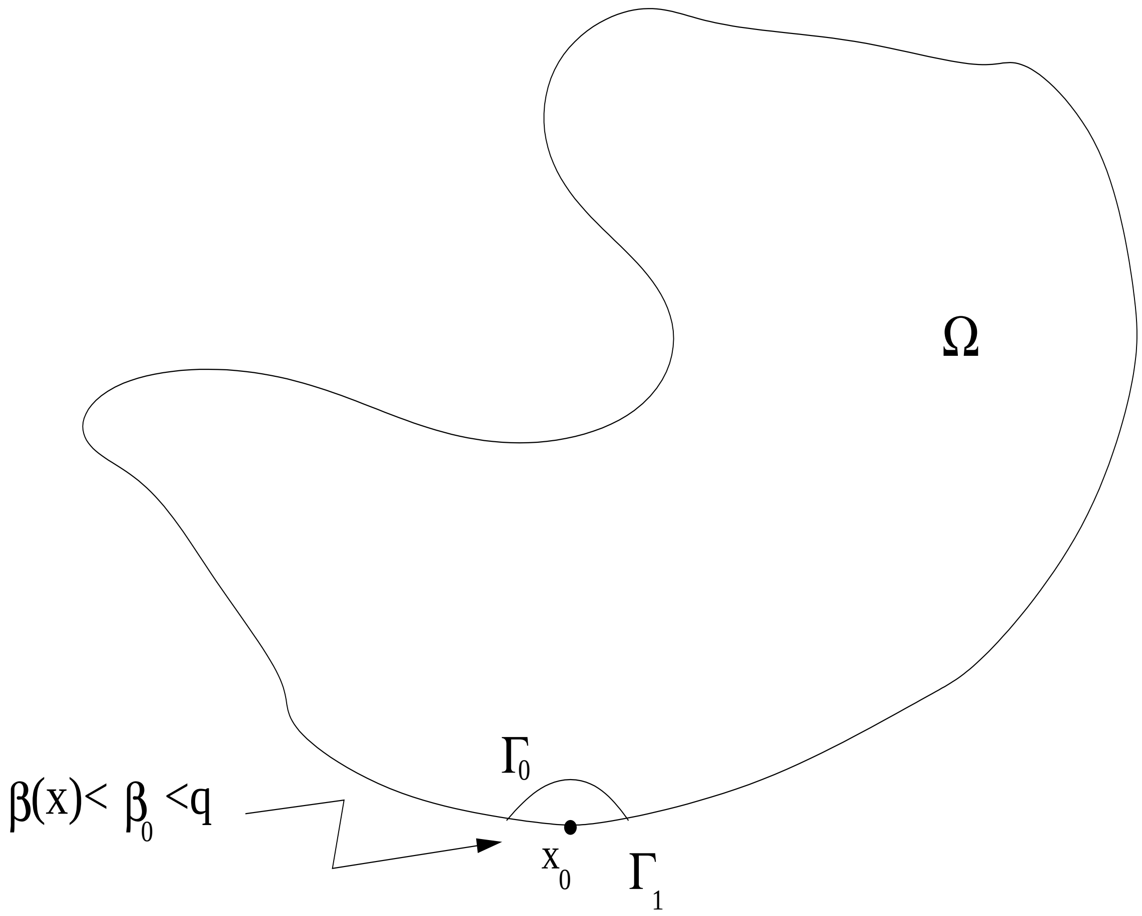
$$v_t - v_{xx} = v^q - qv^{2q-1} = (v^{q-p} - q)v^p \leq -\beta_0 v^p$$

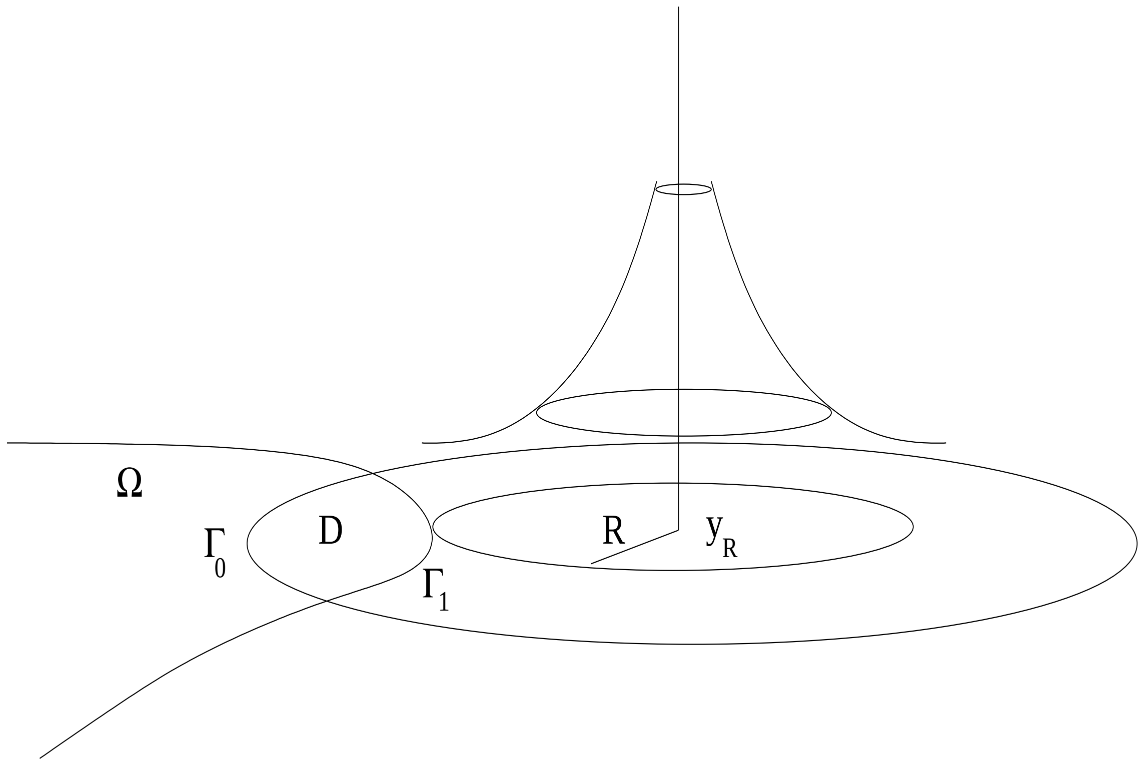
as long as a, b are small enough (so that v is large enough) and $\beta_0 < q$.

