Navier Stokes 2-D Channel Flow: Control and Nonlinear Observer Design

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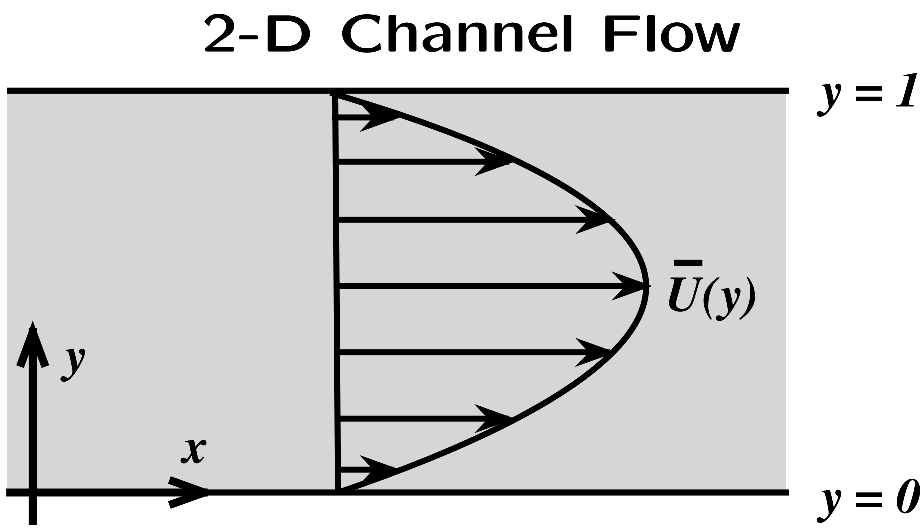


Navier Stokes Equations:

$$U_t = \frac{1}{Re}(U_{xx} + U_{yy}) - UU_x - VU_y - P_x$$
$$V_t = \frac{1}{Re}(V_{xx} + V_{yy}) - UV_x - VV_y - P_y$$

Incompressibility:

$$U_x + V_y = 0$$

Remark: Spatially Invariant Plant in the x direction.

Equilibrium profile: $\bar{U} = 4y(1-y)$, $V = 0$. Linearly unstable for $Re \approx 5772$ or larger.

Streamwise velocity fluctuation definition:

$$u = U - \bar{U}$$

Linearized Plant Equations:

$$\begin{aligned} u_t &= \frac{1}{Re}(u_{xx} + u_{yy}) + 4y(y-1)u_x + 4(2y-1)V - p_x \\ V_t &= \frac{1}{Re}(V_{xx} + V_{yy}) + 4y(y-1)V_x - p_y \\ p_{xx} + p_{yy} &= 8(2y-1)V_x \end{aligned}$$

Boundary conditions:

$$\begin{aligned} u(x,0) &= 0 \\ u(x,1) &= U_c(x) \\ V(x,0) &= 0 \\ V(x,1) &= V_c(x) \\ p_y(x,0) &= -\frac{u_{yx}(x,0)}{Re} \\ p_y(x,1) &= \frac{(V_c)_{xx}(x) - u_{yx}(x,1)}{Re} - (V_c)_t(x) \end{aligned}$$

Boundary control can be done by actuation of U_c and V_c.

Eliminating the pressure:

$$
\begin{aligned}
u_t \;=\; & \frac{1}{Re}\left(u_{xx}+u_{yy}\right)+4y(y-1)u_x+4(2y-1)V \\
& +\int_0^y \int_{-\infty}^{\infty} V(\xi,\eta)\int_{-\infty}^{\infty} \mathrm{e}^{2\pi ik(x-\xi)}16\pi k(2\eta-1) \\
& \times \sinh\left(2\pi k(y-\eta)\right) dk\,d\xi\,d\eta \\
& -\int_0^1 \int_{-\infty}^{\infty} V(\xi,\eta)\int_{-\infty}^{\infty} \mathrm{e}^{2\pi ik(x-\xi)}\frac{\cosh\left(2\pi ky\right)}{\sinh\left(2\pi k\right)}(2\eta-1) \\
& \times \cosh\left(2\pi k(1-\eta)\right) 16\pi k\,dk\,d\xi\,d\eta \\
& +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\mathrm{e}^{2\pi ik(x-\xi)}\frac{\cosh\left(2\pi ky\right)}{\sinh\left(2\pi k\right)} \\
& \times \left((V_c)_t(\xi)+\frac{(V_c)_{xx}(x)-u_{yx}(x,1)}{Re}\right)dk\,d\xi \\
& +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\mathrm{e}^{2\pi ik(x-\xi)}\frac{\cosh\left(2\pi k(1-y)\right)}{\sinh\left(2\pi k\right)}\frac{u_{yx}(x,0)}{Re}dk\,d\xi
\end{aligned}
$$

