Controllability of a 1D Schrödinger equation

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Schrödinger equation

A quantum particle is represented by a wave function :

$$\begin{array}{rccccc} \phi : & [0,T] & \times & \mathbb{R} & \to & \mathbb{C} \\ & (t & , & z) & \mapsto & \phi(t,z), \end{array}$$

$$\int_{\mathbb{R}} |\phi(t,z)|^2 dz = 1.$$

When the particle is in a potential $V(z)$ $(\hbar \leftarrow 1, m \leftarrow 1)$, then

$$i\frac{\partial \phi}{\partial t}(t,z) = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2}(t,z) + V(z)\phi(t,z).$$

Particle in a potential well

A fixed potential $V(z)$ is translated,

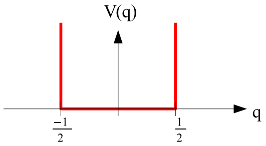
$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2}\frac{\partial^2\phi}{\partial z^2} + V(z - D(t))\phi,$$

where $D(t)$: "position "of the potential.

$$\begin{cases} q := z - D(t) \\ \psi(t,q) := \phi(t,z)e^{i(-z\dot{D}+D\dot{D}-\frac{1}{2}\int_0^t \dot{D}^2)} \end{cases}$$

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial q^2} + (V(q) - u(t)q)\,\psi \qquad \text{where} \qquad u := -\ddot{D}.$$

Potential well :



Question

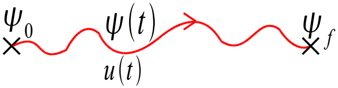
$$(\Sigma) \begin{cases} i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2} - u(t)q\psi, \ \ q \in I := (-1/2, 1/2), \\ \psi(t, \pm 1/2) = 0. \end{cases}$$

State : $\psi \in \mathcal{S}$.

Control : u.

Controllability ?

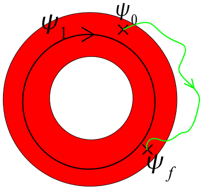
ψ_0, ψ_f fixed.
Does there exist $T > 0$ and a trajectory $(\psi(t), u(t))$ of (Σ) on $[0, T]$ such that $\psi(0) = \psi_0$ and $\psi(T) = \psi_f$?



Result : local controllability

$$(\Sigma) \qquad i\dot{\psi} = -(1/2)\psi'' - u(t)q\psi, \qquad \psi(t, \pm 1/2) = 0.$$

Ground state for $u \equiv 0$: $\qquad \psi_1(t,q) := \varphi_1(q)e^{-i\lambda_1 t}$



Theorem : There exists $\eta > 0$ such that, for every ψ_0, ψ_f in $\mathcal{S} \cap H^7_{(0)}(I, \mathbb{C})$ which satisfy

$$\|\psi_0 - \varphi_1 e^{i\phi_0}\|_{H^7} \leqslant \eta, \quad \|\psi_f - \varphi_1 e^{i\phi_f}\|_{H^7} \leqslant \eta,$$

there exists a trajectory (ψ, u) of (Σ) on an interval $[0, T]$ such that $\psi(0) = \psi_0$, $\psi(T) = \psi_f$, moreover $u \in H^1_0((0,T), \mathbb{R})$.

Classical approach

Let (ψ^*, u^*) be a trajectory of (Σ).

Linearized system around (ψ^*, u^*) controllable in time T.

$\Downarrow$ (often)

Nonlinear system (Σ) locally controllable in a neighborhood of $(\psi^*(0), \psi^*(T))$ in time T.

Proof : Inverse Mapping Theorem on

$$\Theta : (\psi_0, u) \mapsto (\psi(0), \psi(T))$$

where ψ solves (Σ) with control u and initial condition ψ_0.