$$\frac{\partial v}{\partial n}(t,0) = v_x(t,0) = \psi'(t) = v^q(t,0)$$
$$v(0,x) = \psi(x) \leq \psi(0) = a$$
$$v(t,-b) = \psi(t-b) \leq \psi(T_a - b) = \frac{E}{b^{1/(q-1)}}$$

Hence, we just need to choose an initial condition u_0 large enough so that $u_0(x) \geq a$, $x \in (-b,0)$ and $u(t,-b,u_0) \geq \frac{E}{b^{1/(q-1)}}$ for $t \in (0,T_a)$.

For the higher dimensional case we proceed similarly:




Ω
D
Γ_0
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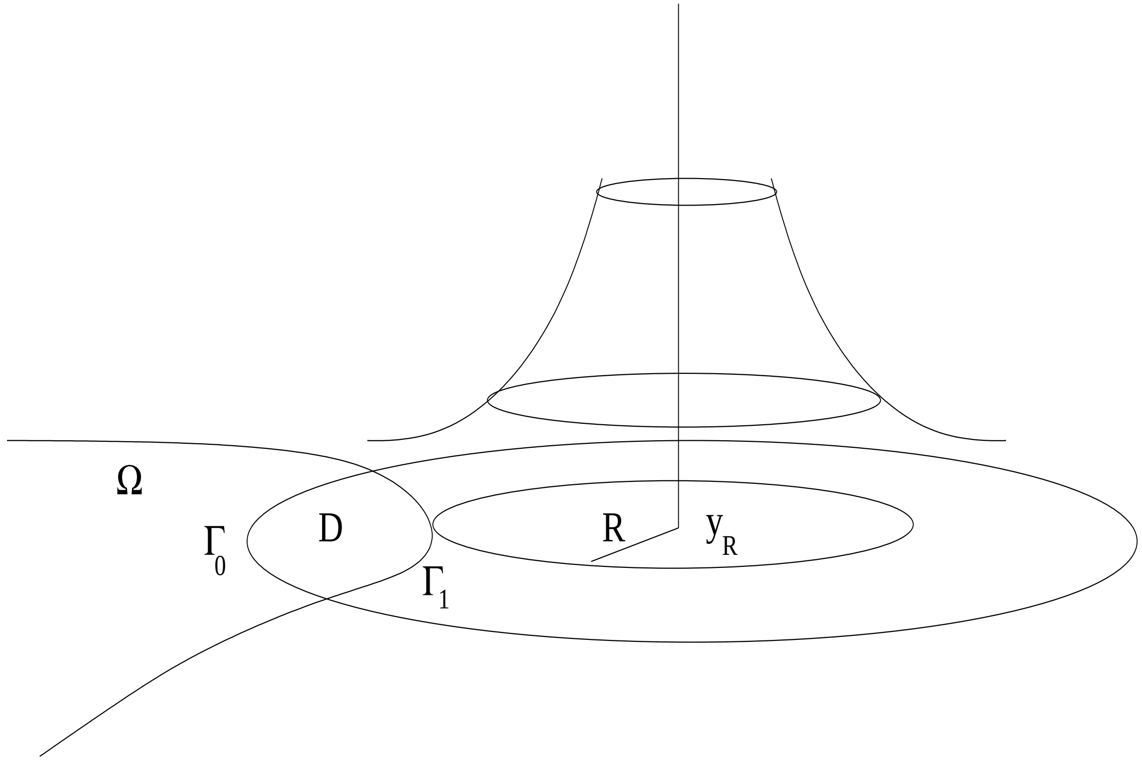
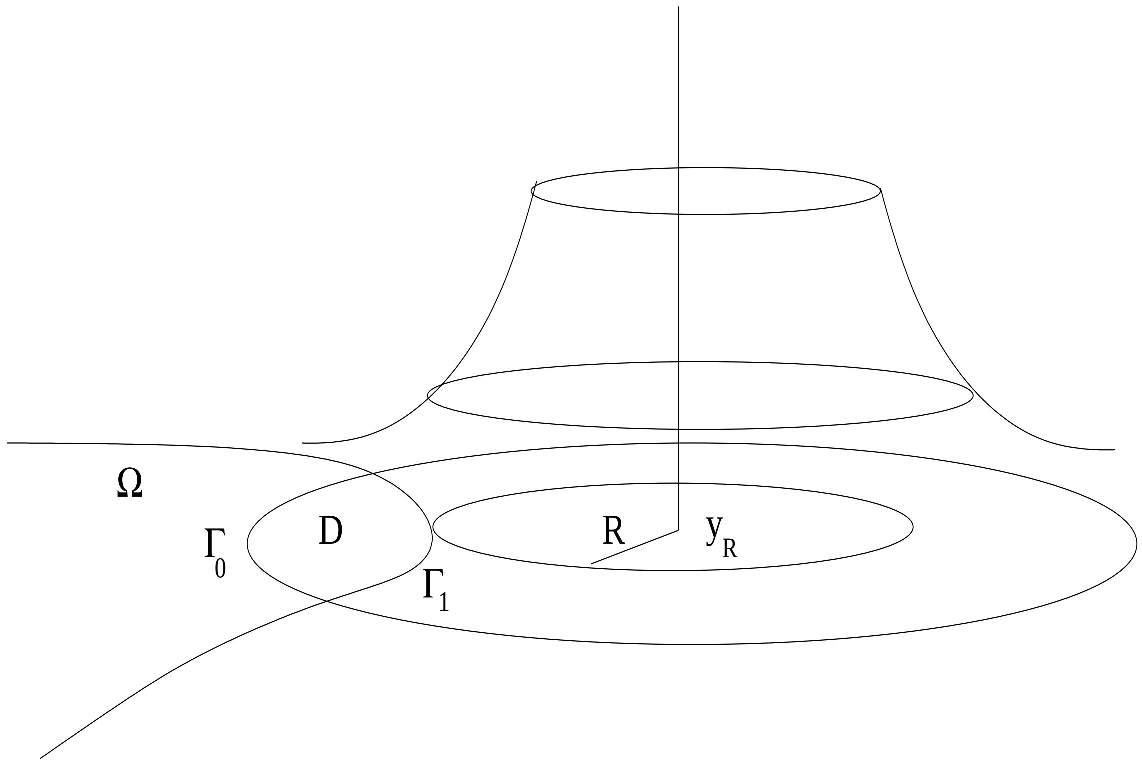
R
y_R



