PDE, Optimal Design and Numerics

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Some problems in control and stabilization of fluid flows

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Plan

Boundary feedback stabilization of the incompressible N.S.E.

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Boundary feedback stabilization of boundary layer equations

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Boundary feedback stabilization of the incompressible N.S.E.

Consider a stationary solution

$$-\Delta \mathbf{z}_s + (\mathbf{z}_s \cdot \nabla)\mathbf{z}_s + \nabla\chi = \mathbf{f}, \qquad \text{in } \Omega,$$

$$\text{div } \mathbf{z}_s = 0 \quad \text{in } \Omega, \quad \mathbf{z}_s = \mathbf{u}_s \text{ on } \Gamma.$$

Assume that $\mathbf{z}_s$ is an unstable solution of the instationary N.S.E.

$$\frac{\partial \mathbf{z}}{\partial t} - \Delta \mathbf{z} + (\mathbf{z} \cdot \nabla)\mathbf{z} + \nabla q = \mathbf{f}, \qquad \text{div } \mathbf{z} = 0 \quad \text{in } Q,$$

$$\mathbf{z} = \mathbf{u}_s \text{ on } \Sigma, \quad \mathbf{z}(0) = \mathbf{z}_s + \mathbf{y}_0 \text{ in } \Omega,$$

$$\text{and } |\mathbf{z}(t) - \mathbf{z}_s|_{\mathbf{L}^2(\Omega)} \longrightarrow \infty \quad \text{as} \quad t \longrightarrow \infty.$$

The stabilization problem

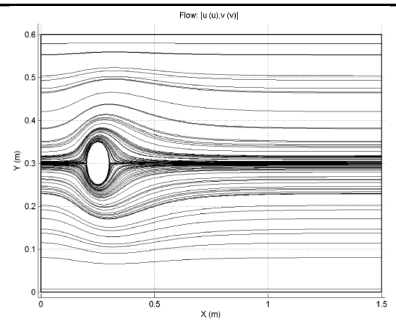
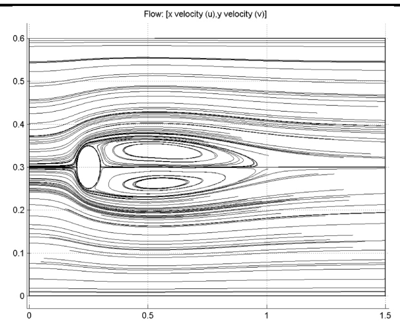
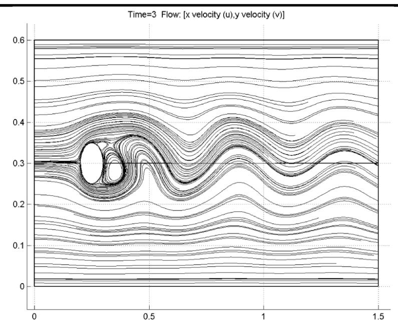
Find $\mathbf{u}$ in feedback form s.t. $|\mathbf{z}(t) - \mathbf{z}_s|_{\mathbf{Y}} \longrightarrow 0 \qquad$ as $t \longrightarrow \infty$,

where

$$\frac{\partial \mathbf{z}}{\partial t} - \Delta \mathbf{z} + (\mathbf{z} \cdot \nabla)\mathbf{z} + \nabla q = 0, \qquad \text{div } \mathbf{z} = 0 \quad \text{in } Q,$$

$$\mathbf{z} = \mathbf{u}_s + M\mathbf{u}, \quad \mathbf{z}(0) = \mathbf{z}_0 \text{ in } \Omega.$$

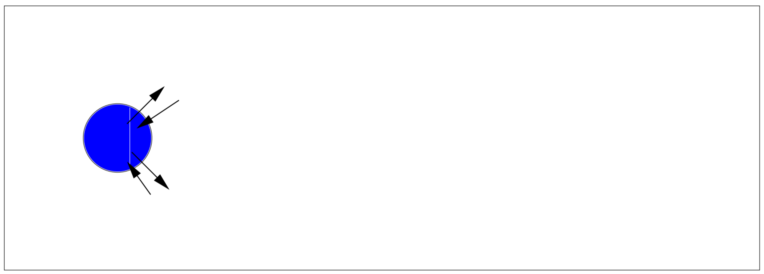
The physical phenomenon $Re = UD/\nu,\ D = 10cm,\ U = 1m/s$

	$Re < 5$	Regime of unseparated flow
	$5 < Re < 50$	A fixed pair of vortices
	$50 < Re < 150$	Vortex street

The stabilization problem

For an initial condition $\mathbf{z}_0$ close to $\mathbf{z}_s$, we look for $\mathbf{u}$ (in a feedback form) so that

$$|\mathbf{z}(t) - \mathbf{z}_s|_Y \longrightarrow 0 \qquad \text{as} \qquad t \to \infty.$$



The method

- Find a feedback law for the linearized equation by minimizing a quadratic functional.

- Prove that the solution to the nonlinear closed loop system exponentially decays.

The difficulties

- The Stokes and of the L.N.S.E. with boundary conditions s. t.

$$\int_\Gamma \mathbf{u}(t)\cdot\mathbf{n} = 0 \qquad \text{but} \quad \mathbf{u}(t)\cdot\mathbf{n} \neq 0.$$

- Choose a Linear-Quadratic control problem so that the feedback law be able to 'control' the nonlinearity of the N.S.E.

The nonlinear and the linearized equation

Set $\mathbf{y} = \mathbf{z} - \mathbf{z}_s$. We look for $\mathbf{u}(t) = K\mathbf{y}(t)$ such that if $|\mathbf{y}_0|_{\mathbf{Y}} = |\mathbf{z}_0 - \mathbf{z}_s|_{\mathbf{Y}} \leq \mu$, then the solution $\mathbf{y}$ of

$$\frac{\partial \mathbf{y}}{\partial t} - \Delta \mathbf{y} + (\mathbf{z}_s \cdot \nabla)\mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{z}_s + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla q = 0,$$

$$\text{div } \mathbf{y} = 0 \quad \text{in } Q, \quad \mathbf{y} = M\mathbf{u} \quad \text{on } \Sigma, \quad \mathbf{y}(0) = \mathbf{y}_0 \quad \text{in } \Omega\,,$$

obeys

$$|\mathbf{y}(t)|_{\mathbf{Y}} = |\mathbf{z}(t) - \mathbf{z}_s|_{\mathbf{Y}} \longrightarrow 0 \quad \text{as} \quad t \to \infty,$$

if

$$|\mathbf{y}_0|_{\mathbf{Y}} = |\mathbf{z}_0 - \mathbf{z}_s|_{\mathbf{Y}} \leq \mu.$$

The Stokes equation with non homogeneous boundary conditions

Known results when $\mathbf{u} \cdot \mathbf{n} = 0$

G. Grubb, V. A. Solonnikov, Math. Scand. 91,

$\mathbf{u} \in \mathbf{H}^{s,s/2}(\Gamma \times (0,T))$, $s \geq 1$, small data for N.S.E.

G. Grubb, J. Math. Fluid. Mech. 01,

$\mathbf{u} \in \mathbf{H}^{s,s/2}(\Gamma \times (0,T))$, $s > 0$ for Stokes.

V. Barbu, I. Lasiecka, R. Triggiani, 2005, s=0.

Known results when $\mathbf{u}\cdot\mathbf{n}\neq 0$ and $\int_\Gamma \mathbf{u}\cdot\mathbf{n}=0$

A. Fursikov, M. D. Gunzburger, L. S. Hou, SICON 98, 2D,

$\mathbf{u} \in L^2(0,T;\mathbf{H}^{1/2}(\Gamma))$, $\gamma_\tau\mathbf{u} \in H^{1/4}(0,T;\mathbf{L}^2(\Gamma))$, $\gamma_n\mathbf{u} \in H^{3/4}(0,T;\mathbf{H}^{-1}(\Gamma))$.

A. V. Fursikov, M. D. Gunzburger, L. S. Hou, Trans. A.M.S. 01,

J. Math. Fluid. Mech. 02, 3D

Stokes: $\mathbf{u} \in L^2(0,T;\mathbf{H}^{1/2}(\Gamma))$, $\gamma_\tau\mathbf{u} \in H^{1/2}(0,T;\mathbf{H}^{-1/2}(\Gamma))$, $\gamma_n\mathbf{u} \in H^{3/4}(0,T;\mathbf{H}^{-1}(\Gamma))$.

For NSE: $\mathbf{u} \in \mathbf{H}^1(\Gamma\times(0,T))$ with small data.

The Helmholtz decomposition

$$\mathbf{V}_n^0(\Omega) = \left\{ \mathbf{y} \in \mathbf{L}^2(\Omega) \mid \; \text{div } \mathbf{y} = 0, \; \mathbf{y} \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma \right\},$$

$$\mathbf{L}^2(\Omega) = \mathbf{V}_n^0(\Omega) \oplus \text{grad}\, H^1(\Omega),$$

$$\mathbf{V}_0^1(\Omega) = \{ \mathbf{y} \in \mathbf{H}_0^1(\Omega) \mid \; \text{div } \mathbf{y} = 0 \; \},$$

$$P \; : \; \mathbf{L}^2(\Omega) \longmapsto \mathbf{V}_n^0(\Omega).$$

The Stokes equation with a Dirichlet control: $\mathbf{u}(t) \cdot \mathbf{n} \neq 0$

Due to the incompressibility condition

$$\int_\Omega \text{div } \mathbf{y} = \int_\Gamma \mathbf{y} \cdot \mathbf{n} = 0.$$

We want to solve the Stokes equation for

$$\mathbf{u} \in \mathbf{V}^0(\Gamma) = \Big\{ \mathbf{y} \in \mathbf{L}^2(\Gamma) \mid \int_\Gamma \mathbf{y} \cdot \mathbf{n} = 0 \Big\}.$$

Therefore we look for $\mathbf{y}(t)$ in

$$\mathbf{V}^0(\Omega) = \Big\{ \mathbf{y} \in \mathbf{L}^2(\Omega) \mid \text{ div } \mathbf{y} = 0, \; \langle \mathbf{y} \cdot \mathbf{n}, 1 \rangle = 0 \Big\}.$$

The transposition method for the Stokes equation

A function $\mathbf{y} \in L^2(0,T;\mathbf{V}^0(\Omega))$ is a weak solution to the Stokes equation if

$$\int_Q \mathbf{y}\mathbf{g} = \int_\Sigma \Big(-\frac{\partial \mathbf{\Phi}}{\partial \mathbf{n}} + \psi \mathbf{n} \Big) \mathbf{u} + \int_\Omega \mathbf{y}_0 \mathbf{\Phi}(0)$$

for every $\mathbf{g} \in L^2(0,T;\mathbf{V}^0(\Omega))$, where $(\mathbf{\Phi}, \psi)$ is the solution to

$$-\frac{\partial \mathbf{\Phi}}{\partial t} - \Delta \mathbf{\Phi} + \nabla \psi = \mathbf{g}, \quad \mathrm{div}\, \mathbf{\Phi} = 0 \quad \text{in } Q,$$
$$\mathbf{\Phi} = 0 \text{ on } \Sigma, \quad \mathbf{\Phi}(T) = 0 \text{ in } \Omega.$$

Drawback: No information neither on the pressure nor on $P\mathbf{y}$.