1^{st} difficulty : the linearized system around $(\psi_1, u \equiv 0)$ is not controllable (P. Rouchon)

$$(\Sigma^L): \qquad i\dot{\Psi} = -\frac{1}{2}\Psi'' - w(t) q \psi_1 \qquad \Psi(t, \pm 1/2) = 0$$

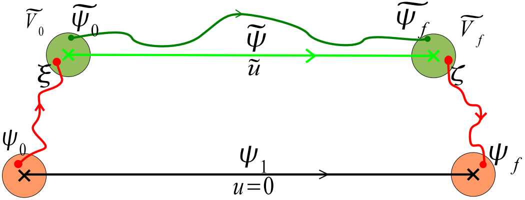
State : $\Psi(t) \in T_{\psi_1(t)}\mathcal{S}$ Control : w

$$\Psi(t) = \sum_{k=1}^{\infty} x_k(t) \varphi_k \quad \text{where} \quad \begin{cases} -\frac{1}{2}\varphi_k'' = \lambda_k \varphi_k \\ \varphi_k(\pm 1/2) = 0 \end{cases}$$

$$i\dot{x}_k(t) = \lambda_k x_k(t) - w(t) < q\varphi_1, \varphi_k > e^{-i\lambda_1 t}$$

when k is odd : $i\dot{x}_k = \lambda_k x_k$

Strategy : return method



1) Find a trajectory $(\widetilde{\psi}, \widetilde{u})$ of (Σ) such that the linearized system around $(\widetilde{\psi}, \widetilde{u})$ is controllable.

2) Construct neighborhoods and trajectories.

Return method, 1^{st} step : the linearized system around the GS for $u \equiv \gamma > 0$ is controllable.

$$(\Sigma) \qquad i\dot{\psi} = -(1/2)\psi'' - u(t)q\psi, \qquad \psi(t, \pm 1/2) = 0.$$

Ground state for $u \equiv \gamma$: $\qquad \psi_1(t,q) := \varphi_1(q) e^{-i\lambda_1 t}$

$$(\Sigma_\gamma^L) \left\{ \begin{array}{l} i\dot{\Psi} = -\frac{1}{2}\Psi'' - \gamma q \Psi - wq\psi_1, \\ \Psi(t, \pm 1/2) = 0 \end{array} \right.$$

$$\Psi(t) = \sum_{k=1}^{\infty} x_k(t) \varphi_k \quad \text{where} \quad \left\{ \begin{array}{l} -\frac{1}{2}\varphi_k'' - \gamma q \varphi_k = \lambda_k \varphi_k \\ \varphi_k(\pm 1/2) = 0 \end{array} \right.$$

$$i\dot{x}_k(t) = \lambda_k x_k(t) - w(t) < q\varphi_1, \varphi_k > e^{-i\lambda_1 t}$$

Controllability of the linearized system around $(\psi_1, u \equiv \gamma)$

$\Psi(T) = \Psi_f$ is equivalent to : $\forall k \in \mathbb{N}^*$,

$$< \Psi_f, \varphi_k > = \left(< \Psi_0, \varphi_k > + i b_k \int_0^T w(t) e^{i(\lambda_k - \lambda_1)t} dt \right) e^{-i\lambda_k T}$$

where $b_k := < q\varphi_1, \varphi_k >$.

Trigonometric moment problem :

$$\int_0^T w(t) e^{i\omega_k t} dt = d_k, \forall k \in \mathbb{N}^*.$$

If $T > 0$ and $d \in l^2(\mathbb{N}^*, \mathbb{C})$, there exists $w \in L^2((0,T), \mathbb{R})$.
(Ingham inequalities)

Controllability of the linearized system around $(\psi_1, u \equiv \gamma)$

- when $\gamma \neq 0$ small enough,
- $\forall T > 0$,
- in $H^3(I, \mathbb{C})$,
- with control $w \in L^2((0, T), \mathbb{R})$.

2^{nd} difficulty : for the local controllability around $(\psi_1(0), \psi_1(T))$, the IMT cannot be applied.

$$\begin{array}{rccc} \Theta : & E & \rightarrow & F \\ & (\psi_0, u) & \mapsto & (\psi(0), \psi(T)) \end{array}$$

Classical situation :
1) $\Theta \in C^1$
2) $d\Theta(\varphi_1, \gamma)$ is surjective

Here, loss of regularity :
$\forall y \in F, \exists x \in \widetilde{E}$ such that $d\Theta(\varphi_{1,\gamma}, \gamma)x = y$ but $\widetilde{E} \supsetneq E$.