Ω
D
R
y_R
Γ_0
Γ_1


Ω
D
R
y_R
Γ_0
Γ_1

LOCALIZATION OF GLOBAL BOUNDEDNESS

Proposition 3. *(J.A.(2005))*

Let $p > 1$, $q \geq 1$ *and* $x_0 \in \Gamma_N$. *If one of the two following conditions hold*

i) $p + 1 > 2q$ *and* $\beta(x_0) > 0$ *or*

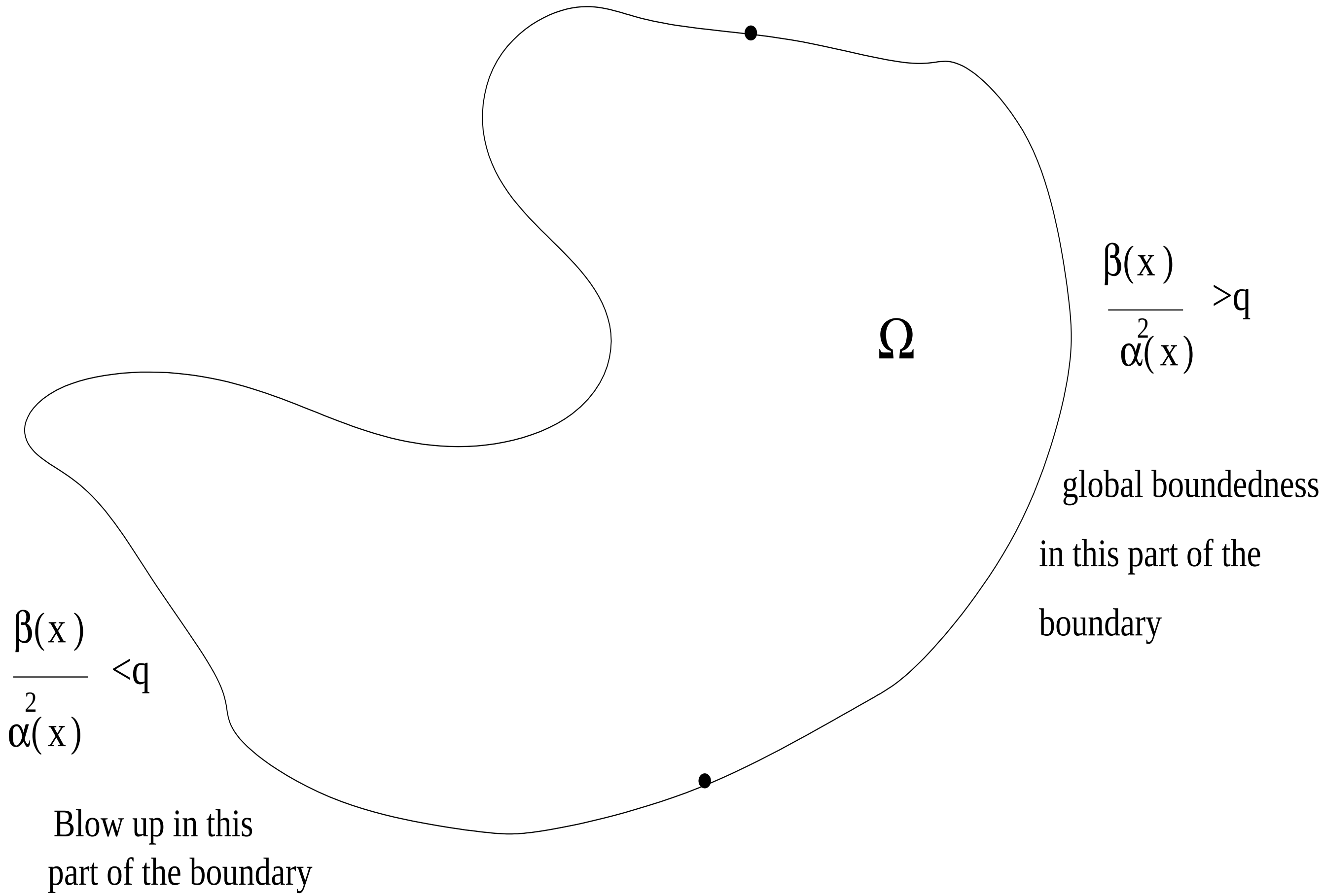
ii) $p + 1 = 2q$ *and* $\beta(x_0) > q$,