A new approach. The Dirichlet operator

$$(D, D_p) \ : \ \mathbf{V}^0(\Gamma) \longmapsto \mathbf{V}^0(\Omega) \times L^2(\Omega)/\mathbb{R},$$
$$\mathbf{u} \longmapsto (\mathbf{w}, q),$$

where

$-\Delta \mathbf{w}(t) + \nabla q(t) = 0$ and div $\mathbf{w}(t) = 0$ in Ω, $\mathbf{w}(t) = \mathbf{u}(t)$ on Γ.

We set

$$\mathbf{y} = \mathbf{z} + \mathbf{w}.$$

Equation satisfied by $\mathbf{z}$:

$$\frac{\partial \mathbf{z}}{\partial t} = \Delta \mathbf{z} + \nabla q - \nabla p + \mathbf{w}', \quad \text{div } \mathbf{z} = 0,$$
$$\mathbf{z} = 0 \quad \text{on } \Sigma, \qquad \mathbf{z}(0) = \mathbf{y}_0 - \mathbf{w}(0).$$

We show that

$$P\mathbf{y}' = AP\mathbf{y} + (-A)PD\mathbf{u}, \qquad P\mathbf{y}(0) = \mathbf{y}_0.$$

(we have to extend the semigroup to $(D(A))'$)

What is the equation satisfied by $(I-P)\mathbf{y}$?

$$(I-P)\mathbf{y}(t) = (I-P)\mathbf{w}(t) = (I-P)D\mathbf{u}(t).$$

The system satisfied by $\mathbf{y}$ is finally :

$$P\mathbf{y}' = AP\mathbf{y} + (-A)PD\mathbf{u}, \qquad P\mathbf{y}(0) = \mathbf{y}_0,$$

$$(I-P)\mathbf{y} = (I-P)D\mathbf{u} = (I-P)D\gamma_n\mathbf{u}.$$

Theorem. Assume that $\mathbf{y}_0 \in \mathbf{V}^0_n(\Omega)$. A function $\mathbf{y} \in L^2(0,T;\mathbf{V}^0(\Omega))$ is a weak solution to the Stokes equation in the sense of the transposition method iff $\mathbf{y}$ satisfies the system:

$$P\mathbf{y}' = AP\mathbf{y} + (-A)PD\mathbf{u}, \qquad P\mathbf{y}(0) = \mathbf{y}_0,$$

$$(I-P)\mathbf{y} = (I-P)D\mathbf{u} = (I-P)D\gamma_n\mathbf{u}.$$

Theorem. For all $\mathbf{y}_0 \in \mathbf{V}^0(\Omega)$ and all $\mathbf{u} \in L^2(0,T;\mathbf{V}^0(\Gamma))$ the Stokes equation admits a unique weak solution in $L^2(0,T;\mathbf{V}^0(\Omega))$.

This solution obeys

$$\|P\mathbf{y}\|_{L^2(0,T;\mathbf{V}^{1/2-\varepsilon}(\Omega))} + \|P\mathbf{y}\|_{H^{1/4-\varepsilon/2}(0,T;\mathbf{V}^0(\Omega))}$$

$$+\|(I-P)\mathbf{y}\|_{L^2(0,T;\mathbf{V}^{1/2}(\Omega))}$$

$$\leq C\left(\|\mathbf{y}_0\|_{\mathbf{V}^0(\Omega)} + \|\mathbf{u}\|_{L^2(0,T;\mathbf{V}^0(\Gamma))}\right) \qquad \text{for all } \varepsilon > 0.$$

If moreover $\mathbf{u}$ belongs to $\mathbf{V}^{s,s/2}(\Sigma)$ then

$$\|(I-P)\mathbf{y}\|_{L^2(0,T;\mathbf{V}^{s+1/2}(\Omega))} + \|(I-P)\mathbf{y}\|_{H^{s/2}(0,T;\mathbf{V}^{1/2}(\Omega))}$$

$$\leq C(\|\mathbf{y}_0\|_{\mathbf{V}^{s-1/2\vee 0}(\Omega)} + \|\mathbf{u}\|_{\mathbf{V}^{s,s/2}(\Sigma)})$$

If $\mathbf{u}$ belongs to $\mathbf{V}^{s,s/2}(\Sigma)$ and $\mathbf{y}_0 \in \mathbf{V}^{s-1/2\vee 0}(\Omega)$, with $0 \leq s < 1$ then

$$\|P\mathbf{y}\|_{L^2(0,T;\mathbf{V}^{s+1/2-\varepsilon}(\Omega))} + \|P\mathbf{y}\|_{H^{s/2+1/4-\varepsilon/2}(0,T;\mathbf{V}^{0}(\Omega))}$$

$$\leq C(\|\mathbf{y}_0\|_{\mathbf{V}^{s-1/2\vee 0}(\Omega)} + \|\mathbf{u}\|_{\mathbf{V}^{s,s/2}(\Sigma)}) \qquad \text{for all } \varepsilon > 0.$$

If $\mathbf{u}$ belongs to $\mathbf{V}^{s,s/2}(\Sigma)$, with $1 < s \leq 2$ and if $\mathbf{y}_0$ et $\mathbf{u}(0)$ obeys the compatibility condition

$$P(\mathbf{y}_0 - D\mathbf{u}(0)) \in \mathbf{V}_0^{s-1/2}(\Omega) \quad \text{if} \quad 1 < s \leq 3/2,$$

$$\text{and}$$

$$P(\mathbf{y}_0 - D\mathbf{u}(0)) \in \mathbf{V}_0^{s-1/2}(\Omega) \cap \mathbf{V}_0^{1}(\Omega) \quad \text{if} \quad 3/2 \leq s \leq 2,$$

then the above estimate is satisfied. If moreover

$$(I - P)(\mathbf{y}_0 - D\mathbf{u}(0)) \in \mathbf{V}^{s-1/2}(\Omega)$$

then $\mathbf{y}(0) = \mathbf{y}_0$.

Other results.

- G. Grubb, 2001. We can take $\varepsilon = 0$ if $\gamma_n \mathbf{u} = 0$.

- Fursikov, Gunzburger, Hou, 2002. If $\gamma_n \mathbf{u} \in L^2(0,T;\mathbf{V}^{1/2}(\Gamma)) \cap H^{3/4}(0,T;\mathbf{V}^{-1}(\Gamma))$ then the solution belongs to

$$\mathcal{V}^1(Q) \subset \mathbf{W}(0,T) = \Big\{ \mathbf{y} \in L^2(0,T;\mathbf{V}^1(\Omega)) \mid \mathbf{y}' \in L^2(0,T;\mathbf{V}^{-1}(\Omega)) \Big\}.$$

- J.P.R., 2005 If $\gamma_n \mathbf{u} \in L^2(0,T;\mathbf{V}^{1/2}(\Gamma)) \cap H^{1/2}(0,T;\mathbf{V}^{-1/2}(\Gamma))$ then the solution belongs to

$$\mathbf{W}(0,T) = \Big\{ \mathbf{y} \in L^2(0,T;\mathbf{V}^1(\Omega)) \mid \mathbf{y}' \in L^2(0,T;\mathbf{V}^{-1}(\Omega)) \Big\}.$$

The Navier-Stokes equations with non homogeneous boundary conditions

G. Grubb (01) Existence of solutions for small data and when $\gamma_n \mathbf{u} = 0$, $\mathbf{u} \in \mathbf{H}^{1,1/2}(\Sigma)$.

Fursikov et al. (02)

If $N = 3$, if $\mathbf{u}$ belongs to

$$\mathbf{V}^{1,1}(\Sigma) = \Big\{ \mathbf{u} \in L^2(0,T;\mathbf{V}^1(\Gamma)) \mid \mathbf{u} \in H^1(0,T;\mathbf{V}^0(\Gamma)) \Big\},$$

and if

$$\|\mathbf{y}_0\|_{\mathbf{V}^0(\Omega)} + \|\mathbf{u}\|_{\mathbf{V}^{1,1}(\Sigma)} \qquad \text{is small enough,}$$

$+$ C.C., then the N.S.E. admits a unique solution belonging to $\mathbf{W}(0,T)$.

J.P.R. (05)

If $N = 3$, if $\mathbf{u}$ belongs to

$$\mathbf{V}^{1,1}(\Sigma) = \Big\{ \mathbf{u} \in L^2(0,T;\mathbf{V}^1(\Gamma)) \mid \mathbf{u} \in H^1(0,T;\mathbf{V}^0(\Gamma)) \Big\},$$

and $\mathbf{y}_0 \in \mathbf{V}^0_n(\Omega)$, then the N.S.E. admits a weak solution belonging to $C_w([0,T];\mathbf{V}^0(\Omega)) \cap L^2(0,T;\mathbf{V}^1(\Omega))$.

Feedback stabilization of the N.S.E.

Optimal control problem

$$(\mathcal{P}) \quad \text{inf}\Big\{J(\mathbf{y},\mathbf{u}) \mid (\mathbf{y},\mathbf{u}) \text{ obeys the L.N.S.E.}\Big\},$$

$$\mathbf{y}' = A\mathbf{y} + F'(\mathbf{z}_s)\mathbf{y} + B\mathbf{u} \text{ in } (0,\infty),\ \ \mathbf{y}(0) = \mathbf{y}_0,$$

$$J(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \Big|C\mathbf{y}\Big|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\int_0^\infty \Big|\mathbf{u}\Big|^2_{\mathbf{U}}.$$

Existence of admissible solutions (Internal null controllability)

O. Yu. Imanuvilov, COCV, 2001

Fernandez-Cara, Guerrero, Imanuvilov, Puel, JMPA 2004

V. Barbu, R. Triggiani, Internal stabilization of the N.S.E., 2004

A. Fursikov, DCDS, 2004.

There exists an operator $\Pi \in \mathcal{L}(\mathbf{V}_n^0(\Omega))$, or $\Pi \in \mathcal{L}(\mathbf{Y}(\Omega), (\mathbf{Y}(\Omega))')$ with

$$\mathbf{Y}(\Omega) \hookrightarrow \mathbf{V}_n^0(\Omega) \hookrightarrow (\mathbf{Y}(\Omega))',$$

satisfying $\Pi = \Pi^* \geq 0$, and such that

$$J(\mathbf{y}_{\mathbf{y}_0}, \mathbf{u}_{\mathbf{y}_0}) = \frac{1}{2}(\Pi \mathbf{y}_0, \mathbf{y}_0)_{\mathbf{V}_n^0(\Omega)} \qquad \text{for all } \mathbf{y}_0 \in \mathbf{Y}(\Omega).$$

Three methods

The Lyapunov function method (B.L.T.), the smoothing observation method (JPR), the method using an extended system (M. Badra).