$$
\begin{aligned}
V_t \;=\; & \frac{1}{Re}(V_{xx}+V_{yy}) + 4y(y-1)V_x \\
& - \int_0^y \int_{-\infty}^{\infty} V(\xi,\eta) \int_{-\infty}^{\infty} \mathrm{e}^{2\pi i k(x-\xi)} 16\pi k i (2\eta - 1) \\
& \times \cosh\left(2\pi k (y-\eta)\right) dk d\xi d\eta \\
& + \int_0^1 \int_{-\infty}^{\infty} V(\xi,\eta) \int_{-\infty}^{\infty} \mathrm{e}^{2\pi i k(x-\xi)} \frac{\sinh(2\pi k y)}{\sinh(2\pi k)} (2\eta - 1) \\
& \times \cosh\left(2\pi k(1-\eta)\right) 16\pi k i dk d\xi d\eta \\
& + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{2\pi i k(x-\xi)} \frac{\sinh(2\pi k y)}{\sinh(2\pi k)} \\
& \times \left((V_c)_t(\xi) + \frac{(V_c)_{xx}(x) - u_{yx}(x,1)}{Re} \right) dk d\xi \\
& - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{2\pi i k(x-\xi)} \frac{\sinh(2\pi k(1-y))}{\sinh(2\pi k)} \frac{u_{yx}(x,0)}{Re} dk d\xi
\end{aligned}
$$

Objective: find U_c and V_c to guarantee stability of the closed loop system, for *arbitrary* Re.

Controller:

$$
\begin{aligned}
V_c(x,t) &= \int_0^t \int_0^1 \int_{-\infty}^{\infty} V(\tau,\xi,\eta) \int_{-\infty}^{\infty} \chi(k)(2\eta-1)16\pi k i \cosh\left(2\pi k(1-\eta)\right) \\
&\quad \times \mathrm{e}^{-\frac{4\pi^2 k^2}{Re}(t-\tau)} \mathrm{e}^{2\pi i k(x-\xi)} dk\, d\xi\, d\eta\, d\tau \\
&\quad + \int_0^t \int_{-\infty}^{\infty} \frac{u_y(\tau,\xi,0)-u_y(\tau,\xi,1)}{Re} \int_{-\infty}^{\infty} \chi(k) 2\pi i k \mathrm{e}^{-\frac{4\pi^2 k^2}{Re}(t-\tau)} \\
&\quad \times \mathrm{e}^{2\pi i k(x-\xi)} dk\, d\xi\, d\tau \\
U_c(x) &= \int_0^1 \int_{-\infty}^{\infty} u(t,\xi,\eta) \int_{-\infty}^{\infty} \chi(k) K(k,1,\eta) \mathrm{e}^{2\pi i k(x-\xi)} dk\, d\xi\, d\eta
\end{aligned}
$$

$$
\chi(k) = \begin{cases} 1, & m < |k| < M \\ 0, & \text{otherwise} \end{cases}
$$

m, M: Design parameters (truncation points of integral in k).

Kernel Definition:

$$K(k,y,\eta) = \lim_{n\to\infty} K_n(k,y,\eta)$$

$$\begin{aligned}
K_0 &= \frac{Re}{3}\pi i k\eta\left(15y^2 - 6y(1+4\eta) + \eta(12+5\eta)\right) \\
&\quad + 2iRe\eta(\eta-1)\sinh\left(2\pi k(y-\eta)\right) - 6\eta i\frac{Re}{\pi k}(1-\cosh\left(2\pi k(y-\eta)\right)) \\
&\quad + 2\pi k\frac{\cosh\left(2\pi k(1-y+\eta)\right) - \cosh\left(2\pi k(y-\eta)\right)}{\sinh\left(2\pi k\right)} \\
K_n &= K_{n-1} - 4iRe\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}\int_{-\delta}^{\delta}\{2\pi k(\gamma-\delta-1)\cosh\left(\pi k(\xi+\delta)\right) \\
&\quad + \sinh\left(\pi k(\xi+\delta)\right) - \pi k(2\xi-1)\}\, K_{n-1}\left(k, \frac{\gamma+\delta}{2}, \frac{\gamma+\xi}{2}\right) d\xi d\delta d\gamma \\
&\quad + \frac{Re}{2}\pi i k\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}(\gamma-\delta)(\gamma-\delta-2) K_{n-1}\left(k, \frac{\gamma+\delta}{2}, \frac{\gamma-\delta}{2}\right) d\delta d\gamma \\
&\quad + 2\pi k\int_0^{y-\eta}\frac{\cosh\left(2\pi k(1-\delta)\right) - \cosh\left(2\pi k\delta\right)}{\sinh\left(2\pi k\right)} K_{n-1}\left(k, y-\eta, \delta\right) d\delta
\end{aligned}$$