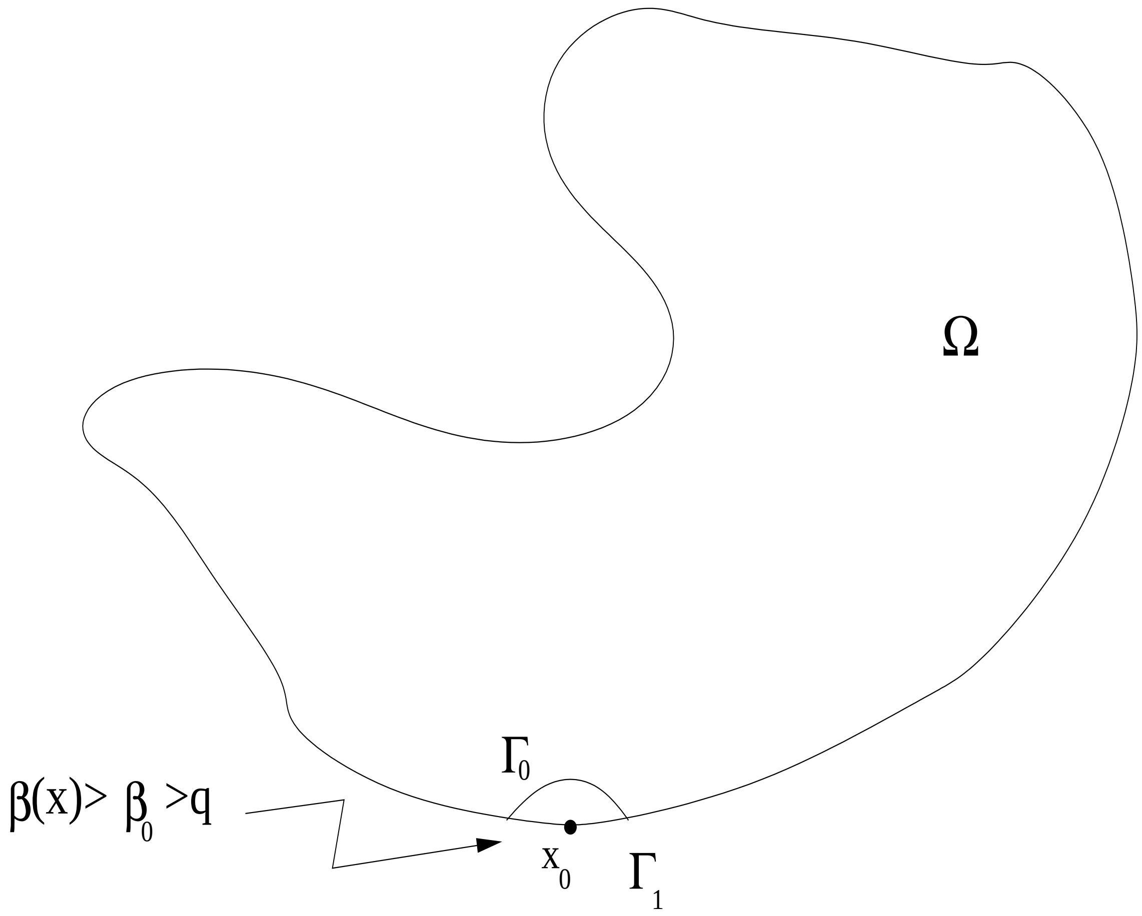
then, for any initial condition $0 \leq u_0 \in L^\infty(\Omega)$ *the proper minimal solution starting at* u_0 *is bounded in a neighborhood of* x_0 *in* $\bar{\Omega}$, *for all* $t > 0$. *That is, there exist* $\delta, M > 0$ *such that*

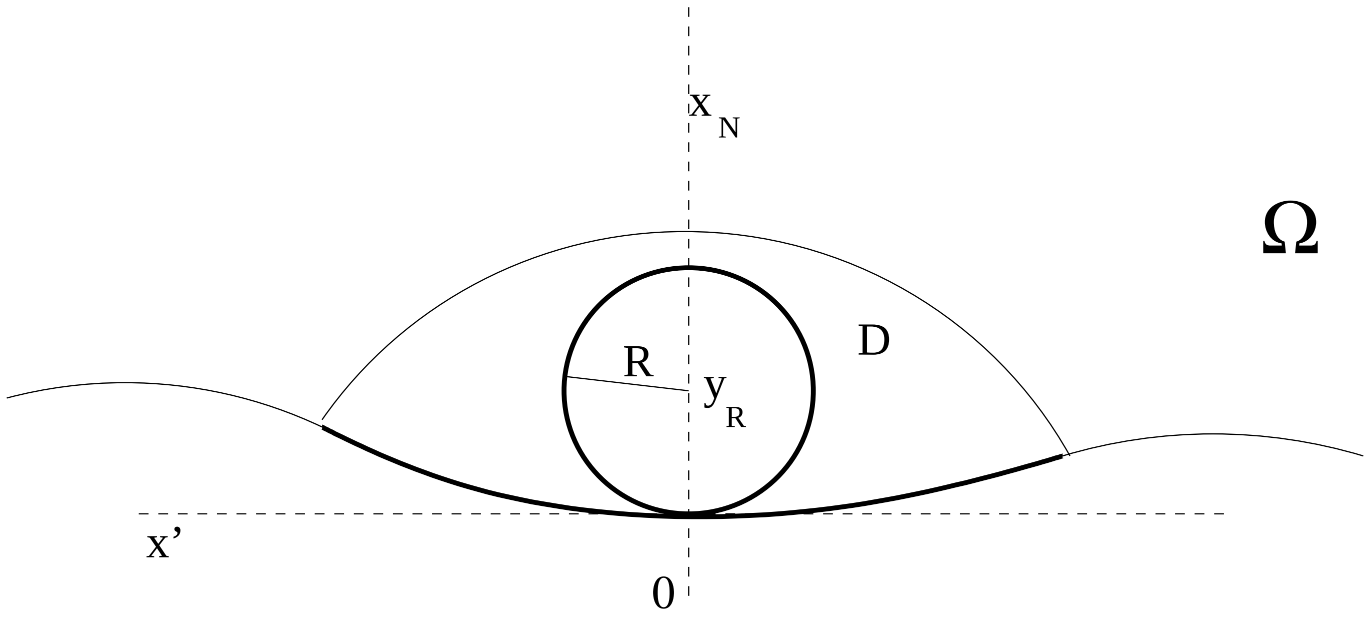
$$\sup_{0 \leq t < \infty,\, x \in B(x_0, \delta) \cap \bar{\Omega}} u(t, x, u_0) \leq M$$

Remark 4. *i) Local nature of the result: the boundedness of the solution is localized around* x_0*, independently of the behavior of* $\beta(x)$ *at other points.*

ii) If the nonlinear boundary condition is $\frac{\partial u}{\partial n} = \alpha(x)u^q$ *with* $\alpha(x_0) > 0$*, then, the condition for boundedness is in case ii) is* $\frac{\beta(x_0)}{\alpha^2(x_0)} > q$.