The Lyapunov function method. Choose J so that

$$\mathbf{y}_0 \longmapsto \Big(\Pi\mathbf{y}_{\mathbf{y}_0}(t), \mathbf{y}_{\mathbf{y}_0}(t)\Big)_{\mathbf{L}^2(\Omega)}$$

be a Lyapunov function – in $\mathbf{L}^2(\Omega)$ – for the closed loop nonlinear system, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{y}_{\mathbf{y}_0}(t), \Pi\mathbf{y}_{\mathbf{y}_0}(t)\Big) + \gamma\Big(\mathbf{y}_{\mathbf{y}_0}(t), \Pi\mathbf{y}_{\mathbf{y}_0}(t)\Big) \leq 0 \qquad t \geq 0,\ \gamma > 0.$$

The smoothing observation method. Choose J so that Π be a smoothing operator. Prove some regularizing properties for $(A_\Pi, D(A_\Pi))$. Use these regularity results to prove that the solution to the the closed loop nonlinear system exponentially decays.

V. Barbu, COCV, 2003 $N = 3$, Internal control, $B = \chi_\omega$,

$$J(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \left|(-P\Delta)^{3/4}\mathbf{y}\right|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\int_0^\infty \left|\mathbf{u}\right|^2_{\mathbf{L}^2(\omega)}.$$

The equation satisfied by Π is

$$\Pi A_{\mathbf{z}_s} + A^*_{\mathbf{z}_s}\Pi - \Pi B B^*\Pi + (P(-\Delta))^{3/2} = 0, \qquad \Pi = \Pi^* \geq 0.$$

V. Barbu, I. Lasiecka, R. Triggiani, AMS, 2005, $N = 3$, with a boundary control acting everywhere on Γ and satisfying

$$\mathbf{u}(t) \cdot \mathbf{n} = 0,$$

choose the functional

$$J(\mathbf{y}, \mathbf{u}) = \frac{1}{2} \int_0^\infty \left| C\mathbf{y} \right|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2} \int_0^\infty \left| \mathbf{u} \right|^2_{\mathbf{L}^2(\omega)}$$

with

$$\left| C\mathbf{y} \right|_{\mathbf{L}^2(\Omega)} \approx \left| \mathbf{y} \right|_{\mathbf{V}^{3/2+\varepsilon}(\Omega)}$$

The equation A.R.E. satisfied by Π is only satified on $D(A_\Pi)$.

Feedback boundary stabilization of the Linearized N.S.E.

The optimal control problem

$$\text{Minimize} \quad J(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \int_\Omega |\mathbf{y}|^2 + \frac{1}{2}\int_0^\infty \int_\Gamma |\mathbf{u}|^2$$

$$\frac{\partial \mathbf{y}}{\partial t} - \Delta \mathbf{y} + (\mathbf{z}_s \cdot \nabla)\mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{z}_s + \nabla q = 0,$$

$$\text{div } \mathbf{y} = 0 \quad \text{in } Q, \quad \mathbf{y} = M\mathbf{u} \quad \text{on } \Sigma, \quad \mathbf{y}(0) = \mathbf{y}_0 \quad \text{in } \Omega\,.$$

For simplicity, take $\mathbf{z}_s = 0$ and $M = I$, that is

$$P\mathbf{y}' = AP\mathbf{y} + (-A)PD\mathbf{u} = AP\mathbf{y} + B\mathbf{u}, \qquad P\mathbf{y}(0) = \mathbf{y}_0,$$

$$(I - P)\mathbf{y} = (I - P)D\mathbf{u}.$$

We transform the functional

$$J(\mathbf{y}, \mathbf{u})$$
$$= \frac{1}{2} \int_0^\infty \int_\Omega |\mathbf{y}|^2 + \frac{1}{2} \int_0^\infty \int_\Gamma |\mathbf{u}|^2$$
$$= \frac{1}{2} \int_0^\infty \int_\Omega |P\mathbf{y}|^2 + \frac{1}{2} \int_0^\infty \int_\Omega |(I-P)\mathbf{y}|^2 + \frac{1}{2} \int_0^\infty \int_\Gamma |\mathbf{u}|^2$$

and

$$|(I-P)\mathbf{y}| = |(I-P)D\mathbf{u}| = |(I-P)D\gamma_n \mathbf{u}|.$$

Setting

$$R = D^*(I - P)D + I,$$

we have

$$J(\mathbf{y}, \mathbf{u}) = \frac{1}{2}\int_0^\infty \int_\Omega |P\mathbf{y}|^2 + \frac{1}{2}\int_0^\infty \int_\Gamma (|R^{1/2}\mathbf{u}_n|^2 + |\mathbf{u}_\tau|^2)$$

$$= \frac{1}{2}\int_0^\infty \int_\Omega |P\mathbf{y}|^2 + \frac{1}{2}\int_0^\infty \int_\Gamma |R^{1/2}\mathbf{u}|^2.$$

The control problem is now

Minimize

$$I(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \int_\Omega |P\mathbf{y}|^2 + \frac{1}{2}\int_0^\infty \int_\Gamma |R^{1/2}\mathbf{u}|^2$$

$$P\mathbf{y}' = AP\mathbf{y} + (-A)PD\mathbf{u}_n + (-A)D\mathbf{u}_\tau = AP\mathbf{y} + B_n\mathbf{u}_n + B_\tau\mathbf{u}_\tau,$$

$$P\mathbf{y}(0) = \mathbf{y}_0, \qquad (I-P)\mathbf{y} = (I-P)D\mathbf{u}.$$

The value function is

$$\frac{1}{2}\Big(\Pi\mathbf{y}_0, \mathbf{y}_0\Big)_{\mathbf{V}_n^0(\Omega)}$$

The optimal solution is characterized by

$$P\mathbf{y}' = AP\mathbf{y} + B_n\mathbf{u}_n + B_\tau\mathbf{u}_\tau, \qquad P\mathbf{y}(0) = \mathbf{y}_0,$$
$$(I - P)\mathbf{y} = (I - P)D\mathbf{u},$$

$$\mathbf{\Phi}(t) = \Pi P\mathbf{y}(t),$$

$$-\mathbf{\Phi}' = A\mathbf{\Phi} + P\mathbf{y}, \quad \mathbf{\Phi}(\infty) = 0,$$
$$\mathbf{u}_n = -R^{-1}B_n^*\mathbf{\Phi}, \qquad \mathbf{u}_\tau = -B_\tau^*\mathbf{\Phi},$$

which is equivalent to

$$P\mathbf{y}' = AP\mathbf{y} + B\mathbf{u}, \qquad P\mathbf{y}(0) = \mathbf{y}_0,$$

$$(I-P)\mathbf{y} = (I-P)D\mathbf{u},$$

$$\mathbf{\Phi}(t) = \Pi P\mathbf{y}(t),$$

$$-\frac{\partial \mathbf{\Phi}}{\partial t} - \Delta\mathbf{\Phi} + \nabla\psi = P\mathbf{y}, \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in } Q,$$

$$\mathbf{\Phi} = 0 \text{ on } \Sigma, \quad \mathbf{\Phi}(\infty) = 0 \text{ in } \Omega,$$

$$\mathbf{u} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{n}} - \psi\,\mathbf{n} + c(\mathbf{\Phi},\psi)\mathbf{n}$$

$$c(\mathbf{\Phi},\psi) = -\frac{1}{|\Gamma|}\int_\Gamma \Big(\frac{\partial \mathbf{\Phi}}{\partial n} - \psi\mathbf{n}\Big)\cdot\mathbf{n}\,.$$

We can prove that Π is the solution to the ARE

$$\Pi A + A^*\Pi - \Pi B_\tau B_\tau^*\Pi - \Pi B_n R^{-1} B_n^*\Pi + I = 0, \qquad \Pi = \Pi^* \geq 0.$$

$$B_\tau = (-A)D\gamma_\tau, \qquad B_\tau^* = \gamma_\tau D^*(-A),$$

$$B_n = (-A)PD\gamma_n, \qquad B_n^* = \gamma_n D^*(-A),$$

$$D^*\mathbf{g} = -\frac{\partial \mathbf{w}}{\partial \mathbf{n}} + \pi\mathbf{n} - c(\mathbf{w}, \pi)\mathbf{n},$$

$$-\Delta\mathbf{w} + \nabla\pi = \mathbf{g}, \quad \text{div}\,\mathbf{w} = 0, \quad \mathbf{w} = 0 \quad \text{on } \Gamma.$$

Studying the regularity of the optimal pair, we prove

$$\Pi \ : \ \mathbf{y}_0 \longmapsto \mathbf{\Phi}(0),$$

$$\Pi \in \mathcal{L}(\mathbf{V}^0_n(\Omega); \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega)),$$

$$\Pi \in \mathcal{L}(\mathbf{V}^{1/2-\varepsilon}_n(\Omega); \mathbf{V}^{5/2-\varepsilon}(\Omega) \cap \mathbf{V}^1_0(\Omega)), \quad 0 < \varepsilon \leq 1/2,$$

$$B^*\Pi \in \mathcal{L}(\mathbf{V}^{1/2-\varepsilon}_n(\Omega); \mathbf{V}^{1-\varepsilon}(\Gamma)), \quad 0 < \varepsilon \leq 1/2,$$

where

$$B^*\Pi\mathbf{y}_0 = \frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}}(0) - \psi(0)\mathbf{n} + c\mathbf{n}.$$

The semigroup $(e^{tA_\Pi})_{t\geq 0}$ is exponentially stable in $\mathbf{V}^0_n(\Omega)$ and in $\mathbf{V}^{1/2-\varepsilon}_n(\Omega)$.