Explicit Closed Loop Solutions:

$$\begin{aligned} V(t,x,y) &= V^*(t,x,y) + \varepsilon_0(t,x,y) \\ u(t,x,y) &= u^*(t,\xi,y) + \varepsilon_1(t,x,y) \end{aligned}$$

$$\begin{aligned} u^* &= 2\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\chi(k)\mathrm{e}^{-t\frac{4k^2\pi^2+\pi^2j^2}{Re}}\left[\sin(\pi j y) + \int_0^y L(k,y,\eta)\sin(\pi j\eta)\,d\eta\right] \\ &\quad \times\int_{-\infty}^{\infty}\mathrm{e}^{2\pi ik(x-\xi)}\int_0^1\left[\sin(\pi j\eta) - \int_\eta^1 K(k,\sigma,\eta)\sin(\pi j\sigma)\,d\sigma\right] \\ &\quad \times u(0,\xi,\eta)d\eta d\xi dk \\ V^* &= -2\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\chi(k)\mathrm{e}^{-t\frac{4k^2\pi^2+\pi^2j^2}{Re}}\left[\frac{1-\cos(\pi j y)}{\pi j} + \int_0^y\left(\int_\eta^y L(k,\sigma,\eta)d\sigma\right)\right. \\ &\quad \left.\times\sin(\pi j\eta)\,d\eta\right]\int_{-\infty}^{\infty}\mathrm{e}^{2\pi ik(x-\xi)}\int_0^1\left[\pi j\cos(\pi j\eta)\right. \\ &\quad \left.+K(k,\eta,\eta)\sin(\pi j\eta) - \int_\eta^1 K_\eta(k,\sigma,\eta)\sin(\pi j\sigma)\,d\sigma\right]V(0,\xi,\eta)d\eta d\xi dk \end{aligned}$$

The errors ε_0, ε_1 decay exponentially.

Inverse Kernel Definition:

$$L(k,y,\eta) = \lim_{n\to\infty} L_n(k,y,\eta)$$

$$\begin{aligned}
L_0 \;=\;& \frac{Re}{3}\pi i k\eta\left(15y^2 - 6y(1+4\eta) + \eta(12+5\eta)\right) \\
&+2iRe\eta(\eta-1)\sinh\left(2\pi k(y-\eta)\right) - 6\eta i\frac{Re}{\pi k}\left(1-\cosh\left(2\pi k(y-\eta)\right)\right) \\
&+2\pi k\frac{\cosh\left(2\pi k(1-y+\eta)\right) - \cosh\left(2\pi k(y-\eta)\right)}{\sinh\left(2\pi k\right)} \\
L_n \;=\;& L_{n-1} + 4iRe\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}\int_{-\delta}^{\delta}\{2\pi k(\gamma+\xi-1)\cosh\left(\pi k(\xi-\delta)\right) \\
&+\sinh\left(\pi k(\xi-\delta)\right) - \pi k(2\delta-1)\} \\
&\times L_{n-1}\left(k,\frac{\gamma+\xi}{2},\frac{\gamma-\delta}{2}\right)d\xi d\delta d\gamma \\
&-\frac{Re}{2}\pi i k\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}(\gamma+\delta)(\gamma+\delta-2)L_{n-1}\left(k,\frac{\gamma+\delta}{2},\frac{\gamma-\delta}{2}\right)d\delta d\gamma
\end{aligned}$$

Main Result

Theorem: *The equilibrium* $u(x,y) \equiv V(x,y) \equiv 0$ *is exp. stable.*

Proof: Let $V(k,y,t), u(k,y,t)$ denote the Fourier transf. w.r.t. x.

Large and small wave numbers: For $|k| > M$ and $|k| < m$ the uncontrolled (u,V) system is exp. stable. (Conservative Lyapunov bounds: $M \geq \frac{1}{\pi}\sqrt{\frac{Re}{2}}$, $m \leq \frac{1}{32\pi Re}$.)

Intermediate wave numbers: for $m \leq |k| \leq M$, use backstepping to stabilize u.