Ω
$\frac{\beta(x)}{\alpha^2(x)} > q$
global boundedness
in this part of the
boundary
$\frac{\beta(x)}{\alpha^2(x)} < q$
Blow up in this
part of the boundary


Ω
Γ_0
$\beta(x) > \beta_0 > q$
x_0
Γ_1



x_N
Ω
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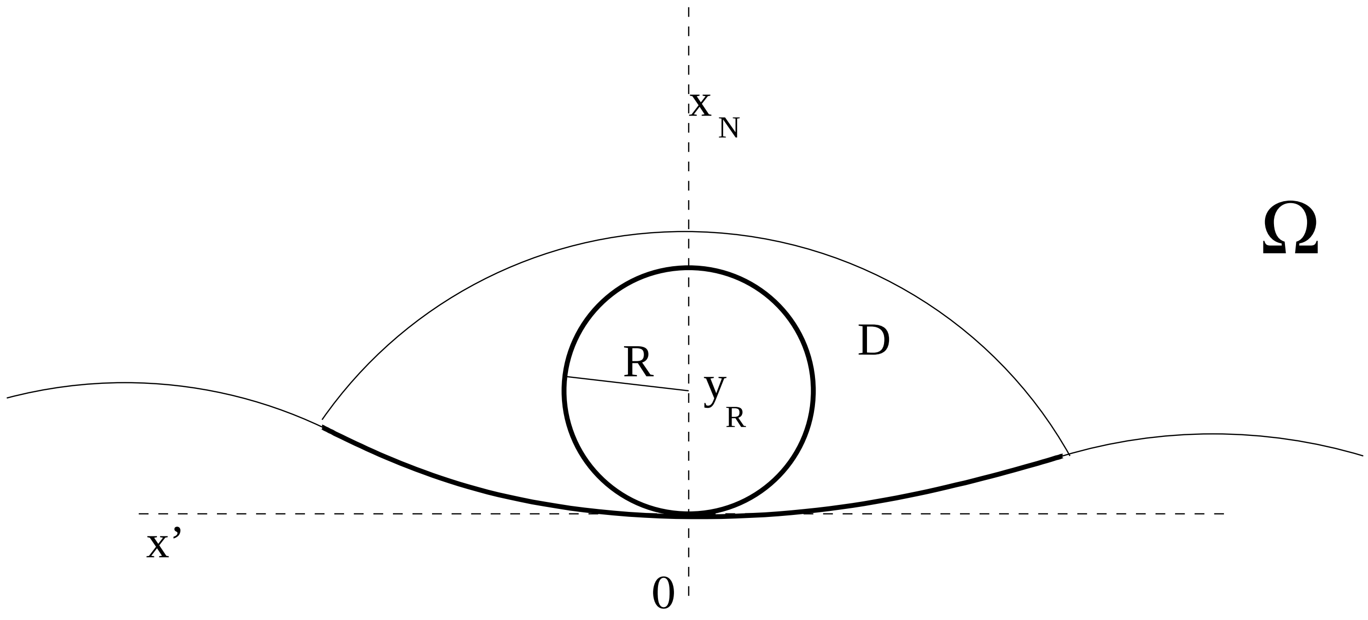
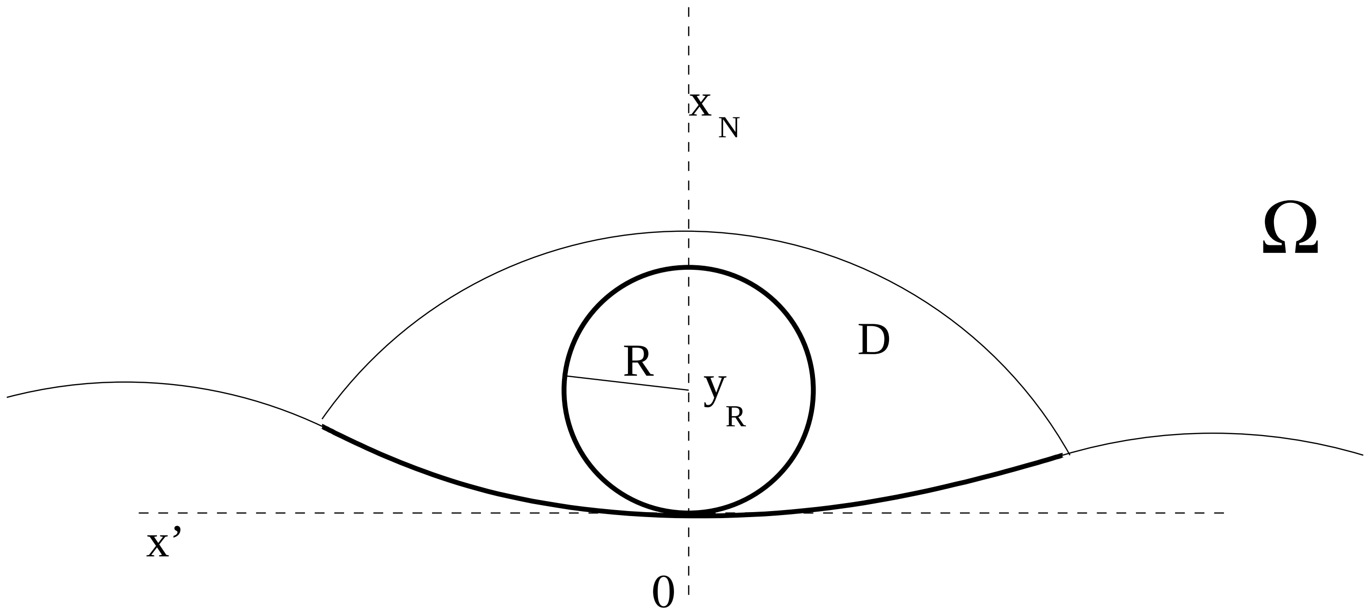
0