Stabilization of the Navier-Stokes equations

Find $\mathbf{u}$ s.t. $|\mathbf{y}(t)|_{\mathbf{V}^{1/2-\varepsilon}(\Omega)} = |\mathbf{z}(t) - \mathbf{z}_s|_{\mathbf{V}^{1/2-\varepsilon}(\Omega)} \longrightarrow 0$ as $t \longrightarrow \infty$,

where

$$\frac{\partial \mathbf{y}}{\partial t} - \Delta \mathbf{y} + (\mathbf{z}_s \cdot \nabla)\mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{z}_s + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = 0, \quad \text{in } Q,$$

$$\text{div } \mathbf{y} = 0 \quad \text{in } Q, \quad \mathbf{y} = \mathbf{u} \quad \text{on } \Sigma, \quad \mathbf{y}(0) = \mathbf{y}_0 = \mathbf{z}_0 - \mathbf{z}_s \text{ in } \Omega.$$

We set

$$A_{\mathbf{z}_s}\mathbf{y} = \Delta \mathbf{y} - (\mathbf{z}_s \cdot \nabla)\mathbf{y} - (\mathbf{y} \cdot \nabla)\mathbf{z}_s, \qquad D(A_{\mathbf{z}_s}) = \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega),$$

$$F(\mathbf{y}) = -P(\mathbf{y} \cdot \nabla)\mathbf{y},$$

$$B_n = (\lambda_0 I - A_{\mathbf{z}_s})PD, \qquad B_\tau = (\lambda_0 I - A_{\mathbf{z}_s})D,$$

where $\mathbf{w} = D\mathbf{u}$ is the solution to

$$\lambda_0\mathbf{w} - \Delta\mathbf{w} - (\mathbf{z}_s \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{z}_s + \nabla q = 0 \quad \text{in } \Omega,$$
$$\text{div } \mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} = \mathbf{u} \quad \text{on } \Gamma.$$

We rewrite the equation in the form

$$P\mathbf{y}' = A_{\mathbf{z}_s}P\mathbf{y} + F(\mathbf{y}) + B_n\mathbf{u}_n + B_\tau\mathbf{u}_\tau, \qquad \mathbf{y}(0) = \mathbf{y}_0,$$
$$(I - P)\mathbf{y} = (I - P)D\mathbf{u}_n.$$

Setting

$$R = D^*(I - P)D + I,$$

we prove that the algebraic Riccati equation

$$\Pi A_{\mathbf{z}_s} + A_{\mathbf{z}_s}^* \Pi - \Pi B_\tau B_\tau^* \Pi - \Pi B_n \, R^{-1} \, B_n^* \Pi + I = 0, \qquad \Pi = \Pi^* \geq 0,$$

admits a unique solution $\Pi \in \mathcal{L}(\mathbf{V}_n^0(\Omega); D((-A_{\mathbf{z}_s}^*)))$.

We set

$$A_\Pi = A_{\mathbf{z}_s} - B_\tau B_\tau^* \Pi - B_n R^{-1} B_n^* \Pi \qquad \text{and} \qquad S_\Pi(t) = e^{t\, A_\Pi}.$$

The semigroup

$$S_\Pi(t) \qquad \text{is analytic and exponentially stable on } \mathbf{V}_n^0(\Omega).$$

Consider the Navier-Stokes equations with the linear feedback law:

$$P\mathbf{y}' = A_\Pi P\mathbf{y} + F(\mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{y}_0,$$
$$(I - P)\mathbf{y} = -(I - P)DR^{-1}B_n^*\Pi P\mathbf{y},$$

that is

$$\frac{\partial \mathbf{y}}{\partial t} - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = 0 \quad \text{and } \operatorname{div}\mathbf{y} = 0 \quad \text{in } Q,$$
$$\mathbf{y} = -DR^{-1}B^*\Pi P\mathbf{y} \quad \text{on } \Sigma, \quad \mathbf{y}(0) = \mathbf{y}_0.$$

Theorem. For all $0 < \varepsilon < 1/4$, there exists $\mu_0 > 0$ and a nondecreasing function η from $\mathbb{R}^+$ into itself, such that if $\mu \in (0, \mu_0)$ and $\|\mathbf{y}_0\|_{\mathbf{V}_n^{1/2-\varepsilon}(\Omega)} \leq \eta(\mu)$, then the above equation admits a unique solution in the set

$$D_\mu = \Big\{ \mathbf{y} \mid \|\mathbf{y}\|_{\mathbf{V}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q)} \leq \mu \Big\}.$$

In particular

$$\lim_{t\to\infty} \|\mathbf{y}(t)\|_{\mathbf{V}^{1/2-\varepsilon}(\Omega)} = 0,$$

and

$$\|\mathbf{y}(t)\|_{\mathbf{V}^{1/2-\varepsilon}(\Omega)} \leq C(\mathbf{z}_s, \mu).$$

Exponential stabilization

To obtain the exponential stabilization We set

$$\hat{\mathbf{y}} = e^{\omega t}\mathbf{y}, \qquad \hat{\mathbf{u}} = e^{\omega t}\mathbf{u}.$$

If

$$P\mathbf{y}' = A_{\mathbf{z}_s}P\mathbf{y} + F(\mathbf{y}), \qquad P\mathbf{y}(0) = \mathbf{y}_0,$$
$$(I-P)\mathbf{y} = -(I-P)D\mathbf{u},$$

then

$$P\hat{\mathbf{y}}' = A_{\mathbf{z}_s}P\hat{\mathbf{y}} + \omega\hat{\mathbf{y}} + e^{-\omega t}PF(\hat{\mathbf{y}}), \qquad P\hat{\mathbf{y}}(0) = \mathbf{y}_0,$$
$$(I-P)\hat{\mathbf{y}} = -(I-P)D\hat{\mathbf{u}}.$$

The 3D case

$$F(\mathbf{y}) = -P(\mathbf{y}\cdot\nabla)\mathbf{y} = -P\mathsf{div}(\mathbf{y}\otimes\mathbf{y}),$$

$$F\ :\ \mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q) \mapsto L^2(0,\infty;(\mathbf{V}^{1/2-2\varepsilon}(\Omega))') \cap L^1(0,\infty;\mathbf{V}^0(\Omega)$$

$$\left\|\int_0^t e^{(t-s)A}\mathbf{f}(s)ds\right\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q)} \leq C\|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}^{1/2-2\varepsilon}(\Omega))')}.$$

We consider the LQ problem

Minimize

$$I(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \int_\Omega |(-A_0)^{-1/2} P\mathbf{y}|^2$$

$$+\frac{1}{2}\int_0^\infty \int_{\Gamma_c} |R_A^{1/2}(t)\mathbf{u}_n|^2 + \frac{1}{2}\int_0^\infty \int_{\Gamma_c} |\mathbf{u}_\tau|^2$$

$$P\mathbf{y}' = AP\mathbf{y} + \theta(t) B_n \hat{\mathbf{u}}_n + \theta(t) B_\tau \hat{\mathbf{u}}_\tau + \mathbf{f},$$

$$P\mathbf{y}(0) = \mathbf{y}_0, \qquad (I-P)\mathbf{y} = (I-P) D_A \theta \mathbf{u}.$$

$$\widehat{\Pi} A + A^* \widehat{\Pi} - \widehat{\Pi} B_\tau B_\tau^* \widehat{\Pi} - \widehat{\Pi} B_n R_A^{-1} B_n^* \widehat{\Pi} + (-A_0)^{-1} = 0, \qquad \widehat{\Pi} = \widehat{\Pi}^* \geq$$

and for $t \in [0, T]$, Π is the solution of

$$-\Pi'(t) = \Pi A + A^*\Pi - \theta^2 \Pi B_\tau B_\tau^* \Pi - \Pi B_n \, \theta \, R_A^{-1} \theta \, B_n^* \Pi + (-A_0)^{-1},$$

$$\Pi(T) = \widehat{\Pi}.$$

The adjoint equation is now

$$\bar{\phi}(t) = \Pi(t)\bar{\mathbf{y}}(t)$$

$$-\bar{\mathbf{\Phi}}' = A^*\bar{\mathbf{\Phi}} + (-A_0)^{-1}\bar{\mathbf{y}}, \qquad \mathbf{\Phi}(\infty) = 0.$$

Theorem. For all $t \geq 0$ and all $0 \leq \varepsilon \leq 1/2$, we have

$$\|\Pi(t)\|_{\mathcal{L}(\mathbf{V}_n^{1/2-\varepsilon}(\Omega),\mathbf{V}^{9/2-\varepsilon}(\Omega))} \leq C,$$

$$\|B^*\Pi(t)\|_{\mathcal{L}(\mathbf{V}_n^{1/2-\varepsilon}(\Omega),\mathbf{V}^{3-\varepsilon}(\Gamma))} \leq C,$$

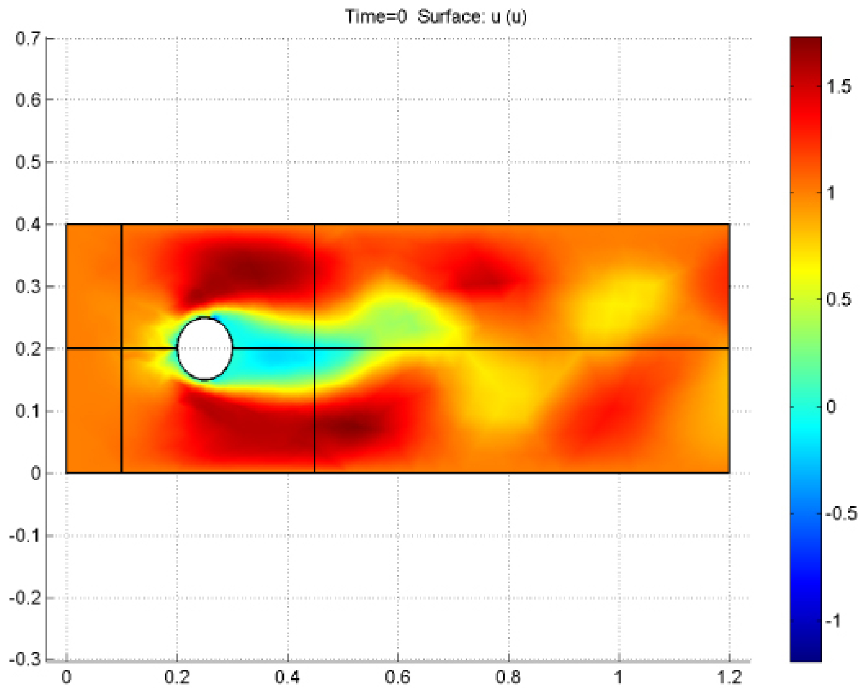
$$\|\Pi'(t)\|_{\mathcal{L}(\mathbf{V}_n^{1/2-\varepsilon}(\Omega),\mathbf{V}^{5/2-\varepsilon}(\Omega))} \leq C.$$