Backstepping transformation:

$$\alpha(k,y) \;=\; u(k,y) - \int_0^y K(k,y,\eta)u(t,k,\eta)d\eta \qquad \text{(invertible)}$$

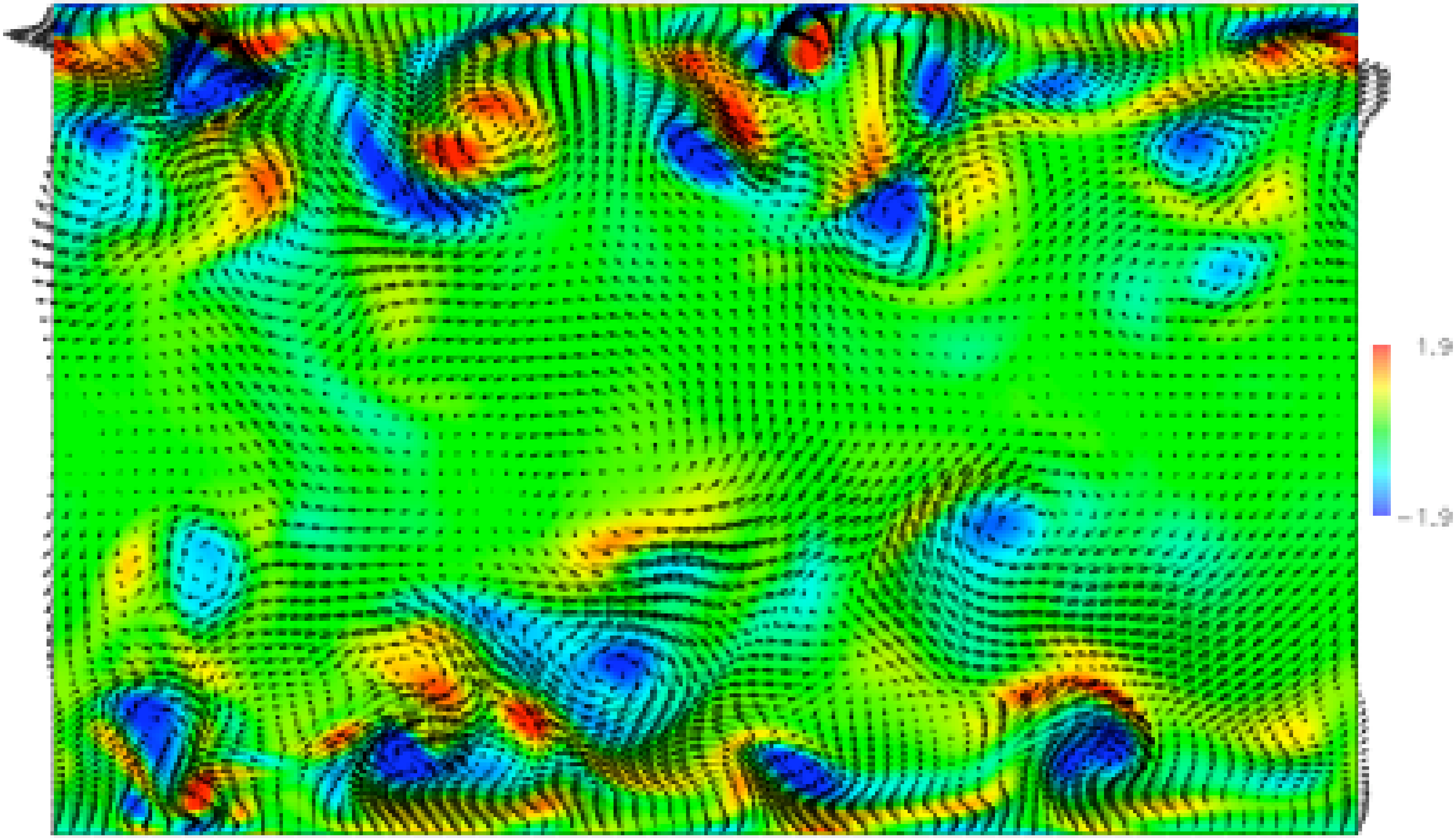
Transformed system:

$$\begin{aligned} \alpha_t &= \frac{1}{Re}\left(-4\pi^2k^2\alpha + \alpha_{yy}\right) \qquad \text{(heat eqn)} \\ \alpha(k,0) &= \alpha(k,1) = 0 \end{aligned}$$

Kernel decay: $1/|x-\xi|$.

3-D Extension: straightforward, no need for additional actuation.