$$\begin{cases} -\Delta w = -\beta_0 w^p, & B(y_R, R) \\ w = +\infty, & \partial B(y_R, R) \end{cases}$$

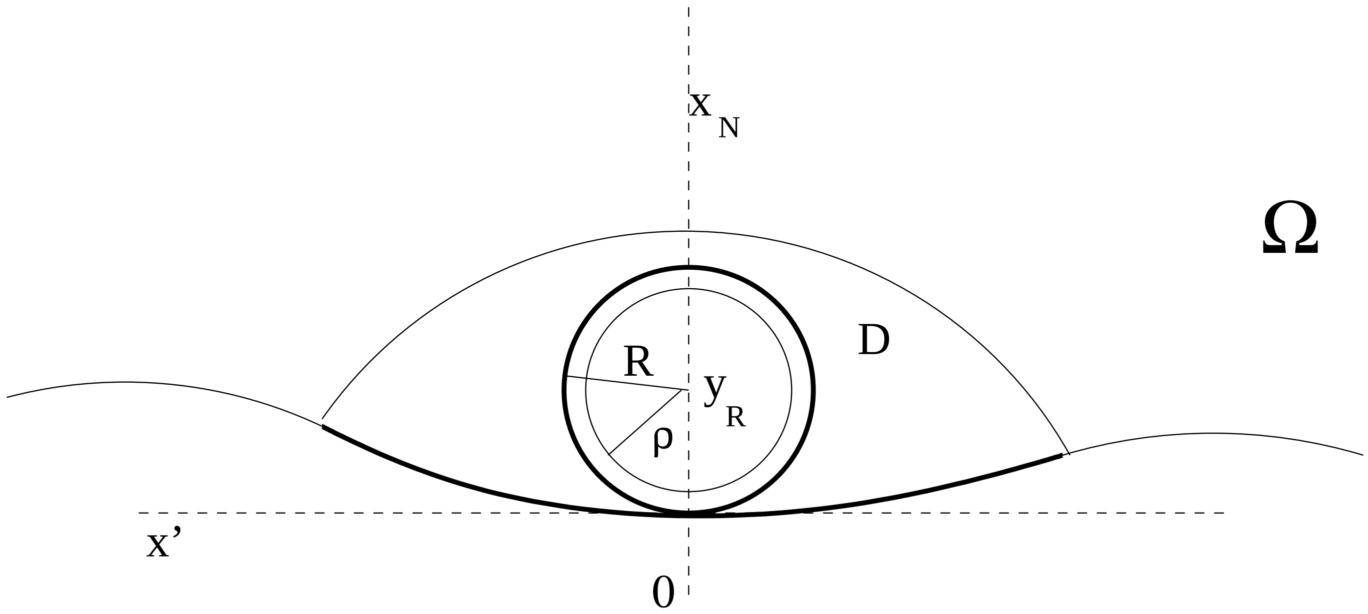
If R is small enough then $u(t, x, u_0) \leq w$ for all $t \geq 0$.



If $r = |x - y_R|$, then the function w is asymptotically for $r \sim R$,

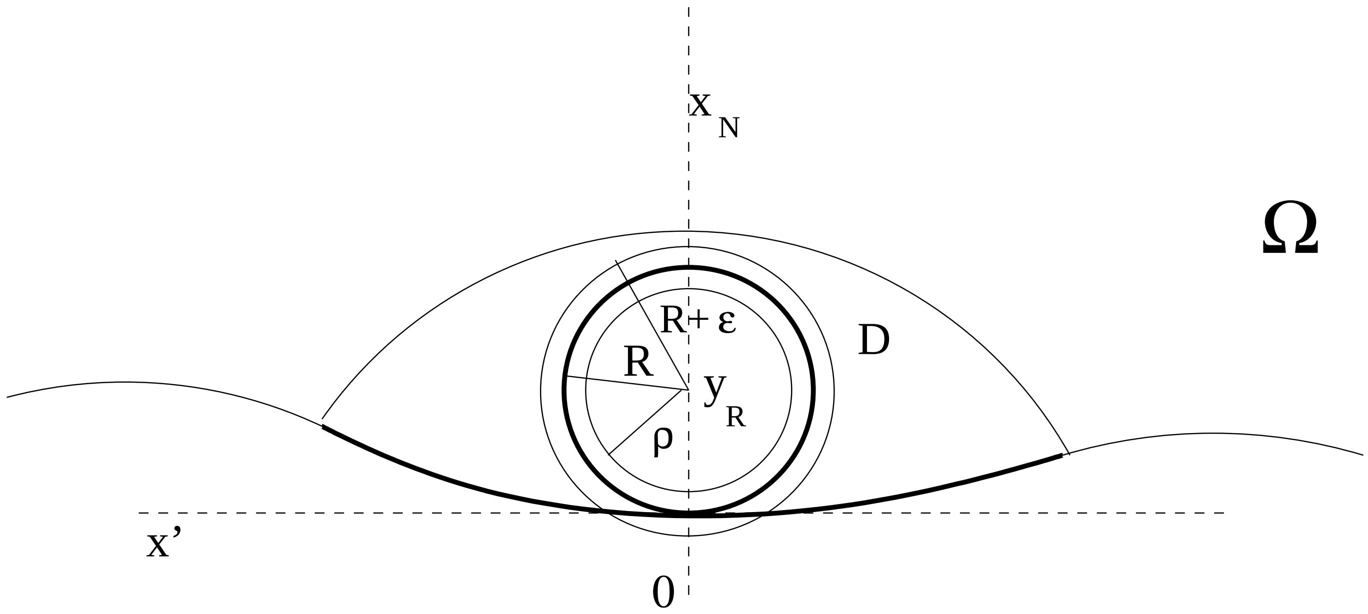
$$w(r) \sim C^* \frac{1}{\beta_0^{\frac{1}{p-1}}} (R - r)^{\frac{-2}{p-1}}$$