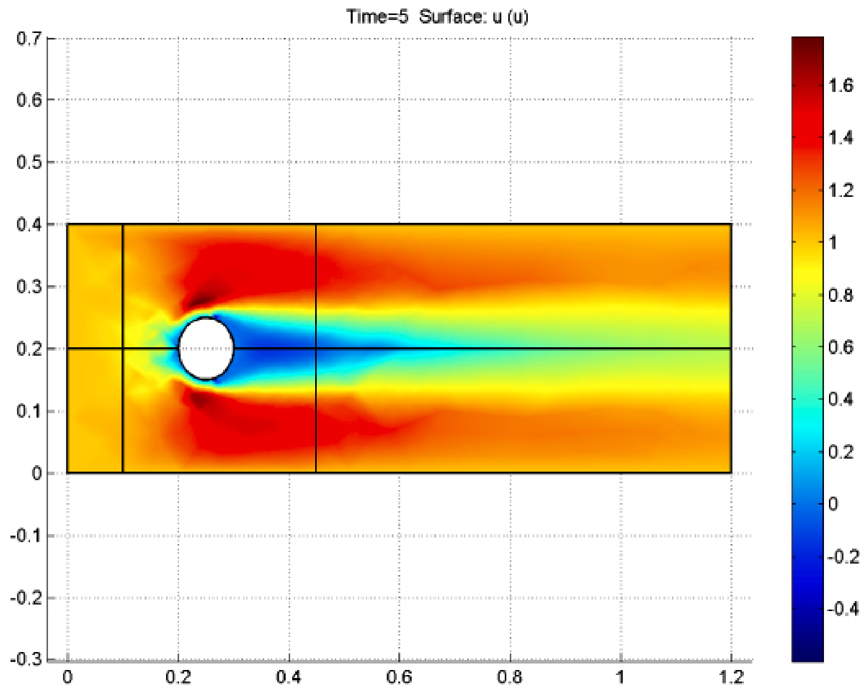
Theorem. If $\mathbf{y}_0 \in \mathbf{V}_0^{1/2+\varepsilon}(\Omega)$, then the linear feedback law

$$\mathbf{u}(t) = -(B_\tau^* + R_A^{-1} B_n^*)\theta(t)\,\Pi(t)\mathbf{y}(t),$$

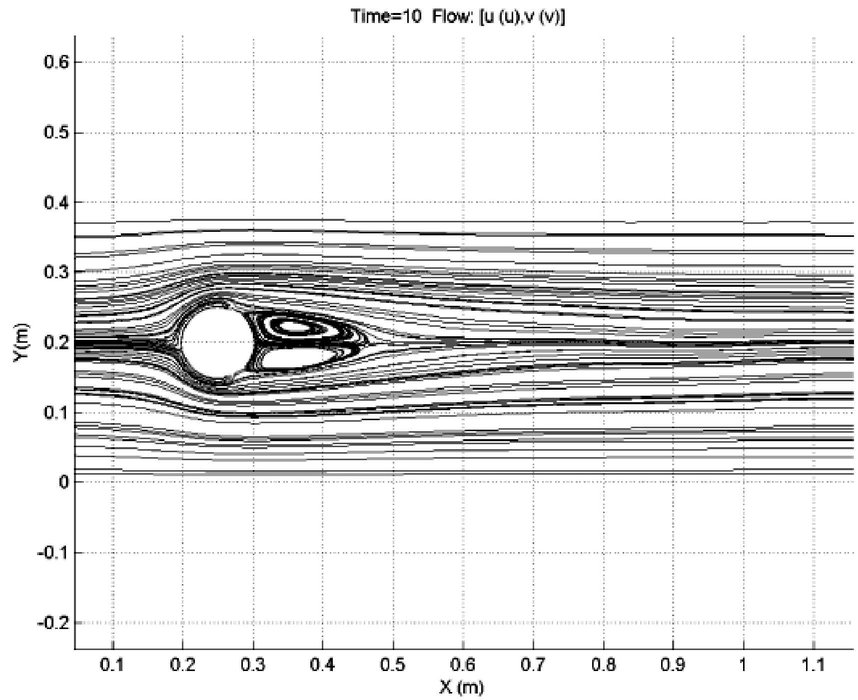
locally stabilizes the 3D N.S.E.

The longitudinal velocity (J.-M. Buchot)





The stream function $10s$ after controlling (J.-M. Buchot)



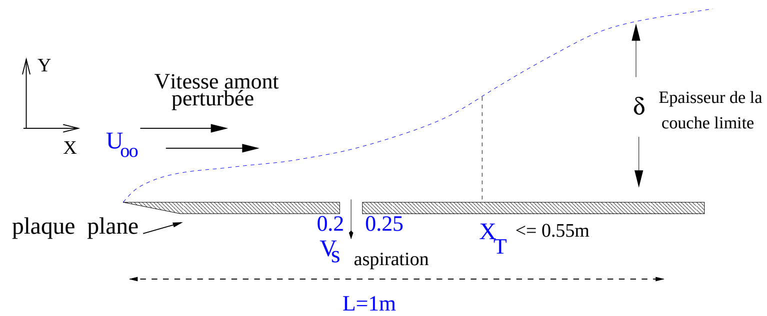
Stabilization and controllability of degenerate parabolic equations

with Jean-Marie Buchot, Patrick Martinez,

Jean-Pierre Raymond and Judith Vancostenoble

The Physical Problem

The laminar boundary layer is described by the Prandtl equations



- $U_\infty(t) = U_\infty^s + u_\infty(t)$: Velocity of the incoming flow,
- $X_T(t) = X_T^s + x_T(t)$: laminar-to-turbulent transition location,
- $v_c(t)$: Suction velocity through the plate,
- $(u(t), v(t))$: Velocity field.

The stabilization problem

- U^s_∞, X^s_T are known.
- We look for v_c (in a feedback form) so that $x_T(t) \longrightarrow 0$ as $t \longrightarrow \infty$, or so that $(u(t) - U^s, v(t) - V^s) \longrightarrow 0$.

The controllability problem

- We look for v_c so that $(u(T) - U^s, v(T) - V^s) = (0, 0)$.

Obstruction

- The Prandtl's equations are stated in an unbounded domain.

The method

The Prandtl equations are transformed into the Crocco equation (a nonlinear degenerate parabolic equation) by using the so-called Crocco transformation

A Linear-Quadratic control problem is stated by linearizing the Crocco equation, and the observation (the transition location) around the steady state solution

The feedback law is determined by solving a LQR or a LQG control problem

The mathematical results

Existence and uniqueness result for the linearized Crocco equation (a degenerate parabolic equation)

Existence and uniqueness result for the Riccati equation associated with the linearized Crocco operator

Feedback formulation of the optimal control

Exact controllability results for a simplified model (Martinez-Vancostenoble-JPR)

Method of matched asymptotic expansions

The regular approximation of the NSE leads to the Euler equations. The singular approximation is obtained with

$$\xi = x, \qquad \eta = \frac{y}{\sqrt{\varepsilon}},$$

as the boundary layer variables, where

$$\varepsilon = \frac{1}{Re}, \qquad Re = \frac{U_\infty L}{\nu}.$$

Prandtl' equations

$$\partial_\xi u + \partial_\eta v = 0, \qquad (0,L)\times(0,\infty)\times(0,T),$$

$$\partial_t u + u\partial_\xi u + v\partial_\eta u - \nu\partial_{\eta\eta} u = U'_\infty,$$

$$u(0,\eta,t) = u_1,$$

$$u(\xi,0,t) = 0, \qquad v(\xi,0,t) = v_c\,\chi_\gamma, \qquad u(\xi,\eta,0) = u_0(\xi,\eta),$$

$$u(\xi,\eta,t) \to U_\infty \qquad \text{when} \qquad \eta \to \infty.$$

The Crocco transformation

$$t = t, \quad x = \xi, \quad y = \frac{u(\xi,\eta,t)}{U_\infty(t)}, \quad w(x,y,t) = \frac{1}{U_\infty(t)}\frac{\partial u}{\partial \eta}(\xi,\eta,t).$$

The transformation is bijective in the domain where $\frac{\partial u}{\partial \eta}(\xi,\eta,t) > 0$.

The Crocco equation

$$w_t + U_\infty \, y \, w_x$$

$$+\tfrac{U'_\infty}{U_\infty}(1-y)w_y - \nu w^2 w_{yy} + \tfrac{U'_\infty}{U_\infty} w = 0, \quad (x,y,t) \in \Omega \times (0,T),$$

$$\nu w w_y(x,0,t) = v_c \, \chi_\gamma \, w - \tfrac{U'_\infty}{U_\infty}, \qquad (x,t) \in (0,L) \times (0,T),$$

$$w(x,1,t) = 0, \qquad (x,t) \in (0,L) \times (0,T),$$

$$w(0,y,t) = w_1(y,t), \qquad (y,t) \in (0,1) \times (0,T),$$

$$w(x,y,0) = w_0(x,y), \qquad (x,y) \in \Omega.$$

Linearized Crocco equation

$$
\begin{aligned}
&z_t + az_x - bz_{yy} + cz = f, && (x,y,t) \in \Omega \times (0,T),\\
&\nu z_y(x,0,t) = g + v_c\,\chi_\gamma && (x,t) \in (0,L) \times (0,T),\\
&bz(x,1,t) = 0, && (x,t) \in (0,L) \times (0,T),\\
&z(0,y,t) = z_1(y,t), && (y,t) \in (0,1) \times (0,T),\\
&z(x,y,0) = z_0(x,y), && (x,y) \in \Omega,
\end{aligned}
$$

where

$$
f = -yu_\infty \partial_x w^s - \frac{u'_\infty}{U^s_\infty}(w^s + (1-y)\partial_y w^s)
$$

$$
g = -\frac{U'_\infty}{w^s U_\infty}.
$$

The coefficients satisfy:

$$a = U^s_\infty y, \qquad b = \nu (w^s)^2, \qquad c = -2\nu w^s w^s_{yy},$$

$$C_1|1-y|^2\sigma(y) \le b(x,y) \le C_2|1-y|^2\sigma(y),$$

$$\left|\frac{\partial b}{\partial y}(x,y)\right| \le C_3\sqrt{\sigma(y)}|1-y|$$

$$\left|\frac{\partial b}{\partial x}(x,y)\right| \le C_4\sigma(y)|1-y|^2,$$

with

$$\sigma(y) = 1$$

or

$$\sigma(y) = (-\ell n(\mu(1-y)))^{1/2}, \quad 0 < \mu < 1.$$

Existence of stationary solutions (Oleinik) in classes of solutions for which

$$\sigma(y) = 1 \quad \text{or} \quad \sigma(y) = (-\ell n(\mu(1-y)))^{1/2}, \quad 0 < \mu < 1.$$

Existence of instationary solutions for the Crocco eq. (Z. Xin, L. Zhang, 2004) if

$$\partial_\xi p(\xi, t) \leq 0, \qquad \partial_\eta u_0 > 0, \qquad \partial_\eta u_1 > 0.$$

The solutions are obtained in the class of solutions for which $\sigma(y) = 1$.