Conrol of Turbulence: Channel Flow



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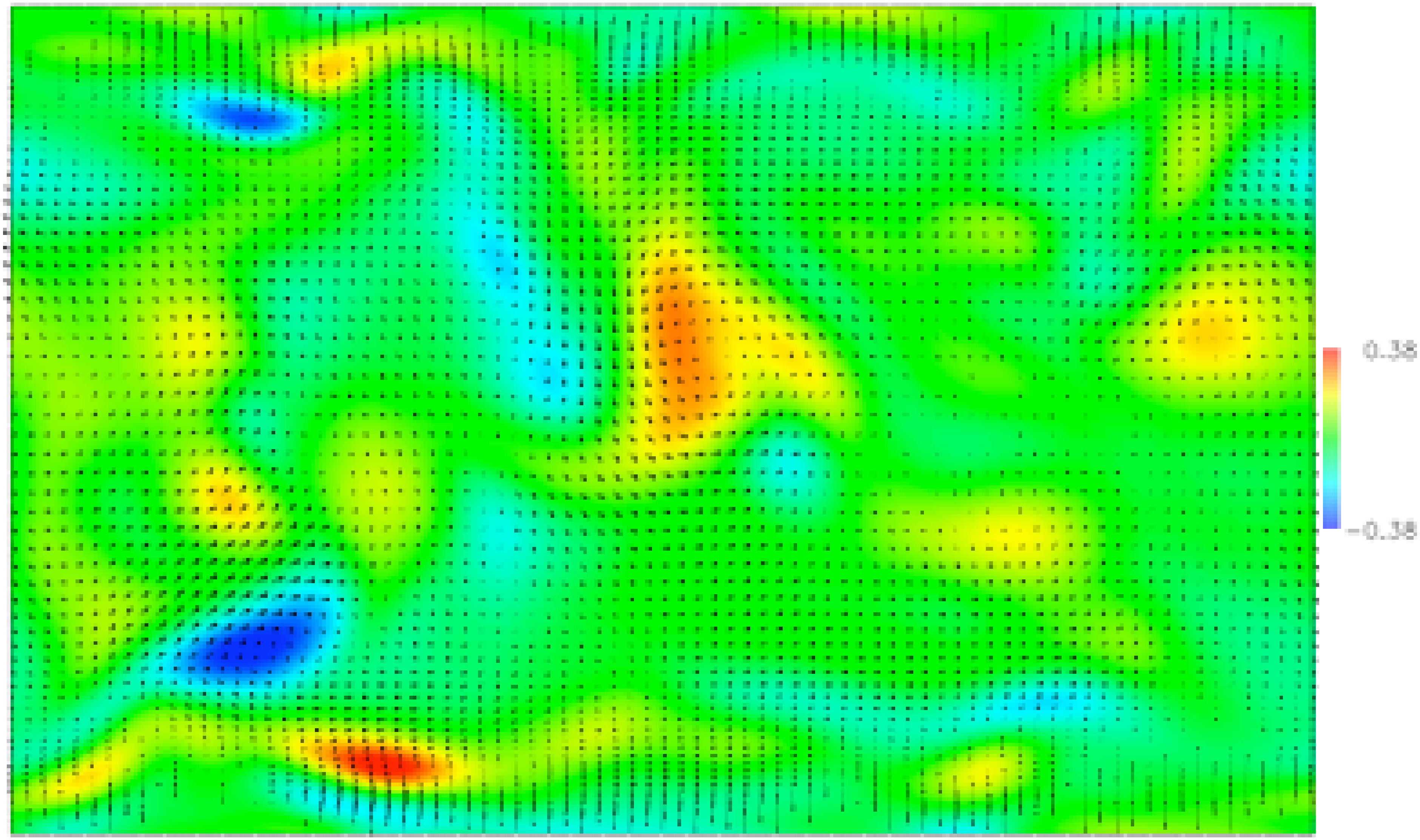
Stabilized Channel Flow



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Nonlinear Observer: (measuring P and U_y at the wall)