$\rightarrow$ Nash-Moser theorem

Main idea of Nash-Moser theorem

$f : \mathbb{R}^n \to \mathbb{R}^n$
$df(\alpha)$ invertible
b fixed, closed to $f(\alpha)$
We search a such that $f(a) = b$.

In the Inverse Mapping Theorem :

$$x_{n+1} = x_n - df(\alpha)^{-1}.[f(x_n) - b]$$

if $x_0 \in H^5 \times H^2$ then $x_1 \in H^3 \times H^1$, $x_2 \in H^1 \times L^2$ etc

In the Nash-Moser Theorem :

$$x_{n+1} = x_n - R_n\{df(x_n)^{-1}.[f(x_n) - b]\}$$

where R_n : smoothing operator.

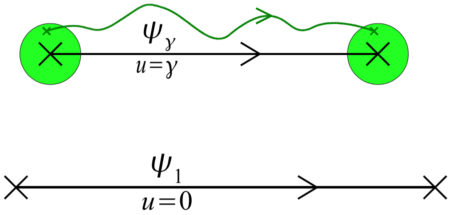
Difficulties in the application of Nash-Moser theorem

$$x_{n+1} = x_n - R_n\{df(x_n)^{-1}.[f(x_n) - b]\}$$

1) Controllability of an infinite number of linear systems : existence of $df(x_n)^{-1}$
2) Tame estimates on the controls $\rightarrow$ convergence
3) Construction of smoothing operators R_n

1),2) “closed ”linear maps (in the sense of tame estimates)
3) for u : convolution, troncature
for ψ : decomposition on a basis, troncature of high frequencies

Local controllability of (Σ) around $(\psi_{1\gamma}, u \equiv \gamma)$

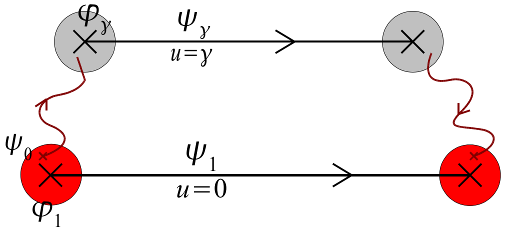


Theorem : Let $T := 4/\pi$ and $\gamma \in (0, \gamma_0)$. There exists $\delta > 0$ such that, for every $\psi_0, \psi_f \in \mathcal{S} \cap H^7_{(\gamma)}(I, \mathbb{C})$ with

$$\|\psi_0 - \psi_{1,\gamma}(0)\|_{H^7} < \delta, \quad \|\psi_f - \psi_{1,\gamma}(T)\|_{H^7} < \delta,$$

there exists $v \in H^1_0((0,T), \mathbb{R})$ such that the solution of (Σ) with control $u := \gamma + v$ and initial condition ψ_0 satisfies $\psi(T) = \psi_f$.

Return method, 2^{nd} step : Quasi-static transformations



$$u(t) = \gamma f(\epsilon t)$$

Remarks, conjectures

With the same proof : local controllability in a $H^{6+\epsilon}$-neighborhood of any eigenstate ψ_k.

Nash-Moser theorem : unavoidable on Sobolev spaces, but the inverse mapping theorem could be sufficient on other spaces (?).

Regularity assumption : $H^{6+\epsilon}$ only technical.
Conjecture : controllable in $H^3(I, \mathbb{C})$ with control functions in L^2.

Time of control : long here (quasi-static transformations).
Open problem : $\exists$ minimal time T_m for controllability?
Conjecture : $T_m > 0$.

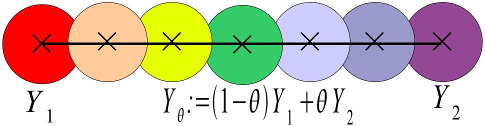
Steady-state controllability of (Σ) (with J.-M. Coron, accepted in J. Funct. Analysis)

$$(\Sigma_0) \begin{cases} i\dot{\psi} = -\frac{1}{2}\psi'' - u(t)q\psi, \\ \psi(t, \pm 1/2) = 0, \\ \dot{s} = u, \\ \dot{d} = s. \end{cases}$$

State : $Y := (\psi, s, d) \in \mathcal{S} \times \mathbb{R} \times \mathbb{R}$, Control : u

Theorem : There exist $T > 0$ and $u \in H_0^1((0,T),\mathbb{R})$ such that the solution of (Σ_0) with $Y(0) = (\varphi_1, 0, 0)$ and control u satisfies $Y(T) = (\varphi_2, 0, 0)$.