$C^* = (2(p+1)/(p-1)^2)^{1/(p-1)}$



Hence, if $q < \beta_1 < \beta_0$, then

$$w(r) \leq C^* \frac{1}{\beta_1^{\frac{1}{p-1}}} (R-r)^{\frac{-2}{p-1}}, \quad \rho < r < R$$



x_N
Ω
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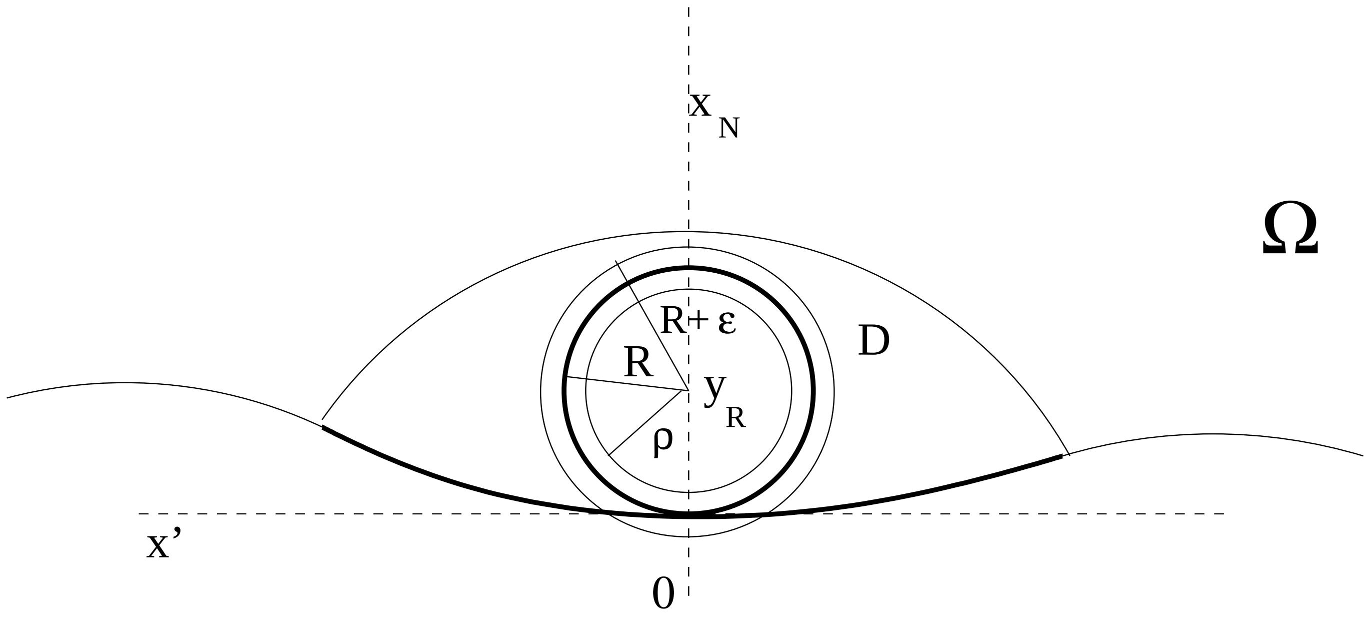
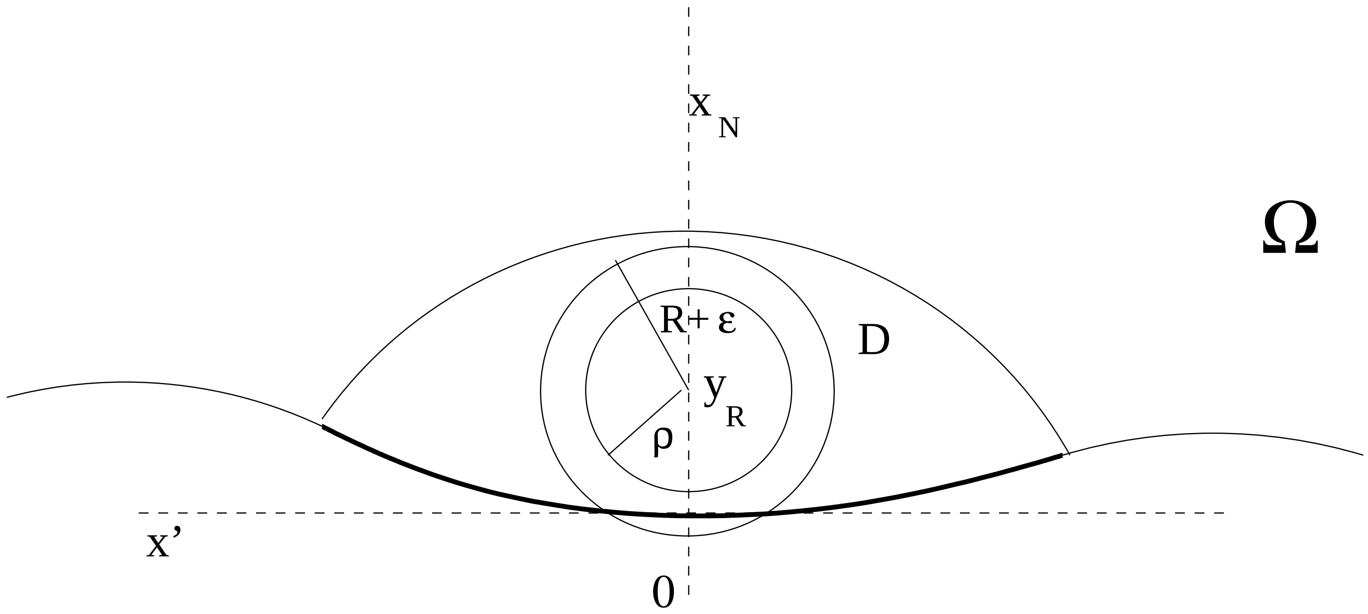