$$\begin{aligned}
\hat{U}_t &= \frac{1}{Re}\left(\hat{U}_{xx} + \hat{U}_{yy}\right) - \hat{P}_x - \hat{U}\hat{U}_x - \hat{V}\hat{U}_y \\
&\quad + \int_{-\infty}^{\infty} Q_2(x-\xi, y)\left(P(\xi,0) - \hat{P}(\xi,0)\right) d\xi \\
&\quad - \frac{1}{Re}\int_{-\infty}^{\infty} Q_1(x-\xi, y)\left(U_y(\xi,0) - \hat{U}_y(\xi,0)\right) d\xi \\
\hat{V}_t &= \frac{1}{Re}\left(\hat{V}_{xx} + \hat{V}_{yy}\right) - \hat{P}_y - \hat{U}\hat{V}_x - \hat{V}\hat{V}_y \\
&\quad - \frac{1}{Re}\int_{-\infty}^{\infty} \hat{V}_y(\xi,0) \int_{-M}^{M} l(k,y,0) \mathrm{e}^{2\pi i k(x-\xi)} dk d\xi \\
&\quad + \int_{-\infty}^{\infty} Q_1(x-\xi, y)\left(P(\xi,0) - \hat{P}(\xi,0)\right) d\xi \\
&\quad + \frac{1}{Re}\int_{-\infty}^{\infty} Q_2(x-\xi, y)\left(U_y(\xi,0) - \hat{U}_y(\xi,0)\right) d\xi \\
\hat{P}_{xx} + \hat{P}_{yy} &= -2(\hat{V}_y)^2 - 2\hat{V}_x\hat{U}_y \\
\hat{U}(t,x,0) &= \hat{U}(t,x,1) = \hat{V}(t,x,0) = \hat{V}(t,x,1) = 0 \\
\hat{P}_y(t,x,0) &= \frac{1}{Re}\hat{U}_{xy}(t,x,0) \\
\hat{P}_y(t,x,1) &= \frac{1}{Re}\hat{U}_{xy}(t,x,1)
\end{aligned}$$

Observer error:

$$\begin{aligned}\tilde{U} &= U - \hat{U}\\ \tilde{V} &= V - \hat{V}\\ \tilde{P} &= P - \hat{P}\end{aligned}$$

Design will be done for the linearized observer error equations and then the kernels will be used for the nonlinear observer, like in the **Extended Kalman Filter** technique.

Remark: Divergence-free condition is not enforced in the observer as an additional degree of freedom, but it is guaranteed to hold when the estimates converge to the real values.

Output Injection Kernels ($\mathcal{M} = 2\pi M$):

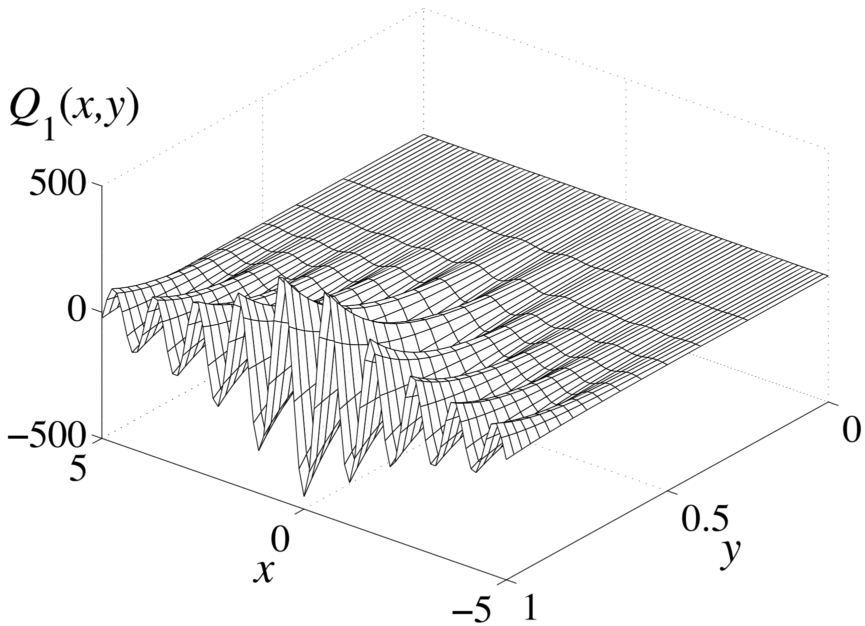
$$
\begin{aligned}
Q_1(x,y) &= \frac{2x}{\pi(x^2+y^2)^2}\left[y \cosh(\mathcal{M}y)\sin(\mathcal{M}x) - x \sinh(\mathcal{M}y)\cos(\mathcal{M}x)\right] \\
&\quad - \frac{2M}{(x^2+y^2)}\left[y \cosh(\mathcal{M}y)\cos(\mathcal{M}x) + x \sinh(\mathcal{M}y)\sin(\mathcal{M}x)\right] \\
&\quad + \frac{\sinh(\mathcal{M}y)\cos(\mathcal{M}x)}{\pi(x^2+y^2)} \\
Q_2(x,y) &= \frac{2x}{\pi(x^2+y^2)^2}\left[x \cosh(\mathcal{M}y)\sin(\mathcal{M}x) + y \sinh(\mathcal{M}y)\cos(\mathcal{M}x)\right] \\
&\quad - \frac{2M}{(x^2+y^2)}\left[x \cosh(\mathcal{M}y)\cos(\mathcal{M}x) - y \sinh(\mathcal{M}y)\sin(\mathcal{M}x)\right] \\
&\quad - \frac{\cosh(\mathcal{M}y)\sin(\mathcal{M}x)}{\pi(x^2+y^2)}
\end{aligned}
$$