Eigenstates of (Σ_0) : $Y_k(t) := (\psi_k(t), s(t) \equiv 0, d(t) \equiv 0)$



Additional difficulty : one direction is missed in the control of the linearized system.

Linearized system around $(Y_\gamma, u \equiv \gamma)$:
$\Psi(T) = \Psi_f \Leftrightarrow \int_0^T w(t) e^{i(\lambda_k - \lambda_1)t} dt = ..., \forall k \in \mathbb{N}^*$,
$S(T) = S_f \Leftrightarrow \int_0^T w(t) dt = ...$
$D(T) = D_f \Leftrightarrow \int_0^T (T-t) w(t) dt = ...$

The directions S and Ψ are linearly dependant.

- Nash-Moser $\Rightarrow$ controllability up to codimension one,
- 2^{nd} order term $d^2\Phi \Rightarrow$ controllability

Prospects

In dimension $N \geqslant 2$: this strategy could be adapted to $N = 2$, but not to $N \geqslant 3$.

Other nonlinearities : this strategy could be adapted to

$$i\dot{\psi} = -\frac{1}{2}\psi'' + \epsilon|\psi|^2\psi - u(t)q\psi, t \in [0, T], q \in I.$$

Nash-Moser theorem on other equations :
1) Rod equation
2) Schrödinger with a potential well of variable length $l(t) > 0$

Non controllability in H^2 (Turinici ; Ball, Marsden, Slemrod)

For every $\psi_0 \in \mathcal{S} \cap H^2 \cap H^1_0(I, \mathbb{C})$, the reachable set

$$\{\psi(T); T > 0, u \in L^r_{loc}(\mathbb{R}_+, \mathbb{R}), r > 1\}$$

has a dense complement in $\mathcal{S} \cap H^2 \cap H^1_0(I, \mathbb{C})$.

But this argument fails with $H^2 \to H^3$.

Rod equation

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)u_{xx} = 0, t \in [0, T], x \in (0, 1), \\ \underline{\text{either}} : u = u_x = 0, \text{ at } x = 0, 1, \\ \underline{\text{or}} : u = u_x = 0 \text{ at } x = 0 \text{ and } u_{xx} = u_{xxx} = 0 \text{ at } x = 1, \end{cases}$$

Local controllability in a $H^{5+} \times H^{3+}((0,1), \mathbb{R})$-neighborhood of

$$u^{ref}(t,x) := \varphi_k(x) \sin(\sqrt{\lambda_k} t) + \varphi_{k+1} \sin(\sqrt{\lambda_{k+1}} t).$$

with control $p \in H^1_0((0,T), \mathbb{R})$ and $T := 8/\pi$.

Schrödinger in a potential of variable length

$$\begin{cases} i\dot{\psi}(t,q) = -\psi''(t,q), t \in \mathbb{R}_+, q \in (0, l(t)), \\ \psi(t,0) = \psi(t,l(t)) = 0. \end{cases}$$

Controllability in a H^{5+}-neighborhood of the ground state for $l \equiv 1$.

Controllability of the linearized system around $(\psi_1, u \equiv \gamma)$

Theorem : There exists $\gamma_0 > 0$ such that, for every $\gamma \in (0, \gamma_0)$, for every $T > 0$, for every $\Psi_0 \in T_{\mathcal{S}}(\psi_{1,\gamma}(0))$, $\Psi_f \in T_{\mathcal{S}}(\psi_{1,\gamma}(T))$, with

$$\Psi_0, \Psi_f \in H^3_{(0)}(I, \mathbb{C})$$

there exists

$$w \in L^2((0, T), \mathbb{R})$$

such that the solution of

$$\begin{cases} i\dot{\Psi} = -\frac{1}{2}\Psi'' - \gamma q \Psi - w(t) q \psi_{1,\gamma}, \\ \Psi(0) = \Psi_0, \\ \Psi(t, \pm 1/2) = 0, \end{cases}$$

satisfies $\Psi(T) = \Psi_f$.