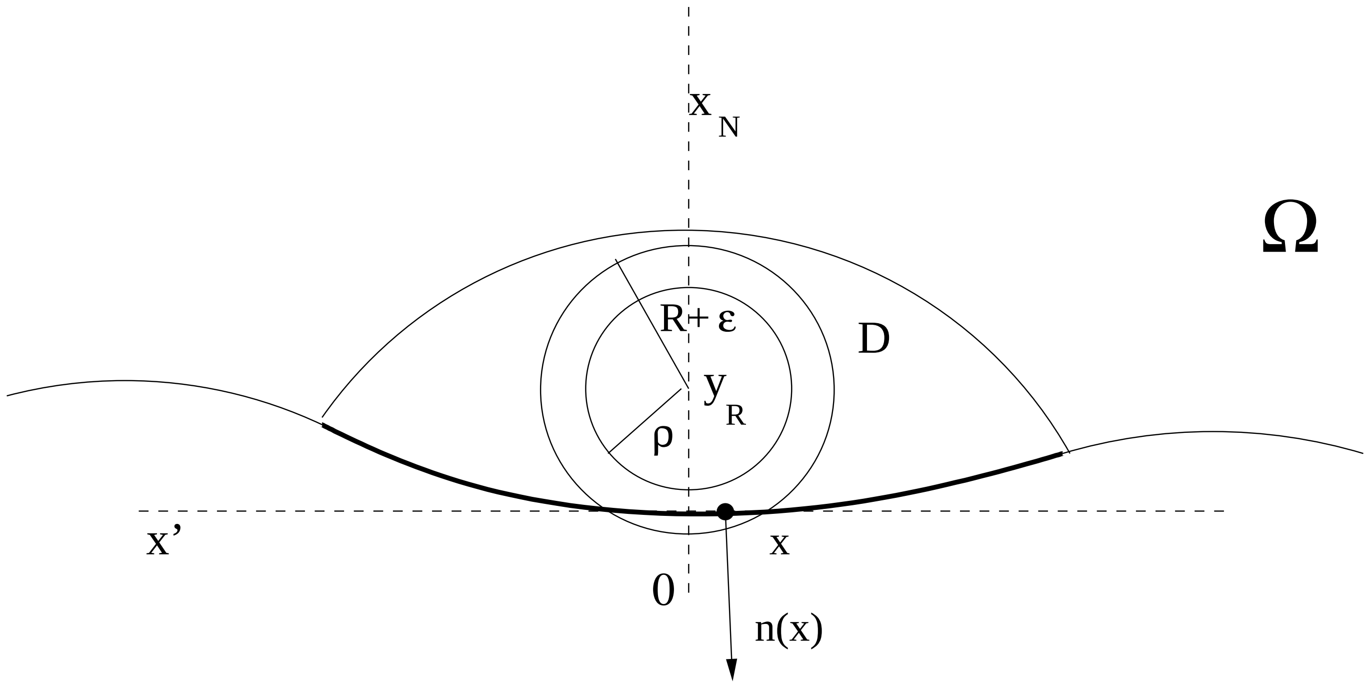
Now, for $q < \beta_2 < \beta_1 < \beta_0$, define the function

$$H(r) = C^* \frac{1}{\beta_2^{\frac{1}{p-1}}} (R + \epsilon - r)^{\frac{-2}{p-1}}, \quad \rho < r < R + \epsilon$$



$\exists \epsilon_0 > 0$ small enough such that for all $0 < \epsilon < \epsilon_0$, we have

$$-\Delta H(r) \geq -\beta_0 H(r)^p, \quad \rho < r < R + \epsilon$$
$$H(\rho) \geq w(|x - y_R|) \geq u(t, x, u_0)$$
$$H(R + \epsilon) = +\infty \geq u(t, x, u_0), x \in \bar{\Omega} \cap \partial B(y_R, R + \epsilon)$$



If $x \in \partial\Omega \cap B(y_R, R+\epsilon)$, then

$$\frac{\partial H(x)}{\partial n} = \nabla H \cdot \vec{n}(x) = H'(r)\frac{x-y_R}{|x-y_R|}\vec{n}(x)$$

But $\frac{x-y_R}{|x-y_R|}\vec{n}(x) \geq 1-\delta$ and $\delta \to 0$ as $\epsilon \to 0$.

Moreover, direct computations, show that (if for instance $p+1=2q$) that

$$H'(r) = \sqrt{\frac{\beta_2}{q}}(H(r))^q$$

Hence,

$$\frac{\partial H}{\partial n} \geq \sqrt{\frac{\beta_2}{q}}(1-\delta)(H(r))^q \geq H(r)^q$$

by choosing ϵ small enough.

Therefore, H is a local supersolution which is bounded in a neighborhood of x_0.

Hence if the equation is

$$\begin{cases} u_t - \Delta u = -\beta(x)u^p & \text{in } \Omega \\ \frac{\partial u}{\partial \vec{n}} = \alpha(x)u^q & \text{on } \Gamma_N \\ u(0,x) = u_0(x) \geq 0 & \text{in } \Omega \end{cases}$$