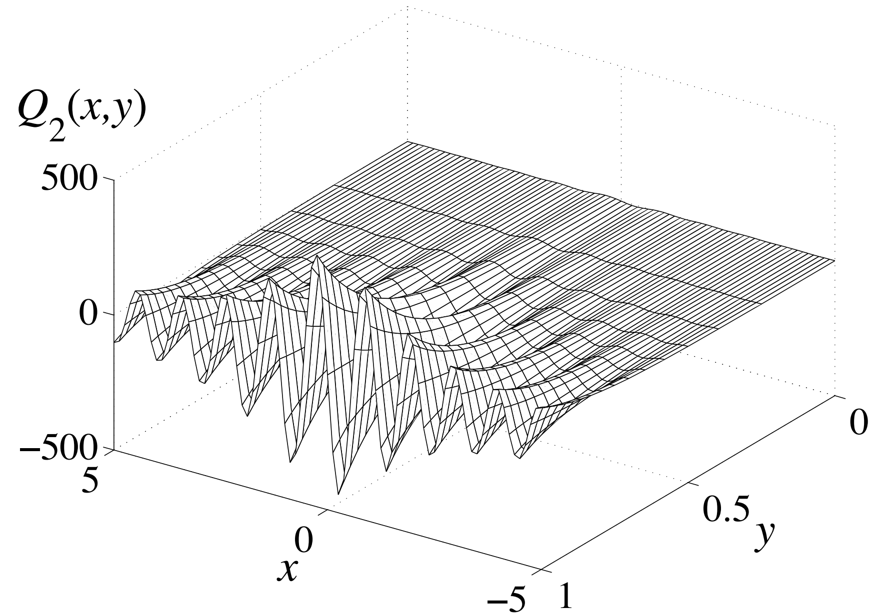
Stabilizing Kernel:

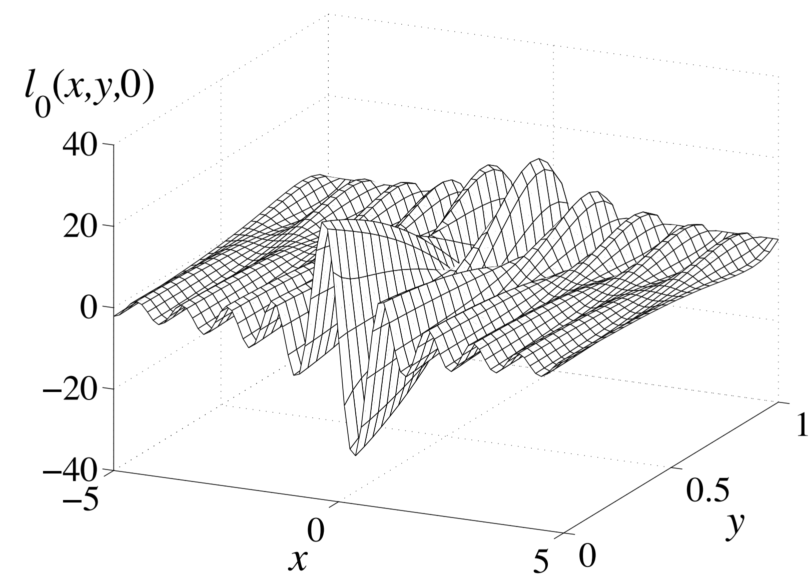
$$l(k,y,\eta) = \lim_{n\to\infty} l_n(k,y,\eta)$$

$$\begin{aligned}
l_0 &= -\frac{Re}{3}\pi ik(1-y)\Big((1-y)^2+3\eta^2-3\Big)+4iRey(y-1)\sinh\left(2\pi k(y-\eta)\right) \\
&\quad +2(1-y)i\frac{Re}{\pi k}(1-\cosh\left(2\pi k(y-\eta)\right)) \\
l_n &= l_{n-1}+8Re\pi ki\int_{y-\eta}^{2-(y+\eta)}\int_0^{y-\eta}\int_{-\delta}^{\delta}\cosh\left(\pi k(\xi+\delta)\right)(\gamma-\delta-1) \\
&\quad \times l_{n-1}\left(k,\frac{\gamma+\delta}{2},\frac{\gamma+\xi}{2}\right)d\xi d\delta d\gamma \\
&\quad +\frac{Re}{2}\pi ik\int_{y-\eta}^{2-(y+\eta)}\int_0^{y-\eta}(\gamma+\delta-2)(\gamma+\delta)l_{n-1}\left(k,\frac{\gamma+\delta}{2},\frac{\gamma-\delta}{2}\right)d\delta d\gamma
\end{aligned}$$