Existence of instationary solutions for the Linearized Crocco eq. (Buchot, R., 2001) if

$$b(x, y) = |1 - y|^2.$$

Existence of instationary solutions for the Linearized Crocco eq. (Buchot, R., 2005, JMFD) if

$$\sigma(y) = 1.$$

Open problem. The case

$$\sigma(y) = (-\ell n(\mu(1-y)))^{1/2}, \quad 0 < \mu < 1,$$

is open for the Linearized Crocco equation. The existence and uniqueness of solution to the Prandtl' equations when the sign condition on the pressure is not satisfied is open.

We introduce $H^1(0,1;d)$ (resp. $H^1_{\{1\}}(0,1;d)$) is the closure of $C^\infty([0,1])$ (resp. $C^\infty_c([0,1))$) in the norm

$$\|z\|^2_{H^1(0,1;d)} = \int_0^1 (|z|^2 + |1-y|^2\sigma^2 z_y^2)\, dy.$$

Theorem The linearized Crocco equation admits a unique weak solution z. Moreover z belongs to $C([0,T];L^2(\Omega)) \cap L^2(0,T;L^2(0,L;H^1(0,1;d)))$ and $\sqrt{a}z \in C_w([0,L];L^2(0,T;L^2(0,1)))$. We have the following estimate

$$\begin{aligned}
&\|z\|_{L^\infty(0,T;L^2(\Omega))} + \|\sqrt{a}z\|_{L^\infty(0,L;L^2(0,T;L^2(0,1)))} \\
&+\|z\|_{L^2(0,T;L^2(0,L;H^1(0,1;d)))} \\
&\le C\Big(\|f\|_{L^2(Q)} + \|g\|_{L^2(0,T;L^2(0,L))} + \|\sqrt{a}z_1\|_{L^2(0,T;L^2(0,1))} + \|z_0\|_{L^2(\Omega)}\Big)
\end{aligned}$$

Stabilization of the laminar-to-turbulent transition location

The transition location depends nonlinearly of (u, v) and u_∞. The linearized transition location is of the form

$$X_T^\ell(t) = X_T^s + \int_\Omega \phi(x,y)\, z(x,y,t)\, dx\, dy + c_0\, u_\infty(t).$$

We consider the control problem

$$\text{Minimize} \qquad J(z_v, v)$$

where

$$J(z,v) = \frac{1}{2}\int_0^\infty |X_T^\ell(t) - X_T^s|^2 dt + \frac{1}{2}\int_0^\infty \int_\gamma |v(x,t)|^2\, dx\, dt.$$

Feedback formulation

The control problem admits a unique solution $(\bar{z}, \bar{v})$ which satisfies the feedback formula

$$\bar{v}(t) = -(\Pi\bar{z}(t) + r(t))|_{\gamma},$$

where $\Pi \in \mathcal{L}(L^2(\Omega))$ is the solution of the algebraic Riccati equation

$$\Pi = \Pi^* \geq 0,\ \Pi\mathcal{A} + \mathcal{A}^*\Pi - \Pi BB^*\Pi + C^*C = 0,$$

and r is the solution to

$$-r' = \mathcal{A}^*r - \Pi BB^*r + c_0 C^*u_\infty + \Pi f + \Pi Bg + \Pi Dz_1, \qquad r(\infty) = 0.$$

$B,\ D$ are boundary control operators,

C is the observation operator.

The operators $\mathcal{A}$ and $\mathcal{A}^*$ are defined by

$$D(\mathcal{A}) = \Big\{ z \in L^2(0,L;H^1(0,1;d)) \mid Az \in L^2(\Omega),\ T_0\Big(az, -b\frac{\partial z}{\partial \eta}\Big) = 0 \Big\}$$

$$\mathcal{A}z = Az = -a\,\frac{\partial z}{\partial \xi} + b\,\frac{\partial^2 z}{\partial \eta^2} - cz\,, \quad \text{for all } z \in D(\mathcal{A}),$$

$$D(\mathcal{A}^*) = \Big\{ z \in L^2(0,L;H^1(0,1;d)) \mid$$

$$A^*z \in L^2(\Omega),\ T_1\Big(-a\phi, -\frac{\partial}{\partial \eta}(b\phi)\Big) = 0 \Big\},$$

$$\mathcal{A}^*z = A^*z = a\frac{\partial z}{\partial \xi} + \frac{\partial^2(bz)}{\partial \eta^2} - cz \quad \text{for all } z \in D(\mathcal{A}^*)\,,$$

T_0 is the normal trace on $\Gamma_0 = \big([0,L) \times \{0\}\big) \cup \big(\{0\} \times (0,1)\big)$,

T_1 is the normal trace on $\Gamma_1 = \big(\{L\} \times (0,1)\big) \cup \big((0,L] \times \{1\}\big)$,

Integral representation of Π

$$\Pi\varphi(x,y) = \int_\Omega \pi(x,y,\xi,\eta)\varphi(\xi,\eta)\,d\xi d\eta,$$

$$\pi \ : \ \mathcal{O} = \Omega_X \times \Omega_\Xi \longmapsto R$$

with

$$X = (x,y) \in \Omega_X, \quad \Xi = (\xi,\eta) \in \Omega_\Xi.$$

Equation satisfied by π

$$A_X^* \pi + A_\Xi^* \pi - \int_\gamma b^2 \, \pi(\Xi)\pi(X) \, ds + \phi(X)\phi(\Xi) = 0 \quad \text{in } \mathcal{O},$$

$$(b\pi)_y(x, 0, \Xi) = 0, \quad (b\pi)(x, 1, \Xi) = 0 \quad \text{in } (0, L) \times \Omega_\Xi,$$

$$(b\pi)_y(X, \xi, 0) = 0, \quad (b\pi)(X, \xi, 1) = 0 \quad \text{in } \Omega_X \times (0, L),$$

$$\sqrt{a}\pi(X, L, y) = 0 \quad \text{in } \Omega_X \times (0, 1),$$

$$\sqrt{a}\pi(L, y, \Xi) = 0 \quad \text{in } (0, 1) \times \Omega_\Xi,$$

$$\pi(X, \Xi) = \pi(\Xi, X) \geq 0 \quad \text{in } \mathcal{O},$$

and r is the solution to

$$-r_t = A^* r - \int_\gamma b^2\, r\, \pi(X) + \int_\Omega \pi(X) f(t) + c_0\, \phi(X)\ u_\infty$$

$$- \int_{(0,L)\times\{0\}} \pi(X) b g(t) + \int_0^1 \pi(X) a z_1(t),$$

$$(br)_y(x,0,t) = 0, \qquad (br)(x,1,t) = 0,$$

$$\sqrt{a} r(L,y,t) = 0,$$

$$r(\infty) = 0.$$

Numerical results - Comparison of solutions of the Prandtl system and of the linearized Crocco equation

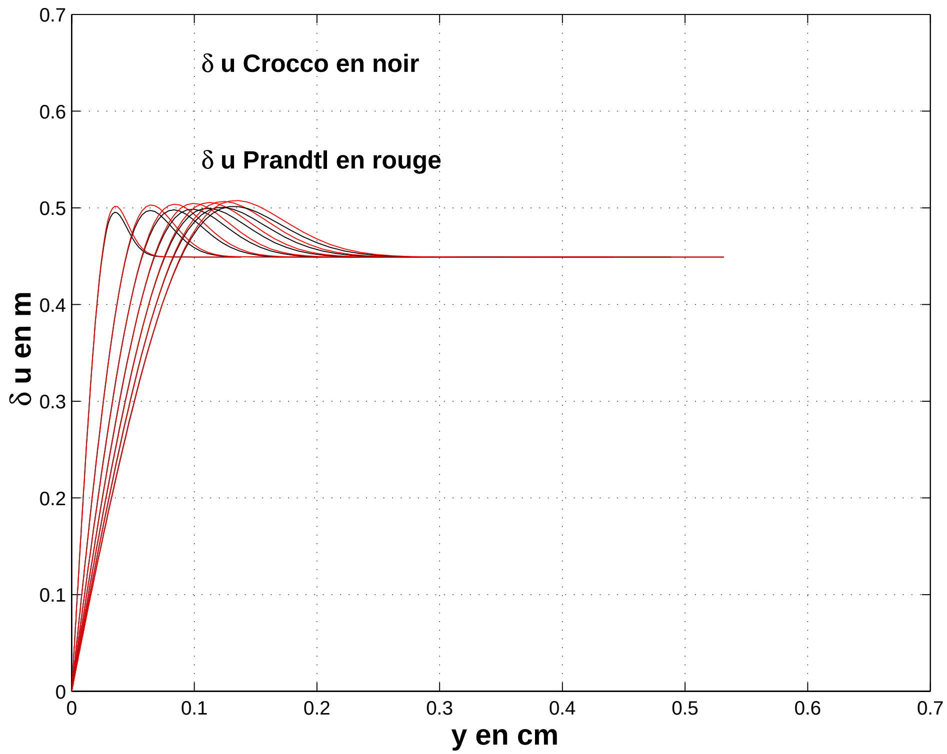
Data.

$U_\infty^0 = 45\, m/s$	$u_\infty = 0.5 sin^2(4\pi t)$
$V_p^0 = 0\, m/s$	$X_T^0 = 36.84\, cm$
$V_p^0 = -0.003\, m/s$	$X_T^0 = 40.5\, cm$

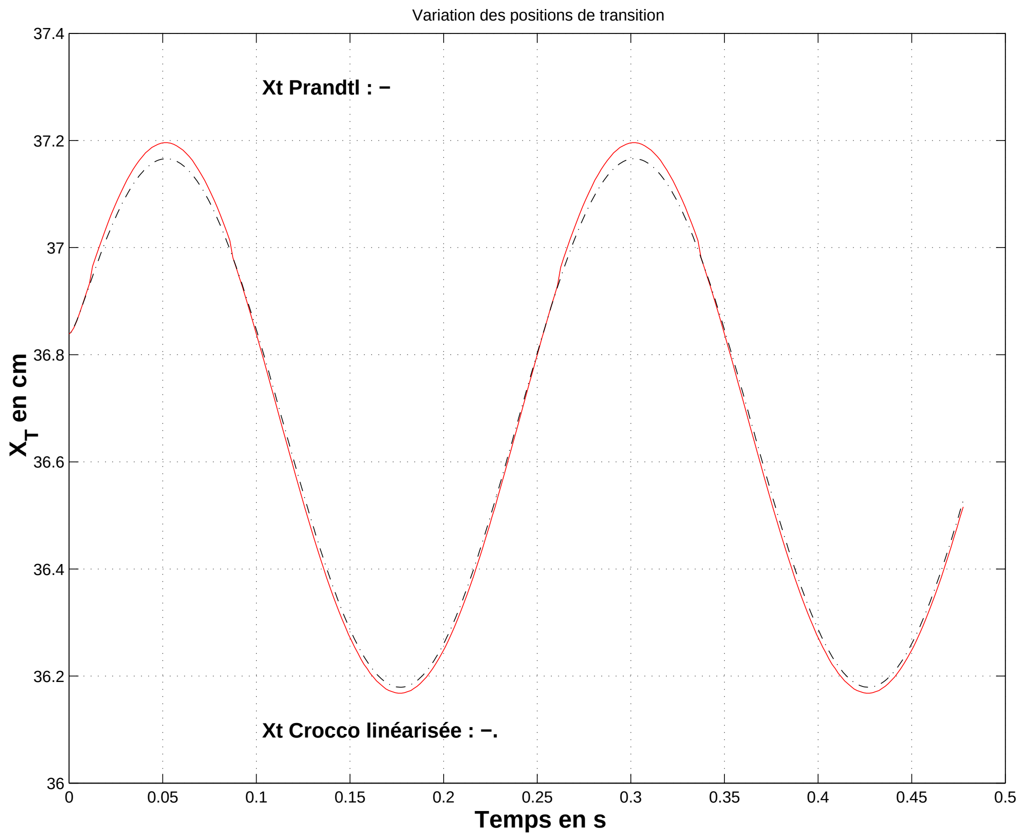
Mesh size for Prandtl and linearized Crocco equations.