Kernel Shapes:







Main Result

Theorem *There exists positive constants* C_1 *and* C_2 *such that, if*

$$\int_0^1 \int_{-\infty}^{\infty} \left(\tilde{U}^2(0,x,y) + \tilde{V}^2(0,x,y) \right) dxdy < C_1,$$

and if $\forall t \geq 0$,

$$\int_0^1 \int_{-\infty}^{\infty} \left(u^2(t,x,y) + V^2(t,x,y) \right) dxdy < C_2,$$

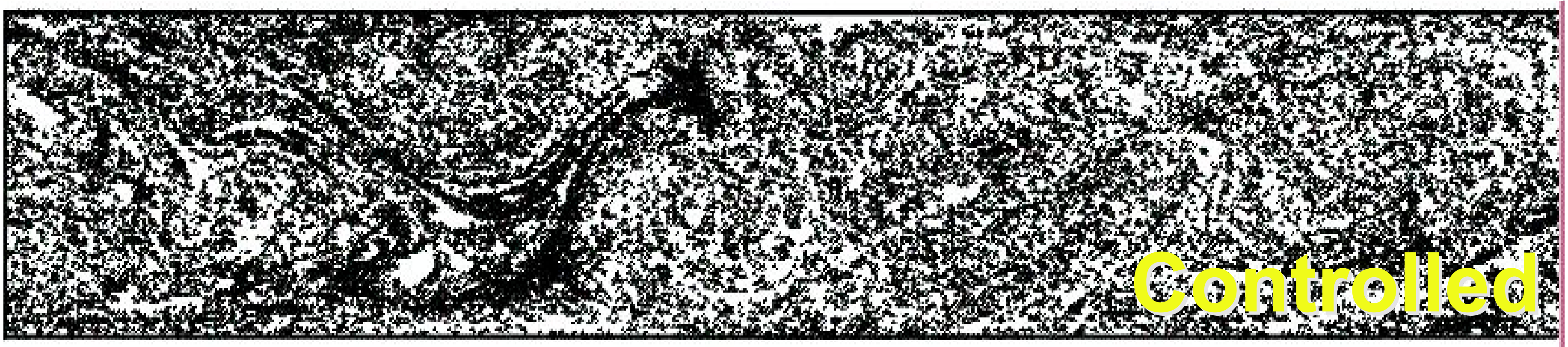
then:

$$\lim_{t \to \infty} \int_0^1 \int_{-\infty}^{\infty} \left(\tilde{U}^2(t,x,y) + \tilde{V}^2(t,x,y) \right) dxdy = 0.$$

Remark: The result can be extended to 3-D using additional measurements of W_y.

Turbulence Enhancement - Mixing

$$V_{\text{bottom wall}}(x) = V_{\text{top wall}}(x) = -k_P\left[P_{\text{top wall}}(x) - P_{\text{bottom wall}}(x)\right]$$



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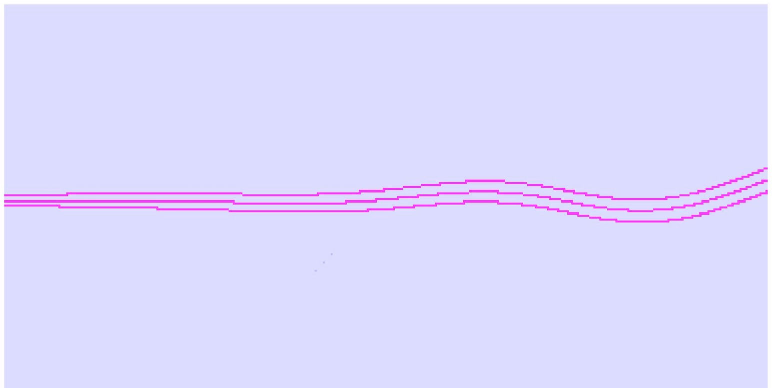
Control of 2D Jet Flow

Uncontrolled



Controlled - light particles

Controlled - heavy particles

Diffusive mixing