$l = 0.003 \leq x \leq L = 0.53\, m$	$n = 200 * 100$	$\Delta x = 0.0025$	$\Delta t = 20\frac{\Delta}{U}$

Comparison of variations of the longitudinal velocity $\delta u(t,x,y) = u(t,x,y) - u^0(x,y)$ for different values of x and for $T = 0.1\,s$

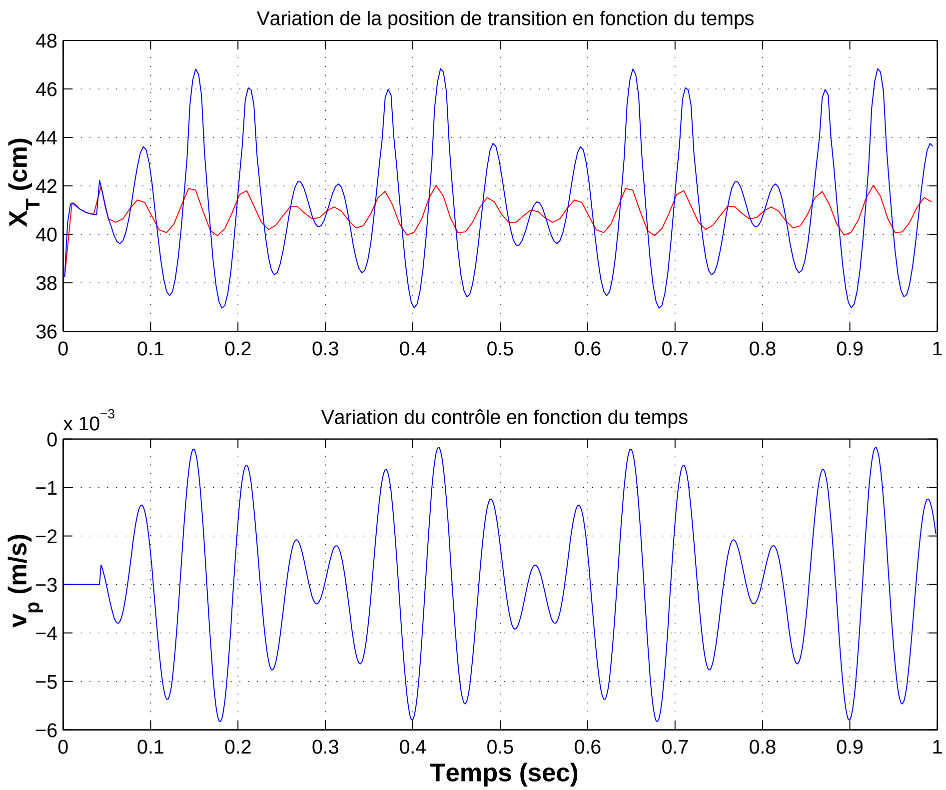


Comparison of variations of the transition location Prandtl and Linearized Crocco equations

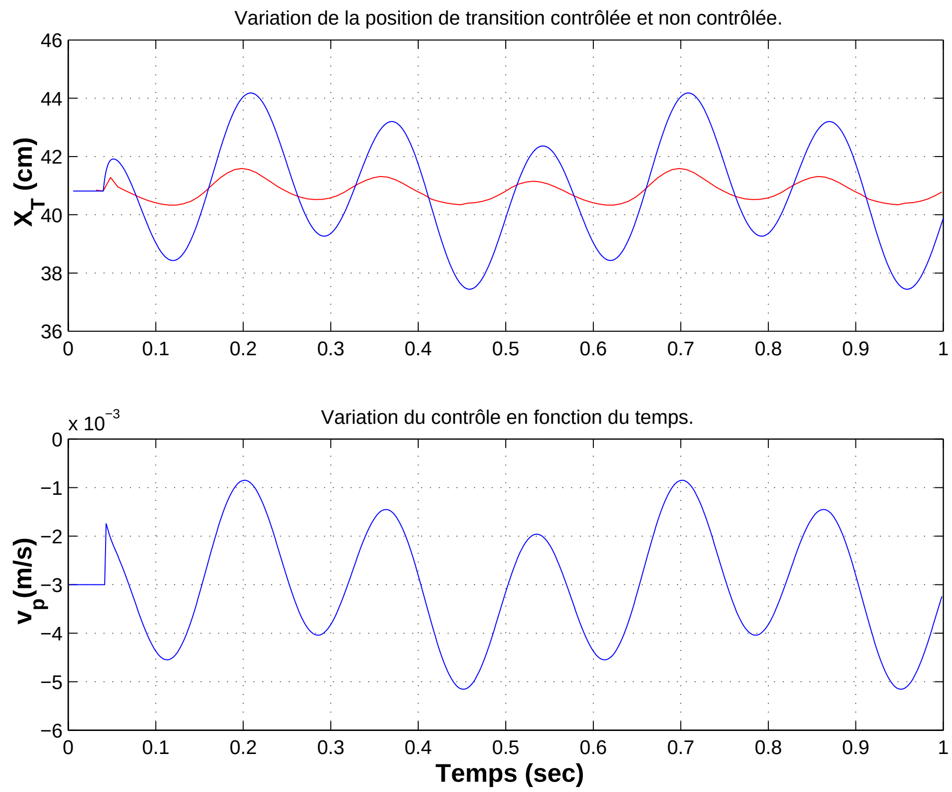


Variation of the controlled and the uncontrolled transition locations

$$u_{\infty}(\tau) = 0.5sin(4\pi\tau)cos(24\pi\tau).$$



$$u_\infty(\tau) = 1.5sin(4\pi\tau)cos(8\pi\tau).$$



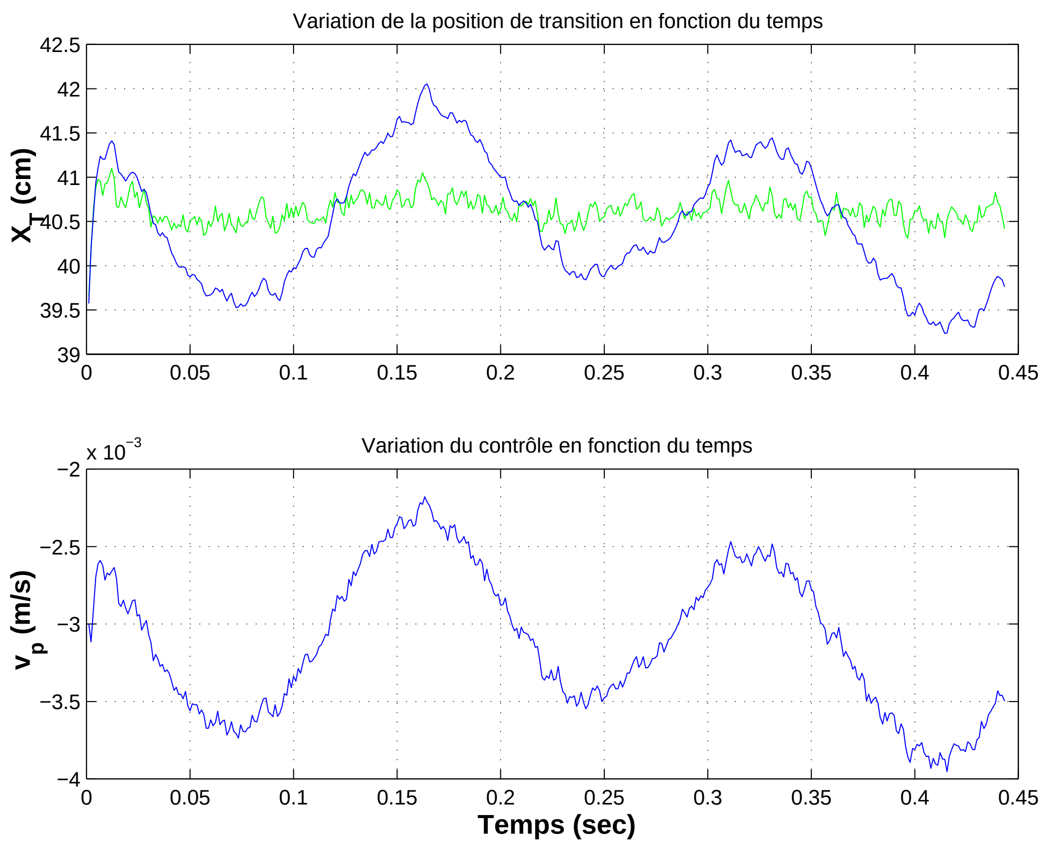
Numerical solution of the LQG problem

Observations:

1. first test: measurement of $x_T(t)$
2. $\frac{\partial u}{\partial y}$ is measured at 6 points of the plate (friction coefficient)

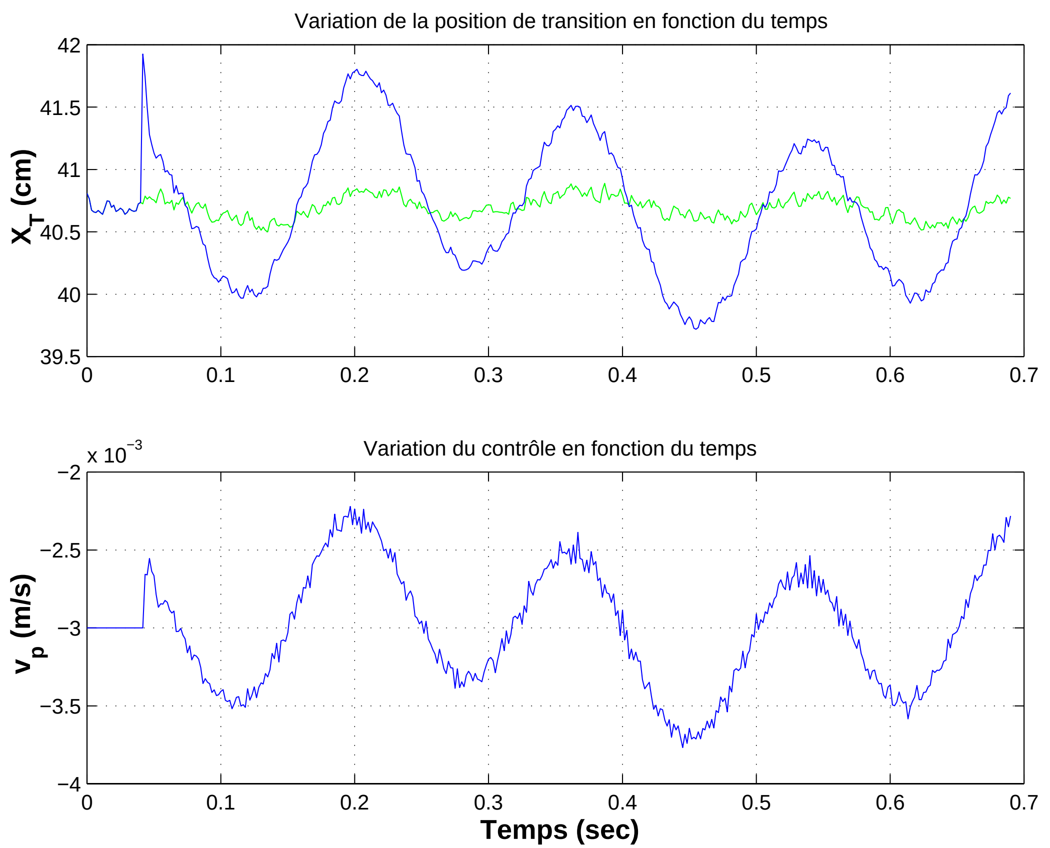
Variation of the controlled and the uncontrolled transition locations for noisy measurement of $x_T(t)$.

$$u_\infty = 0.5sin(4\pi t)cos(8\pi t)$$



Variation of the controlled and the uncontrolled transition locations for noisy measurement of $\frac{\partial u}{\partial y}$.

$$u_\infty = 0.5sin(4\pi t)cos(8\pi t)$$



Null controllability for a linearized Crocco type equation with constant coefficients

Find v so that the solution to

$$z_t + az_x - bz_{yy} + cz = 0,$$

$$z_y(x,0,t) = v(x,t)\chi_{(x_0,x_1)}$$
$$bz(x,1,t) = 0,$$

$$z(0,y,t) = z_1(y,t),$$

$$z(x,y,0) = z_0(x,y),$$

obeys

$$z(T) = 0 \qquad \text{in} \;\; \Omega_C(T,\delta).$$

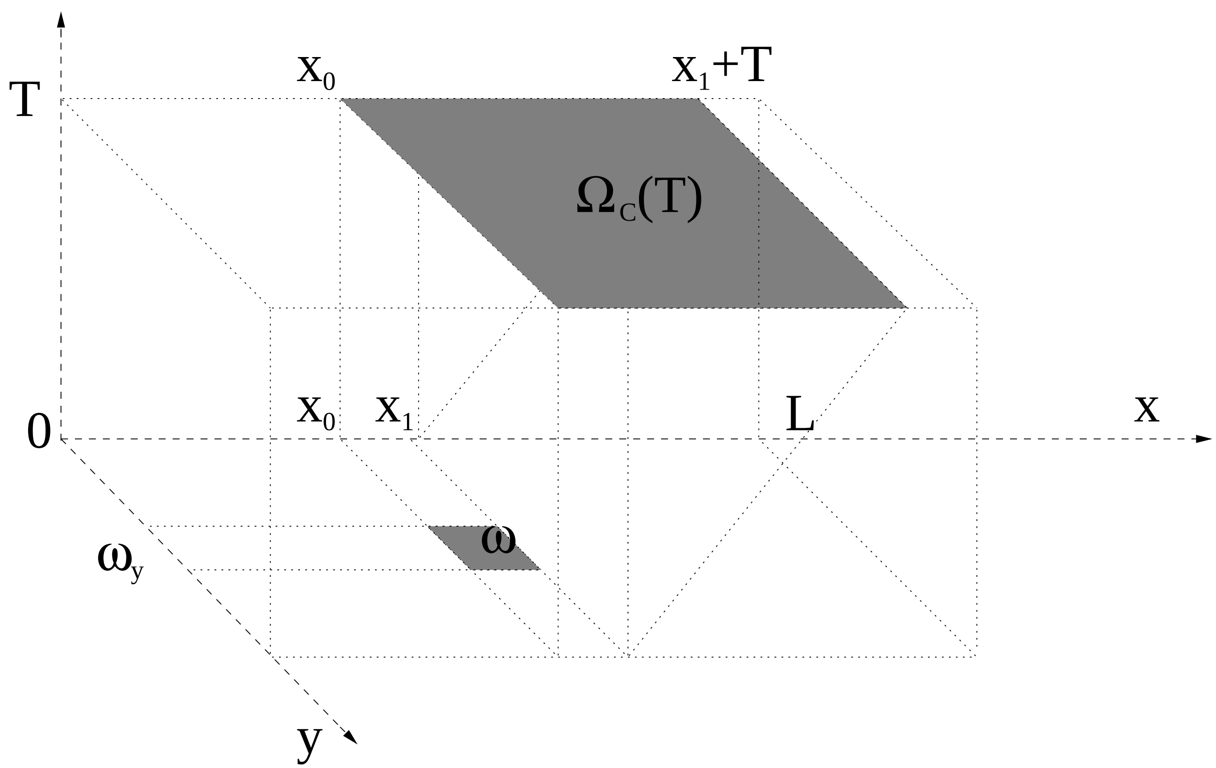
Distributed control

To simplify we set $a = 1$, $b = 1$, $c = 0$.

$$
\begin{aligned}
&z_t + z_x - z_{yy} = f\chi_\omega, && (x, y, t) \in \Omega \times (0, T), \\
&z(x, 0, t) = z(x, 1, t) = 0, && (x, t) \in (0, L) \times (0, T), \\
&z(0, y, t) = z_1(y, t), && (y, t) \in (0, 1) \times (0, T), \\
&z(x, y, 0) = z_0(x, y), && (x, y) \in \Omega.
\end{aligned}
$$

Influence domain of the control

$$\Omega_C(T) := (x_0, x_1 + T) \times (0, 1)$$



Negative result

Let $0 < T < L - x_1$ be fixed and

$$\Omega_C(T) := (x_0, x_1 + T) \times (0, 1).$$

There exists $z_0 \in L^2(\Omega)$, $z_1 \in L^2((0,1) \times (0,T))$ such that, for all $f \in L^2(\omega \times (0,T))$, the solution z of the Crocco type equation is not identically equal to zero in $\Omega \setminus \Omega_C(T)$.

Theorem (M, R, V, 2003)

For all $z_0 \in L^2(\Omega)$, $z_1 \in L^2((0,1) \times (0,T))$, there exists $f \in L^2(\omega \times (0,T))$ such that the solution z_f of the Crocco type equation satisfies

$$z_f(x,y,T) = 0 \quad \text{for } (x,y) \in \Omega_C(T,\delta),$$

for any $0 < \delta < (x_1 - x_0)/2$, where

$$\Omega_C(T,\delta) := (x_0 + \delta, x_1 + T - \delta) \times (0,1).$$

The case of a boundary control

Let $0 < x_0 < x_1 < L$ be fixed. For $z_0 \in L^2(\Omega)$, $z_1 \in L^2((0,1) \times (0,T))$ and $f \in L^2((x_0,x_1) \times (0,T))$, we consider:

$$\begin{cases} z_t + z_x - z_{yy} = 0, & (x,y,t) \in \Omega \times (0,T), \\ z(x,0,t) = 0, & (x,t) \in (0,L) \times (0,T), \\ z(x,1,t) = \chi_{(x_0,x_1)}(x) f(x,t), & (x,t) \in (0,L) \times (0,T), \\ z(0,y,t) = z_1(y,t), & (y,t) \in (0,1) \times (0,T), \\ z(x,y,0) = z_0(x,y), & (x,y) \in \Omega. \end{cases}$$

For $f \in L^2((x_0,x_1) \times (0,T))$, $u \in L^2(\Omega \times (0,T)) \cap C([0,T]; L^2(0,L; H^{-1}(0,1))) \cap C([0,L]; L^2(0,T; H^{-1}(0,1)))$.

Theorem (M, R, V, 2003)

For all $z_0 \in L^2(\Omega)$, $z_1 \in L^2((0,1) \times (0,T))$, there exists $f \in L^2((x_0, x_1) \times (0,T))$ such that the solution z of the Crocco type equation satisfies

$$z(x,y,T) = 0 \quad \text{for } (x,y) \in \Omega_C(T,\delta).$$

Observation inequality

There exists $C(T,\delta) > 0$ such that the solutions p of the adjoint equation

$$p_t + p_x + p_{yy} = 0$$

belonging to $C([0,T]; L^2(0,L; H_0^1(0,1))) \cap C([0,L]; L^2(0,T; H_0^1(0,1))) \cap L^2((0,L) \times (0,T); H^2 \cap H_0^1(0,1))$, satisfy

$$\iint_{(0,L)\times(0,1)} p(x,y,0)^2\,dydx \;+\; \iint_{(0,1)\times(0,T)} p(0,y,t)^2\,dtdy$$
$$\leq\; C\,(T,\delta)\left(\iint_{(0,T)\times(x_0,x_1)} p_y(x,1,t)^2\,dxdt\right.$$
$$\left.+\;\iint_{\Omega\setminus\Omega_C(T,\delta)} p(x,y,T)^2\,dydx \;+\; \iint_{(0,1)\times(0,T)} p(L,y,t)^2\,dtdy\right).$$

Open problem.

If we replace the equation with constant coefficients

$$z_t + z_x - z_{yy} = 0,$$

by

$$z_t + y z_x - z_{yy} = 0,$$

or

$$z_t + z_x - (1-y)^2 z_{yy} = 0,$$

the null controllability is an open problem.

Some results have been obtained by Cannarsa, Martinez and Vancostenoble for the degenerate parabolic equation

$$z_t - ((1-y)^\alpha z_y)_y = u\chi_\omega, \qquad \text{for } 0 \leq \alpha < 2.$$

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